**QUALIFYING EXAMINATION**

Harvard University

Department of Mathematics

Tuesday January 20, 2026 (Day 1)

**1. (Algebra)** In this question, $\mathfrak{g}$ is a non-zero, finite-dimensional Lie algebra over $\mathbb{C}$.

(a) Define the Killing form on $\mathfrak{g}$.

(b) Characterize semisimplicity for $\mathfrak{g}$ in terms of the Killing form.

(c) The Lie algebra $\mathfrak{g}$ is *nilpotent* if its lower central series terminates at $\{0\}$. Show that the Killing form of a nilpotent Lie algebra is zero.

(d) Exhibit a 2-dimensional $\mathfrak{g}$ that is neither semisimple nor nilpotent.

**Solution.**

(a) The Killing form is the symmetric bilinear form $(a, b) = \mathrm{tr}(\mathrm{ad}(a)\mathrm{ad}(b))$.

(b) The Lie algebra is semisimple if and only if the Killing form is non-degenerate.

(c) In a nilpotent Lie algebra, the subalgebra $[\mathfrak{g}, [\mathfrak{g}, \ldots [\mathfrak{g}, \mathfrak{g}] \ldots]]$ is eventually zero. In particular $[x, [y, [x, [y, \ldots, [x, [y, z]] \ldots]$ is zero for all $x$ and $y$ and $z$ eventually; which is to say that $(\mathrm{ad}(x)\mathrm{ad}(y))^n z$ is eventually zero. That is (since $\mathfrak{g}$ is finite-dimensional), $\mathrm{ad}(x)\mathrm{ad}(y)$ is nilpotent. Its trace is therefore zero.

(d) The Lie algebra of traceless uppper-triangular 2-by-2 matrices has a Killing form of rank 1. (Its kernel is the 1-dimensional subalgebra of diagonal, traceless matrices.)

**2. (Algebraic Geometry)** Let $\Lambda \subseteq \mathbb{P}^6$ be a fixed 3-plane and let $\mathbb{G}(4, 6)$ be the Grassmannian of 4-planes in $\mathbb{P}^6$. Let

$$\Sigma = \{\, \Gamma : \dim(\Gamma \cap \Lambda) \geq 2 \,\} \subseteq \mathbb{G}(4, 6).$$

Show that $\Sigma$ is irreducible and compute the dimension of $\Sigma$.

**Solution.** Consider the incidence variety

$$\Phi = \{(A, B) : A \subseteq B\} \subseteq \mathbb{G}(2, 3) \times \mathbb{G}(4, 6)$$

where we have identified $\mathbb{G}(2, 3)$ with the Grassmannian of 2-planes in $\Lambda \cong \mathbb{P}^3$ contained in $\mathbb{P}^6$. The projection $\pi_1 : \Phi \to \mathbb{G}(2, 3)$ is surjective. The fiber $\pi_1^{-1}(A)$ over any $A \in \mathbb{G}(2, 3)$ is the set of 5-dimensional subspaces of a 7-dimensional vector space $V$ containing a 3-dimensional subspace $A$, or the set of 2-dimensional subspaces of the 4-dimensional vector space $V/A$. In other words, $\pi_1$ has fibers $\mathbb{G}(1,3)$ of dimension 4. Because $\mathbb{G}(2,3)$ has dimension 3,

$$\dim(\Phi) = 3 + 4 = 7.$$

Moreover, $\Phi$ is irreducible because $\mathbb{G}(2,3)$ is irreducible, and all fibers of are irreducible of the same dimension.

The second projection $\pi_2 : \Phi \to \mathbb{G}(4,6)$ has image equal to $\Sigma$. There exist $B \in \Sigma \subseteq \mathbb{G}(4,6)$ with $\dim(B \cap \Lambda) = 2$, and the fiber $\pi_2^{-1}(B)$ consists of a single point $(B \cap \Lambda, B) \in \mathbb{G}(2,3) \times \mathbb{G}(4,6)$ for any such $B$. By upper-semicontinuity of fiber dimension, $\pi_2$ is generally one-to-one onto $\Sigma$, and

$$\dim(\Sigma) = \dim(\Phi) = 7.$$

Finally, $\Sigma$ is irreducible because it is the image of an irreducible projective variety $\Phi$.

**3. (Algebraic Topology)** Let $F_2 = \langle a, b \rangle$ denote the free group on two letters $a, b$. Consider the homomorphism $f : F_2 \to \mathbb{Z}/2\mathbb{Z}$ defined by $f(a) = f(b) = 1$.

(a) Draw the cover of $S^1 \vee S^1$ corresponding to the subgroup $\ker(f)$ of $\pi_1(S^1 \vee S^1) \cong F_2$.

(b) There is a group isomorphism $\ker(f) \cong F_r$ for some $r \geq 1$, where $F_r$ denotes the free group on $r$ letters. Determine $r$.

**Solution.**

(a) The subgroup $\ker(f)$ has generators

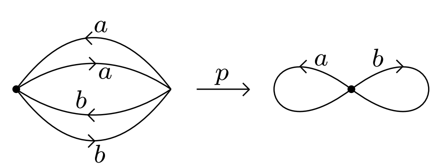
$$\ker(f) = \langle a^2, b^2, ab \rangle.$$

To see this, first observe that $\ker(f)$ is the subgroup of elements of even word length, so it suffices to show that any word $w$ of even length in $F_2$ is contained in $\langle a^2, b^2, ab \rangle$. First, if $w$ has length 2 then $w$ is one of

$$a^2, \quad b^2, \quad ab, \quad ba = b^2(ab)^{-1}a^2, \quad a^{-1}b = a^{-2}(ab), \quad ab^{-1} = (ab)b^{-2},$$

or their inverses. Therefore, $\langle a^2, b^2, ab \rangle$ contains any word of length 2, and hence any word of even length.

