QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Tuesday January 20, 2026 (Day 1)

1. **(Algebra)** In this question, $\mathfrak{g}$ is a non-zero, finite-dimensional Lie algebra over $\mathbb{C}$.
   (a) Define the Killing form on $\mathfrak{g}$.
   (b) Characterize semisimplicity for $\mathfrak{g}$ in terms of the Killing form.
   (c) The Lie algebra $\mathfrak{g}$ is *nilpotent* if its lower central series terminates at $\{0\}$. Show that the Killing form of a nilpotent Lie algebra is zero.
   (d) Exhibit a 2-dimensional $\mathfrak{g}$ that is neither semisimple nor nilpotent.

2. **(Algebraic Geometry)** Let $\Lambda \subseteq \mathbb{P}^6$ be a fixed 3-plane and let $\mathbb{G}(4,6)$ be the Grassmannian of 4-planes in $\mathbb{P}^6$. Let

$$\Sigma = \{\, \Gamma : \dim(\Gamma \cap \Lambda) \geq 2 \,\} \subseteq \mathbb{G}(4,6).$$

   Show that Σ is irreducible and compute the dimension of Σ.

3. **(Algebraic Topology)** Let $F_2 = \langle a, b \rangle$ denote the free group on two letters a, b. Consider the homomorphism $f : F_2 \to \mathbb{Z}/2\mathbb{Z}$ defined by $f(a) = f(b) = 1$.
   (a) Draw the cover of $S^1 \vee S^1$ corresponding to the subgroup $\ker(f)$ of $\pi_1(S^1 \vee S^1) \cong F_2$.
   (b) There is a group isomorphism $\ker(f) \cong F_r$ for some $r \geq 1$, where F_r denotes the free group on r letters. Determine r.

4. **(Complex Analysis)** Prove that

$$\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$$

   by applying the residue theorem to the meromorphic function

$$f(z) = \frac{\pi \cot(\pi z)}{z^2}$$

   integrated over the boundary of the rectangle R_N with vertices $\pm(N + \frac{1}{2}) \pm i(N + \frac{1}{2})$, and letting $N \to \infty$.

5. **(Differential Geometry)** Prove that

$$M := \{x_1^2 + x_2^2 - x_3^2 - x_4^4 = 0\} \cap \{x_1^2 + x_2^2 + x_3^2 + x_4^4 = 4\}$$

   is a 2-dimensional submanifold of $\mathbb{R}^4$. Compute the tangent space of M at the point $(1, 1, -1, -1)$.

6. (Real Analysis) Let $n \geq 3$ be an integer and ω be the volume of the unit sphere in $\mathbb{R}^n$. Let

$$K(x) = \frac{-1}{(n-2)\omega} \frac{1}{|x|^{n-2}}.$$

Let δ_0 be the Dirac delta in $\mathbb{R}^n$ which means that the value of δ_0 at a C^∞ function f with compact support on $\mathbb{R}^N$ is equal to $f(0)$. Let

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

the Laplacian on $\mathbb{R}^n$ with coordinates $x_1, \cdots, x_n$. Prove the identity

$$\Delta K = \delta_0$$

as distributions on $\mathbb{R}^n$. In other words, for any C^∞ function f on $\mathbb{R}^n$ with compact support the identity

$$\int_{\mathbb{R}^n} K(x)(\Delta f)(x) = f(0)$$

holds.

QUALIFYING EXAMINATION

Harvard University

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Wednesday January 21, 2026 (Day 2)

1. (Algebra) Let G be a non-abelian group of order 12. Show that G has either 4 or 6 irreducible complex representations, and show that both of these possibilities do occur.

2. (Algebraic Geometry)

(a) For each ring R below, determine whether R is the coordinate ring of an affine variety (not necessarily irreducible).

- $R = \mathbb{C}[x]/(x^3 - 2x^2 + x)$.
- $R = \mathbb{C}[x]/(x^3 - 1)$.

(b) Consider the following affine varieties

$$X = V(xy(x-y)) \subseteq \mathbb{A}^2_{\mathbb{C}}, \qquad Y = V(xy, yz, xz) \subseteq \mathbb{A}^3_{\mathbb{C}}.$$

Are X and Y isomorphic varieties?

3. (Algebraic Topology) Consider $S^2 \times S^2$ with the product orientation. Let $u \in H^2(S^2;\mathbb{Z}) \cong \mathbb{Z}$ be the positive generator, and set

$$x := \pi_1^* u, \quad y := \pi_2^* u \in H^2(S^2 \times S^2;\mathbb{Z}),$$

where $\pi_i : S^2 \times S^2 \to S^2$ denotes the projection to the i-th factor. Suppose $f : S^2 \times S^2 \to S^2 \times S^2$ is a continuous map of degree 1 with no fixed points. Prove that

$$f^* x = -x, \quad f^* y = -y$$

in $H^2(S^2 \times S^2;\mathbb{Z})$.