Consider the following cover $p : X \to S^1 \vee S^1$



Contracting one of the edges of $X$ shows that $X$ is homotopy equivalent to a wedge of three circles and that three generators of $\pi_1(X) \cong F_3$ have images $a^2$, $ab$, and $b^2$. Therefore, $p_*(\pi_1(X)) = \ker(f)$.

(b) Let $p : X \to S^1 \vee S^1$ be the cover coresponding to $\ker(f)$. The solution to part (a) shows that $X$ is homotopy equivalent to $S^1 \vee S^1 \vee S^1$, and so $\pi_1(X) \cong F_3$. Recall that $p_* : \pi_1(X) \to \pi_1(S^1 \vee S^1)$ is injective and has image $\ker(f)$ by construction, and hence

$$\ker(f) \cong F_3.$$

**4. (Complex Analysis)** Prove that

$$\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$$

by applying the residue theorem to the meromorphic function

$$f(z) = \frac{\pi \cot(\pi z)}{z^2}$$

integrated over the boundary of the rectangle $R_N$ with vertices $\pm(N + \frac{1}{2}) \pm i(N + \frac{1}{2})$, and letting $N \to \infty$.

**Solution.** The function $\pi \cot(\pi z)$ has simple poles at all integers $n \in \mathbb{Z}$ with residue 1, so for any non-zero integer $n$,

$$\mathrm{Res}_{z=n} \frac{\pi \cot(\pi z)}{z^2} = \frac{1}{n^2}.$$

At $z = 0$,

$$\frac{\pi \cot(\pi z)}{z^2} = z^{-3} - \frac{\pi^2}{3} z^{-1} + O(z),$$

so

$$\mathrm{Res}_{z=0} \frac{\pi \cot(\pi z)}{z^2} = -\frac{\pi^2}{3}.$$

Therefore, applying the residue theorem, we have

$$\frac{1}{2\pi i} \int_{\partial R_N} f(z) dz = \sum_{n=-N}^{N} \mathrm{Res}_{z=n} f(z) = -\frac{\pi^2}{3} + 2 \left( \sum_{n=1}^{N} \frac{1}{n^2} \right),$$

and thus it suffices to show that

$$\lim_{N \to \infty} \int_{\partial R_N} f(z) dz = 0$$

to conclude the proof.

For this, observe that there is a uniform upper bound $C$ of $|\pi \cot(\pi z)|$ on $\partial R_N$, independent of $N$. For instance, we can take

$$C = \pi \frac{1 + e^{-\pi}}{1 - e^{-\pi}},$$

since

- on the vertical sides,

$$\left|\pi \cot\left(\pi\big(\pm(N+\frac{1}{2})+it\big)\right)\right| = |\pi\cot(\frac{\pi}{2}+i\pi t)| = |\pi\tanh(\pi t)| \le \pi,$$

- and on the horizontal sides,

$$\left|\pi \cot\left(\pi\big(t \pm i(N+\frac{1}{2})\big)\right)\right| = \pi\left|\frac{e^{2\pi i(t\pm i(N+\frac{1}{2}))}+1}{e^{2\pi i(t\pm i(N+\frac{1}{2}))}-1}\right| \le \pi\frac{1+e^{-\pi(2N+1)}}{1-e^{-\pi(2N+1)}} \le \pi\frac{1+e^{-\pi}}{1-e^{-\pi}}.$$

Hence, since the perimeter of $\partial R_N$ is $8(N+\frac{1}{2})$,

$$\left|\int_{\partial R_N} f(z)dz\right| \le \int_{\partial R_N}|f(z)| \le \frac{C}{(N+\frac{1}{2})^2}\cdot 8(N+\frac{1}{2}) = \frac{8C}{N+\frac{1}{2}} \xrightarrow{N\to\infty} 0.$$

**5. (Differential Geometry)** Prove that

$$M := \{x_1^2+x_2^2-x_3^2-x_4^4=0\}\cap\{x_1^2+x_2^2+x_3^2+x_4^4=4\}$$

is a 2-dimensional submanifold of $\mathbb{R}^4$. Compute the tangent space of $M$ at the point $(1,1,-1,-1)$.

**Solution.** Let $F:\mathbb{R}^4\to\mathbb{R}^2$ be the smooth function given by

$$F(x_1,x_2,x_3,x_4) = (x_1^2+x_2^2-x_3^2-x_4^4, x_1^2+x_2^2+x_3^2+x_4^4-4)\,.$$

Observe that $M=F^{-1}(0)$, so that it will be enough to show that 0 is a regular value of $F$. We have

$$\nabla F(x_1,x_2,x_3,x_4) = \begin{bmatrix} 2x_1 & 2x_2 & -2x_3 & -4x_4^3 \\ 2x_1 & 2x_2 & 2x_3 & 4x_4^3 \end{bmatrix}.$$

But, if $(x_1,x_2,x_3,x_4)\in M$, then $x_1^2+x_2^2=2$ so that at most one of $x_1$ and $x_2$ vanishes. Likewise, we have $x_3^2+x_4^4=2$, so that at most one of $x_3$ and $x_4$ vanishes. It follows that the above matrix has full rank on every point of $M$, which proves the first part.

The tangent space of the submanifold $M$ at the point $p=(1,1,-1,-1)$ is the kernel of the linear map

$$\nabla F(1,1,-1,-1) = \begin{bmatrix} 2 & 2 & 2 & 4 \\ 2 & 2 & -2 & -4 \end{bmatrix},$$

which is spanned by the two vectors

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix},\quad \begin{bmatrix} 0 \\ 0 \\ 2 \\ -1 \end{bmatrix} \in T_p\mathbb{R}^4 \cong \mathbb{R}^4\,.$$

**6. (Real Analysis)** Let $n \geq 3$ be an integer and $\omega$ be the volume of the unit sphere in $\mathbb{R}^n$. Let

$$K(x) = \frac{-1}{(n-2)\omega} \frac{1}{|x|^{n-2}}.$$

Let $\delta_0$ be the Dirac delta in $\mathbb{R}^n$ which means that the value of $\delta_0$ at a $C^\infty$ function $f$ with compact support on $\mathbb{R}^N$ is equal to $f(0)$. Let

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

the Laplacian on $\mathbb{R}^n$ with coordinates $x_1, \cdots, x_n$. Prove the identity

$$\Delta K = \delta_0$$

as distributions on $\mathbb{R}^n$. In other words, for any $C^\infty$ function $f$ on $\mathbb{R}^n$ with compact support the identity

$$\int_{\mathbb{R}^n} K(x)(\Delta f)(x) = f(0)$$

holds.

**Solution.** First, straightforwardly verify that

$$\Delta \frac{1}{|x|^{n-2}} \equiv 0$$

on $\mathbb{R}^n - \{0\}$ as follows. From

$$\frac{\partial}{\partial x_j} \frac{1}{|x|^{n-2}} = \frac{\partial}{\partial x_j} \frac{1}{(|x|^2)^{\frac{n-2}{2}}} = -\frac{n-1}{2} \frac{2x_j}{(|x|^2)^{\frac{n}{2}}}$$

and

$$\frac{\partial^2}{\partial x_j^2} \frac{1}{|x|^{n-2}} = -\frac{n-1}{2} \frac{2}{(|x|^2)^{\frac{n}{2}}} + \frac{(n-1)n}{4} \frac{(2x_j)^2}{(|x|^2)^{\frac{n+2}{2}}}$$

on $\mathbb{R}^n - \{0\}$ it follows that

$$\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \frac{1}{|x|^{n-2}} = -\frac{(n-1)n}{(|x|^2)^{\frac{n}{2}}} + \frac{(n-1)n}{(|x|^2)^{\frac{n}{2}}} = 0$$

on $\mathbb{R}^n - \{0\}$. For $\eta > 0$ let $B(\eta)$ be the closed ball of radius $\eta$ in $\mathbb{R}^n$ centered at the origin. Apply the divergence theorem to

$$\operatorname{div}(f \operatorname{grad} K) - \operatorname{div}(K \operatorname{grad} f) = (f\Delta K) - (K\Delta f) = -(K\Delta f)$$

on $\mathbb{R}^n - B(\eta)$, where div is the divergence operator and grad is the gradient operator. Let $\vec{\nu}$ be the unit outward-pointing normal vector of the boundary $\partial B(\eta)$ of $B(\eta)$. Then

$$\int_{\partial B(\eta)} f(\operatorname{grad} K) \cdot \vec{\nu} - \int_{\partial B(\eta)} K(\operatorname{grad} f) \cdot \vec{\nu} = -\int_{\mathbb{R}^n - B(\eta)} K\Delta f.$$

Since $K = O\left(\frac{1}{|z|^{n-2}}\right)$ and the volume of $\partial B(\eta)$ is $0\left(\eta^{n-1}\right)$ and $f$ is $C^\infty$, as $\eta \to 0$ the term

$$\int_{\partial B(\eta)} K(\text{grad } f) \cdot \vec{\nu}$$

goes to zero. Since

$$(\text{grad } K) \cdot \vec{\nu} = \frac{-1}{\omega \eta^{n-1}} + (\text{lower order terms})$$

and the volume of $\partial B(\eta)$ is $\omega \eta^{n-1}$, it follows that as $\eta \to 0$ the term

$$\int_{\partial B(\eta)} f(\text{grad } K) \cdot \vec{\nu}$$

approaches $-f(0)$. This finishes the proof that

$$f(0) = \int_{\mathbb{R}^n} K \Delta f.$$

# QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday January 21, 2026 (Day 2)

**1. (Algebra)** Let $G$ be a non-abelian group of order 12. Show that $G$ has either 4 or 6 irreducible complex representations, and show that both of these possibilities do occur.

**Solution.** The order of $G$ is the sum of the squares of the dimensions of the irreducible representations; and the number of 1-dimensional representations is the order of the abelianization $G/[G,G]$, which must divide $|G|$. Going systematically we find that the only ways to write 12 as a sum of squares with the constraint that the number of 1's divides 12 are the following:

$$12 = 1^2 + \cdots + 1^2, \quad \text{(12 times)},$$
$$12 = 3^2 + 1^2 + 1^2 + 1^2$$
$$12 = 2^2 + 2^2 + 1^2 + 1^2 + 1^2 + 1^2.$$

In the first case, $G$ is abelian. So for non-abelian $G$, only the last two possibilities are feasible, and there are either 4 or 6 irreducible representations, whose dimensions are either $(3,1,1,1)$ or $(2,2,1,1,1,1)$ respectively.