4. (Complex Analysis) Let $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disk. Suppose $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic function with two distinct fixed points $a \neq b \in \mathbb{D}$. Prove that $f(z) = z$ for all $z \in \mathbb{D}$.

5. (Differential Geometry) Consider the disk $D^2 = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$ with the metric

$$g = \frac{1}{1 - (x^2 + y^2)} (dx \otimes dx + dy \otimes dy) .$$

Compute the Levi-Civita connection of the corresponding Riemann manifold.

6. (Real Analysis) Suppose that f_j ($j = 1, 2, \dots$) and f are real functions on $[0,1]$. We say that $f_j \to f$ *in measure* if and only if for any $\varepsilon > 0$ we have

$$\lim_{j\to\infty} \mu\,\{\, x \in [0,1] : |f_j(x) - f(x)| > \varepsilon \,\} = 0,$$

where μ is the Lebesgue measure on $[0,1]$. In this problems, all functions are assumed to be in $L^1[0,1]$.

(a) Suppose that $f_j \to f$ in measure. Does it follow that

$$\lim_{j\to\infty} \int |f_j(x) - f(x)|\,\mathrm{d}x = 0.$$

Prove it or give a counterexample.

(b) Suppose that $f_j \to f$ in measure. Does it follow that $f_j(x) \to f(x)$ almost everywhere in $[0,1]$? Prove it or give a counter example.

(c) Suppose that $f_j(x) \to f(x)$ almost everywhere in $[0,1]$. Does it follow that $f_j \to f$ in measure? Prove it or give a counter example.

QUALIFYING EXAMINATION

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Thursday January 22, 2026 (Day 3)

1. **(Algebra)** Let k be a field. Let K/k be a finite separable extension, and L/k be an arbitrary extension. Prove that the commutative k-algebra $K \otimes_k L$ splits as a finite product of finite separable extensions of L.

   Hint. You may find it useful to apply the theorem of the primitive element.

2. **(Algebraic Geometry)** Let $X = \mathrm{Bl}_0(\mathbb{A}^2)$ be the blow-up of $\mathbb{A}^2$ at the origin.

   (a) Using local coordinates, identify the exceptional divisor E and show that $E \simeq \mathbb{P}^1$.

   (b) Show that the strict transform of the curve $C = \{(x, y) \in \mathbb{A}^2 \,|\, y^2 = x^3\}$ is smooth.

3. **(Algebraic Topology)** Let $n \geq 1$. Compute the homotopy groups $\pi_k(\mathbb{CP}^n)$, for each $1 \leq k \leq 2n$.

4. **(Complex Analysis)** Suppose f is a doubly-periodic meromorphic function on $\mathbb{C}$ with periods ω_1, ω_2 which are $\mathbb{R}$-linearly independent. Let $a \in \mathbb{C}$ such that the sides of the parallelogram Ω with vertices $a, a+\omega_1, a+\omega_2, a+\omega_1+\omega_2$ do not contain any zeroes or poles of f. Let $b_1, \cdots, b_p$ (respectively $a_1, \cdots, a_q$) be the zeroes (respectively the poles) of f with multiplicities $k_1, \cdots, k_p$ (respectively $\ell_1, \cdots, \ell_q$) inside Ω. By considering the residues of the function

$$\frac{1}{2\pi i} \frac{w f'(w)}{f(w)} dw$$

   or otherwise, prove that

$$\left(\sum_{\mu=1}^{p} k_\mu b_\mu \right) - \left(\sum_{\nu=1}^{q} \ell_\nu c_\nu \right)$$

   belongs to $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. In other words, in a fundamental parallelogram the sum of the coordinates of the zeroes of an elliptic function equals the sum of the coordinates of its poles modulo a period.

5. **(Differential Geometry)**

   (a) Compute $H^k_{\mathrm{dR}}(\mathbb{R}^n \setminus \{0\})$ for all k.

(b) Show that the $(n-1)$-form

$$\eta = \frac{1}{||x||^n} \sum_{i=1}^{n} (-1)^{i-1} x_i dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n$$

is closed on $\mathbb{R}^n \setminus \{0\}$ and $\int_{S^{n-1}} \eta = \mathrm{Vol}(S^{n-1})$.

(c) Deduce that $[\eta]$ generates $H^{n-1}_{\mathrm{dR}}(\mathbb{R}^n \setminus \{0\})$.

6. (Real Analysis) Let f be a bounded real-valued function on $X = [0,1] \subset \mathbb{R}$, and define a function $\phi : [1, \infty) \to \mathbb{R}$ by

$$\phi(p) = \|f\|^p_{L^p(X)}.$$

Prove that ϕ is convex.