Consider now the group $A_4$, which has order 12. It has four conjugacy classes, represented by the elements $e$, $(12)(34)$, $(123)$ and $(132)$. This group therefore has four irreducible representations, because the number of irreducible representations is equal to the number of conjugacy classes. Alternatively, $A_4$ is the group of rotational symmetries of the regular tetrahedron, so it has an irreducible 3-dimensional representation, and therefore the representations must be $(3,1,1,1)$ by the above classification. Alternatively again, the abelianization has order 3, so there must be exactly three 1-dimensional representations, again implying that it must be $(3,1,1,1)$.

Consider next the group dihedral group $D$ of order 12, presented as $\langle\, r, s \mid r^6 = s^2 = e, srs = r^{-1}\,\rangle$ (so that $r$ is a rotation through $2\pi/6$ in the plane and $s$ is a reflection, in the usual way). There are 6 conjugacy classes, represented by the elements $e$, $r$, $r^2$, $r^3$, $s$ and $rs$. (As symmetries of the hexagon, the latter two are reflections in a line through vertices and a line through midpoints of edges, respectively.) The group therefore has 6 irreducible representations. Alternatively, the abelianization has order 4, implying that there are 4 abalian characters and we must be in the case $(2,2,1,1,1,1)$ by the classification.

**2. (Algebraic Geometry)**

(a) For each ring $R$ below, determine whether $R$ is the coordinate ring of an affine variety (not necessarily irreducible).

- $R = \mathbb{C}[x]/(x^3 - 2x^2 + x)$.
- $R = \mathbb{C}[x]/(x^3 - 1)$.

(b) Consider the following affine varieties

$$X = V(xy(x-y)) \subseteq \mathbb{A}^2_{\mathbb{C}}, \qquad Y = V(xy, yz, xz) \subseteq \mathbb{A}^3_{\mathbb{C}}.$$

Are $X$ and $Y$ isomorphic varieties?

**Solution.**

(a) The ring $\mathbb{C}[x]/(x^3 - 2x^2 + x)$ is not the coordinate ring of any affine variety. Factor

$$x^3 - 2x^2 + x = x(x-1)^2$$

and observe that $x(x(x-1)^2) = (x^2 - x)^2$ is contained in the ideal $(x^3 - 2x^2 + x) \subseteq \mathbb{C}[x]$, while $x^2 - x$ is not. Therefore, $\mathbb{C}[x]/(x^3 - 2x^2 + x)$ has nilpotent elements, and so is not the coordinate ring of any affine variety.

For the ring $\mathbb{C}[x]/(x^3 - 1)$, consider the factorization

$$x^3 - 1 = (x-1)(x-\zeta_3)(x-\zeta_3^2)$$

where $\zeta_3 = e^{\frac{2\pi i}{3}} \in \mathbb{C}$. Because $x^3 - 1$ is square-free, $\mathbb{C}[x]/(x^3-1)$ has no nilpotent elements; for example, there is a ring isomorphism

$$\mathbb{C}[x]/(x^3 - 1) \cong \mathbb{C}^3$$

by the Chinese remainder theorem. By the Nullstellensatz, $\mathbb{C}[x]/(x^3 - 1)$ is the coordinate ring of an affine variety. (In particular, it is the coordinate ring of the set of three points $\{1, \zeta_3, \zeta_3^2\}$ in $\mathbb{A}^1_{\mathbb{C}}$.)

(b) Although both $X, Y$ are unions of three lines intersecting at one point, the varieties $X$ and $Y$ are not isomorphic. To see this, observe that both $X$ and $Y$ have unique singular points, at $(0,0)$ and $(0,0,0)$ respectively. We will show that the Zariski tangent space $T_{(0,0)}X$ is 2-dimensional, while $T_{(0,0,0)}Y$ is 3-dimensional. (In fact, the latter computation shows that $Y$ has no embedding into $\mathbb{A}^2_{\mathbb{C}}$ at all.)

Consider the maximal ideal of functions of $X$ vanishing at $(0,0)$

$$\mathfrak{m} = (x, y) \subseteq \mathbb{C}[x,y]/(xy(x-y)).$$

Then $\mathfrak{m}^2 = (x^2, xy, y^2) \subseteq \mathbb{C}[x,y]/(xy(x-y))$. The elements $x, y$ are linearly independent in $\mathfrak{m}/\mathfrak{m}^2$: if $ax + by \in \mathfrak{m}^2$ for some $a, b \in \mathbb{C}$ then $ax + by$ is contained in the ideal $(x^2, xy, y^2)$ as an element of $\mathbb{C}[x,y]$, meaning $a = b = 0$. Therefore, $\dim(\mathfrak{m}/\mathfrak{m}^2) = 2$.

Now consider the maximal ideal of functions of $Y$ vanishing at $(0,0,0)$

$$\mathfrak{m} = (x, y, z) \subseteq \mathbb{C}[x,y,z]/(xy, yz, xz).$$

Then $\mathfrak{m}^2 = (x^2, y^2, z^2) \subseteq \mathbb{C}[x,y,z]/(xy, yz, xz)$. The elements $x, y, z$ are linearly independent in $\mathfrak{m}/\mathfrak{m}^2$: if $ax+by+cz \in \mathfrak{m}^2$ for some $a,b,c \in \mathbb{C}$ then $ax+by+cz$ is contained in the ideal $(x,y,z)^2$ as an element of $\mathbb{C}[x,y,z]$, meaning $a=b=c=0$. Therefore, $\dim(\mathfrak{m}/\mathfrak{m}^2) = 3$.

**3. (Algebraic Topology)** Consider $S^2 \times S^2$ with the product orientation. Let $u \in H^2(S^2;\mathbb{Z}) \cong \mathbb{Z}$ be the positive generator, and set

$$x := \pi_1^* u, \quad y := \pi_2^* u \in H^2(S^2 \times S^2; \mathbb{Z}),$$

where $\pi_i : S^2 \times S^2 \to S^2$ denotes the projection to the $i$-th factor. Suppose $f : S^2 \times S^2 \to S^2 \times S^2$ is a continuous map of degree 1 with no fixed points. Prove that

$$f^* x = -x, \quad f^* y = -y$$

in $H^2(S^2 \times S^2; \mathbb{Z})$.

**Solution.** By Künneth formula, the cohomology ring of $S^2 \times S^2$ is

$$H^*(S^2 \times S^2; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^2) \otimes \mathbb{Z}[y]/(y^2).$$

In particular, its cohomology groups are

$$H^0(S^2 \times S^2; \mathbb{Z}) \cong \mathbb{Z}\langle 1 \rangle, \quad H^2(S^2 \times S^2; \mathbb{Z}) \cong \mathbb{Z}\langle x \rangle \oplus \mathbb{Z}\langle y \rangle, \quad H^4(S^2 \times S^2; \mathbb{Z}) \cong \mathbb{Z}\langle xy \rangle,$$

and zero otherwise. Write

$$f^* x = ax + by, \quad f^* y = cx + dy, \quad a, b, c, d \in \mathbb{Z}.$$

Then,

$$\begin{aligned} 0 &= f^*(x^2) = (ax+by)(ax+by) = 2abxy, \\ 0 &= f^*(y^2) = (cx+dy)(cx+dy) = 2cdxy, \end{aligned}$$

so

$$ab = 0, \quad cd = 0. \tag{1}$$

Moreover,

$$f^*(xy) = (ax+by)(cx+dy) = (ad+bc)xy,$$

so from the degree-1 assumption,

$$ad + bc = 1. \tag{2}$$

It follows from (1) and (2) that, either

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}.$$

Finally, the Lefschetz number of $f$ is

$$\begin{aligned}L(f) &= \sum_{k\geq 0}(-1)^k \mathrm{Tr}(f^*|_{H^k(S^2\times S^2;\mathbb{Q})}) \\ &= 1 + (a+d) + (ad+bc) = 2 + a + d,\end{aligned}$$

so from the fixed-point-free assumption,

$$a + d = -2,$$

meaning

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

as desired.

**4. (Complex Analysis)** Let $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disk. Suppose $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic function with two distinct fixed points $a \neq b \in \mathbb{D}$. Prove that $f(z) = z$ for all $z \in \mathbb{D}$.

**Solution.** Let

$$\begin{aligned}\phi_a : \mathbb{D} &\to \mathbb{D} \\ z &\mapsto \frac{z-a}{1-\bar{a}z}\end{aligned}$$

be the standard biholomorphic automorphism of $\mathbb{D}$ sending $a$ to 0, with inverse $\phi_a^{-1} : w \mapsto \frac{w+a}{1+\bar{a}w}$. Define

$$f_0 := \phi_a \circ f \circ \phi_a^{-1}.$$

Then $f_0(0) = 0$ and $f_0(\phi_a(b)) = \phi_a(b) \neq 0$.

Recall that Schwarz lemma states that any holomorphic map $g : \mathbb{D} \to \mathbb{D}$ with $g(0) = 0$ must satisfy

$$|g(z)| \leq |z| \quad \text{for all } z \in \mathbb{D},$$

and moreover, if the equality holds at some nonzero point, then $g$ must be a rotation $g(z) = e^{i\theta}z$, for some $\theta$. It follows that $f_0$ is the identity map, and so is $f$.

**5. (Differential Geometry)** Consider the disk $D^2 = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$ with the metric

$$g = \frac{1}{1-(x^2+y^2)}(dx \otimes dx + dy \otimes dy).$$

Compute the Levi-Civita connection of the corresponding Riemann manifold.

**Solution.** Recall that $\nabla_{\partial_i}\partial_j = \sum_k \Gamma_{ij}^k \partial_k$, where

$$\Gamma_{ij}^k = \frac{1}{2} g^{kk}(\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}),$$

as $g$ is diagonal. We find

$$\Gamma^x_{xx} = \frac{x}{1-(x^2+y^2)}, \quad \Gamma^y_{yy} = \frac{y}{1-(x^2+y^2)},$$

$$\Gamma^y_{xx} = \frac{-y}{1-(x^2+y^2)}, \quad \Gamma^x_{yy} = \frac{-x}{1-(x^2+y^2)},$$

$$\Gamma^x_{xy} = \Gamma^x_{yx} = \frac{y}{1-(x^2+y^2)}, \quad \Gamma^y_{xy} = \Gamma^y_{yx} = \frac{x}{1-(x^2+y^2)}.$$

**6. (Real Analysis)** Suppose that $f_j$ $(j = 1, 2, \ldots)$ and $f$ are real functions on $[0, 1]$. We say that $f_j \to f$ *in measure* if and only if for any $\varepsilon > 0$ we have

$$\lim_{j\to\infty} \mu\{\, x \in [0,1] : |f_j(x) - f(x)| > \varepsilon \,\} = 0,$$

where $\mu$ is the Lebesgue measure on $[0, 1]$. In this problems, all functions are assumed to be in $L^1[0, 1]$.

(a) Suppose that $f_j \to f$ in measure. Does it follow that

$$\lim_{j\to\infty} \int |f_j(x) - f(x)|\, \mathrm{d}x = 0.$$

Prove it or give a counterexample.

(b) Suppose that $f_j \to f$ in measure. Does it follow that $f_j(x) \to f(x)$ almost everywhere in $[0, 1]$? Prove it or give a counter example.

(c) Suppose that $f_j(x) \to f(x)$ almost everywhere in $[0, 1]$. Does it follow that $f_j \to f$ in measure? Prove it or give a counter example.

**Solution.**

(a) No. For a counterexample, take $f = 0$ and take $f_j$ to be $j$ times the characteristic function of $[0, 1/j]$. Then $f_j - f$ is non-zero on a set of measure $1/j$ while $\int |f_j - f| = 1$.

(b) No. For $n \geq 0$ and $j = 2^n + k$ with $0 \leq k < 2^n$, let $f_j$ be the characteristic function of the interval $2^{-n}[k, k+1]$. Let $f = 0$. Then $|f - f_j|$ is supported in an interval of measure at most $2^{-n}$ for $j \geq 2^n$, so $f_j$ converges to $f$ in measure. But $f_j$ does not converge almost everywhere to 0, because for every $x \in [0, 1]$, the value $f_j(x)$ is 1 infinitely often.

(c) Yes. For clarity, we can take $f = 0$ (subtracting the original $f$ from everthing) and we may assume $f_j(x) \to 0$ for *all* $x$ by modifying the functions on a null set. To show convergence in measure, fix any $\epsilon > 0$. Let $E_j = \{\, x : |f_j(x)| > \epsilon \,\}$, let

$$S_j = \bigcup_{k\geq j} E_k,$$

and let $s_j$ be the characteristic function of $S_j$. Pointwise convergence implies that, for all $x$, the value $s_j(x)$ is eventually 0. By the dominated convergence theorem then, $\int |s_j| \to 0$. In other words, the measure of $S_j$ tends to zero as $j \to \infty$, and *a fortiori* the same holds of the measure of $E_j$. Since $\epsilon$ was arbitrary, it follows that $f_j \to 0$ in measure.

**QUALIFYING EXAMINATION**

HARVARD UNIVERSITY

Department of Mathematics

Thursday January 22, 2026 (Day 3)

**1. (Algebra)** Let $k$ be a field. Let $K/k$ be a finite separable extension, and $L/k$ be an arbitrary extension. Prove that the commutative $k$-algebra $K \otimes_k L$ splits as a finite product of finite separable extensions of $L$.

*Hint.* You may find it useful to apply the theorem of the primitive element.

**Solution.**

By the primitive element theory, there exists an irreducible separable polynomial $f \in k[x]$ such that $K \cong k[x]/f$ as fields. It is therefore enough to analyze the structure of the commutative $k$-algebra $K \otimes_k L \cong L[x]/f$. In order to do so, let

$$f(x) = f_1(x) \cdot \cdots \cdot f_r(x)$$

be the factorization of $f$ into monic irreducible polynomials in $L[x]$. As $f$ is separable, the polynomials $f_1, \ldots, f_r$ are pairwise relatively prime. It follows from the Chinese remainder theorem that

$$L[x]/f \cong \prod_{i=1}^{r} L[x]/f_i$$

as commutative $L$-algebras. But $L[x]/f_i$ is a finite separable extensions of $L$ as $f_i$ is irreducible and separable. This concludes the proof.

**2. (Algebraic Geometry)** Let $X = \mathrm{Bl}_0(\mathbb{A}^2)$ be the blow-up of $\mathbb{A}^2$ at the origin.

(a) Using local coordinates, identify the exceptional divisor $E$ and show that $E \simeq \mathbb{P}^1$.

(b) Show that the strict transform of the curve $C = \{(x, y) \in \mathbb{A}^2 \mid y^2 = x^3\}$ is smooth.

**Solution.**

(a) We have

$$X \subset \mathbb{A}^2 \times \mathbb{P}^1, \qquad X = \{((x, y), [s : t]) \mid xt = ys\}.$$

It follows that the fiber over $(0, 0) \in \mathbb{A}^2$ is $\mathbb{P}^1$.

(b) The *strict transform* $\widetilde{C} \subset X$ is the closure of $\pi^{-1}(C \setminus \{0\})$, where $\pi\colon \mathrm{Bl}_0(\mathbb{A}^2) \to \mathbb{A}^2$ is the natural morphism.

Let us cover $X$ by two open charts and verify that the intersection of $\widetilde{C}$ with both of them is smooth. We have

$$X = U_s \cup U_t,$$

where $U_s$ consists of $((x,y),[s:t]) \in X$ such that $s \neq 0$ and $U_t$ consists of $((x,y),[s:t]) \in X$ such that $t \neq 0$.

On $U_s$ we have $y = \frac{xt}{s}$ and setting $v := \frac{t}{s}$ we get $y = xv$. Now $C$ is given by the equation $f = y^2 - x^3 = 0$. Substituting $y = xv$ we see that $f$ becomes $x^2v^2 - x^3 = x^2(v^2 - x)$, so the strict transform of $C$ being intersected with $U_s$ is given (in coordinates $(x,y,v)$) by the equation $v^2 - x = 0$ so is indeed smooth.

On $U_t$ we have $x = y\frac{s}{t}$ so $x = yu$, where $u = \frac{s}{t}$. Substituting $x = yu$ in the equation defining $f$ we obtain $y^2 - y^3u^3$, so the strict transform of $C$ being intersected with $U_t$ is given by the equation $1 - yu^3 = 0$ and also defines a smooth variety.

3. **(Algebraic Topology)** Let $n \geq 1$. Compute the homotopy groups $\pi_k(\mathbb{CP}^n)$, for each $1 \leq k \leq 2n$.

**Solution.** Using the Serre fibration

$$S^1 \hookrightarrow S^{2n+1} \to \mathbb{C}P^n,$$

we get a long exact sequence in homotopy groups

$$\cdots \to \pi_k(S^{2n+1}) \to \pi_k(\mathbb{C}P^n) \to \pi_{k-1}(S^1) \to \pi_{k-1}(S^{2n+1}) \to \cdots .$$

Since $k < 2n+1$, the portion of the long exact sequence shown above becomes

$$0 \to \pi_k(\mathbb{C}P^n) \to \pi_{k-1}(S^1) \to 0,$$

and thus

$$\pi_k(\mathbb{C}P^n) \cong \pi_{k-1}(S^1) \cong \begin{cases} \mathbb{Z} & \text{if } k = 2, \\ 0 & \text{if } k = 1 \text{ or } 3 \leq k \leq 2n. \end{cases}$$

4. **(Complex Analysis)** Suppose $f$ is a doubly-periodic meromorphic function on $\mathbb{C}$ with periods $\omega_1, \omega_2$ which are $\mathbb{R}$-linearly independent. Let $a \in \mathbb{C}$ such that the sides of the parallelogram $\Omega$ with vertices $a, a+\omega_1, a+\omega_2, a+\omega_1+\omega_2$ do not contain any zeroes or poles of $f$. Let $b_1, \cdots, b_p$ (respectively $a_1, \cdots, a_q$) be the zeroes (respectively the poles) of $f$ with multiplicities $k_1, \cdots, k_p$ (respectively $\ell_1, \cdots, \ell_q$) inside $\Omega$. By considering the residues of the function

$$\frac{1}{2\pi i}\frac{wf'(w)}{f(w)}dw$$

or otherwise, prove that

$$\left(\sum_{\mu=1}^{p} k_\mu b_\mu\right) - \left(\sum_{\nu=1}^{q} \ell_\nu c_\nu\right)$$

belongs to $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. In other words, in a fundamental parallelogram the sum of the coordinates of the zeroes of an elliptic function equals the sum of the coordinates of its poles modulo a period.

**Solution.** It follows from the computation of residues that the integral of

$$\frac{1}{2\pi i}\frac{wf'(w)}{f(w)}dw$$

over the boundary $\partial\Omega$ of the fundamental parallelogram $\Omega$ is equal to

$$\left(\sum_{\mu=1}^{p} k_\mu b_\mu\right) - \left(\sum_{\nu=1}^{q} \ell_\nu c_\nu\right).$$

We compute the boundary integral over $\partial\Omega$ by integrating over the two pairs of opposite sides of $\Omega$. The sum of the integrals over the opposite sides $[a+\omega_1, a+\omega_1+\omega_2]$ and $[a, a+\omega_2]$ is

$$\begin{aligned}\frac{1}{2\pi i}\int_{[a+\omega_1,a+\omega_1+\omega_2]}\frac{wf'(w)}{f(w)}dw - \frac{1}{2\pi i}\int_{[a,a+\omega_2]}\frac{wf'(w)}{f(w)}dw \\ = \omega_1\frac{1}{2\pi i}\int_{[a,a+\omega_2]}\frac{f'(w)}{f(w)}dw.\end{aligned} \tag{3}$$

Note that

$$\frac{1}{2\pi i}\int_{[a,a+\omega_2]}\frac{f'(w)}{f(w)}dw \tag{4}$$

equals $1/2\pi i$ times the difference of the value of $\log f(w)$ at $a+\omega_2$ and at $a$ when $w$ runs along $[a, a+\omega_2]$. Since $f(w)$ has the same value at $a$ as at $a+\omega_2$, the difference of the value of $\log f(w)$ at $a+\omega_2$ and at $a$ when $z$ runs along $[a, a+\omega_2]$ must be $2\pi i$ times an integer. Therefore (4) is an integer and (3) is a period of $f$. Likewise

$$\frac{1}{2\pi i}\int_{[a,a+\omega_1]}\frac{wf'(w)}{f(w)}dw - \frac{1}{2\pi i}\int_{[a+\omega_2,a+\omega_1+\omega_2]}\frac{wf'(w)}{f(w)}dw$$

is also a period of $f$.

**5. (Differential Geometry)**

(a) Compute $H^k_{\mathrm{dR}}(\mathbb{R}^n \setminus \{0\})$ for all $k$.

(b) Show that the $(n-1)$-form

$$\eta = \frac{1}{||x||^n} \sum_{i=1}^{n} (-1)^{i-1} x_i dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n$$

is closed on $\mathbb{R}^n \setminus \{0\}$ and $\int_{S^{n-1}} \eta = \mathrm{Vol}(S^{n-1})$.

(c) Deduce that $[\eta]$ generates $H^{n-1}_{\mathrm{dR}}(\mathbb{R}^n \setminus \{0\})$.

**Solution.**

(a) The space $U := \mathbb{R}^n \setminus \{0\}$ deformation retracts onto the unit sphere $S^{n-1}$ via the radial retraction

$$r : U \to S^{n-1}, \qquad r(x) = \frac{x}{\|x\|}.$$

Hence $U$ is homotopy equivalent to $S^{n-1}$, so de Rham cohomology agrees:

$$H^k_{\mathrm{dR}}(U) \cong H^k_{\mathrm{dR}}(S^{n-1}).$$

Since

$$H^k_{\mathrm{dR}}(S^{n-1}) \cong \begin{cases} \mathbb{R} & k = 0, \\ \mathbb{R} & k = n-1, \\ 0 & \text{otherwise}, \end{cases}$$

the same holds for $U$.

(b) Let $\Omega = dx_1 \wedge \cdots \wedge dx_n$ be the standard volume form on $\mathbb{R}^n$, and let

$$R = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$$

be the radial vector field. Then

$$\iota_R \Omega = \sum_{i=1}^{n} (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n,$$

so

$$\eta = \|x\|^{-n} \, \iota_R \Omega.$$

Write $r = \|x\|$. Then

$$d\eta = d(r^{-n}) \wedge \iota_R \Omega + r^{-n} \, d(\iota_R \Omega).$$

By Cartan's formula, $d(\iota_R \Omega) = \mathcal{L}_R \Omega - \iota_R(d\Omega) = \mathcal{L}_R \Omega$ since $d\Omega = 0$. Moreover $\mathcal{L}_R \Omega = (\mathrm{div}\, R)\Omega = n\Omega$, so $d(\iota_R \Omega) = n\Omega$.
Next, $d(r^{-n}) = -nr^{-n-1} dr$. Since $dr = r^{-1} \sum_{i=1}^{n} x_i \, dx_i$, we get

$$d(r^{-n}) = -nr^{-n-2} \sum_{i=1}^{n} x_i \, dx_i.$$

Let $\alpha = \sum_{i=1}^{n} x_i \, dx_i$. It's easy to see that

$$\alpha \wedge \iota_R \Omega = r^2 \, \Omega.$$

Therefore

$$d(r^{-n}) \wedge \iota_R \Omega = -nr^{-n-2} \alpha \wedge \iota_R \Omega = -nr^{-n-2} \, r^2 \, \Omega = -nr^{-n} \Omega,$$

while

$$r^{-n} d(\iota_R \Omega) = r^{-n} \, n\Omega = nr^{-n} \Omega.$$

These cancel, hence $d\eta = 0$ on $U$.
On $S^{n-1}$ we have $r = 1$, so $\eta|_{S^{n-1}} = \iota_R \Omega|_{S^{n-1}}$. Along the sphere, $R$ is the outward normal vector field, and contracting the ambient volume form with the outward unit normal gives the induced oriented volume form on the hypersurface. Hence $\eta|_{S^{n-1}}$ is the standard volume form, so

$$\int_{S^{n-1}} \eta = \mathrm{Vol}(S^{n-1}).$$

(c) We know $H^{n-1}_{\mathrm{dR}}(U) \cong \mathbb{R}$, so it is one-dimensional. To show $[\eta] \neq 0$, note that if $\eta = d\beta$ on $U$, then by Stokes theorem

$$\int_{S^{n-1}} \eta = \int_{S^{n-1}} d\beta = \int_{\partial(B^n)} d\beta = \int_{B^n} d(d\beta) = 0,$$

contradicting $\int_{S^{n-1}} \eta = \mathrm{Vol}(S^{n-1}) \neq 0$. Hence $[\eta] \neq 0$, and in a one-dimensional vector space this means $[\eta]$ is a generator.

**6. (Real Analysis)** Let $f$ be a bounded real-valued function on $X = [0,1] \subset \mathbb{R}$, and define a function $\phi : [1, \infty) \to \mathbb{R}$ by

$$\phi(p) = \|f\|^p_{L^p(X)}.$$

Prove that $\phi$ is convex.

**Solution.** The exponential function is convex, so for any fixed $a \geq 0$, the function $\psi(p) = a^p$ is a convex function of $p$. This means that if we take any $p_0, p_1 \in [1, \infty)$ and any $t \in [0,1]$, and set

$$p = (1-t)p_0 + tp_1,$$

then

$$|f(x)|^p \leq (1-t)|f(x)|^{p_0} + t|f(x)|^{p_1}$$

for all $x \in [0,1]$. (Take $a = |f(x)|$ in the above.) Integrating this inequality over $[0,1]$ gives

$$\phi(p) \leq (1-t)\phi(p_0) + t\phi(p_1),$$

which says that $\phi$ is convex.