QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday September 1, 2026 (Day 1)

1. (Algebra) Let A be a commutative ring with unit, and M a module over A.

(a) Define what it means for M to be a *Noetherian* A-module, and what it means for A to be a *Noetherian ring*.

(b) Now suppose A is the polynomial ring $\mathbb{C}[z]$, and M its fraction field $\mathbb{C}(z)$ considered as an A-module. Show that A is a Noetherian ring but M is not a Noetherian A-module.

Solution.

(a) M is Noetherian if its submodules satisfy the ascending chain condition: every infinite ascending chain $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ stabilizes, i.e. there is some N such that $M_N = M_n$ for all $n \geq N$; equivalently, there is no strictly ascending infinite chain (with each M_n a proper submodule of M_{n+1}). The ring A is Noetherian if it is Noetherian as an A-module, i.e. if the ideals of A satisfy the ascending chain condition.

(b) To show that A is Noetherian, recall that it is a principal ideal domain (for example, by the Euclidean algorithm), so an ascending chain of ideals can be written as a sequence $(P_0), (P_1), (P_2), \ldots$ of principal ideals for some $P_n \in A$ with each $P_n \in (P_{n+1})$. The containment is strict iff $P_{n+1} \notin (P_n)$. If $P_n \neq 0$, this means $P_n = p_n P_{n+1}$ for some nonconstant polynomial p_n; and if $P_n = 0$ then $P_{n+1} \neq 0$. Thus $P_n \neq 0$ for all $n > 0$, and $\deg P_n > \deg P_{n+1}$ for all $n > 0$. This is impossible because once $n > 1 + \deg P_1$ the degree of P_n would be negative.

M is not Noetherian because there are infinite strictly ascending chains such as $M_n = z^{-n} A$.

2. (Algebraic Geometry) Consider the cubic curve

$$C := V(y^2 z - x^3) \subset \mathbb{P}^2_{\mathbb{C}}.$$

Show that C is birational to $\mathbb{P}^1_{\mathbb{C}}$ and exhibit an explicit birational equivalence. Prove that C is not isomorphic to $\mathbb{P}^1_{\mathbb{C}}$.

Solution. We note that we have a natural morphism

$$\nu : \mathbb{P}^1_{\mathbb{C}} \to C$$

$$[s : t] \mapsto [s^2 t : s^3 : t^3].$$

This map admits a rational inverse induced by the projection map

$$\nu^{-1} : C \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$$

$$[x : y : z] \mapsto [y : x]$$

which is well-defined on the open subset of C given by intersecting with the open subset $\{x \neq 0\} \subset \mathbb{P}^2_{\mathbb{C}}$. Hence, C is birational to $\mathbb{P}^1_{\mathbb{C}}$. (Alternatively: note that the open subset of C where $z = 1$ identifies with the affine curve $V(y^2 = x^3) \subset \mathbb{A}^2_{\mathbb{C}}$ which has coordinate ring isomorphic to $\mathbb{C}[t^2, t^3]$ then identify this with $\mathbb{C}[t, t^{-1}]$ on the locus where t is invertible).

On the other hand, we note that the curve C is not smooth. Indeed, it is singular at the point $p := [0 : 0 : 1]$. The partial derivatives of $F := y^2 z - x^3$ are given by

$$F_x := -3x^2 \ , \ F_y = 2yz \ , \ F_z = y^2,$$

which all vanish at p. Since $\mathbb{P}^1_{\mathbb{C}}$ is smooth, the two are not isomorphic.

3. (Algebraic Topology) Let $n \geq 1$ and let $i : S^n \to S^n$ denote the antipodal map.

(a) Suppose n is even. For any continuous map $f : S^n \to S^n$, show that there exists a point $x \in S^n$ such that $f(x) = x$ or $f(x) = i(x)$.

(b) For every odd integer n, find a continuous map $f : S^n \to S^n$ such that $f(x) \neq x$ and $f(x) \neq i(x)$ for all $x \in S^n$.

Solution.

(a) Let $d \in \mathbb{Z}$ denote the degree of f. In other words,

$$f_*([S^n]) = d[S^n] \in H_n(S^n; \mathbb{Z})$$

where $[S^n]$ denotes the fundamental class of S^n. The Lefschetz number of f is

$$\tau(f) = 1 + (-1)^n d = 1 + d$$

because n is even. The Lefschetz fixed point theorem implies that there exists some $x \in S^n$ such that $f(x) = x$ if $d \neq -1$.
If $d = -1$ then consider $i \circ f : S^n \to S^n$ instead. Observe that

$$(i \circ f)_*([S^n]) = i_*(-[S^n]) = [S^n]$$

since $i_*([S^n]) = -[S^n]$ if n is even. Therefore the Lefschetz number of $i \circ f$ is $\tau(i \circ f) = 2$, and there exists some $x \in S^n$ such that $i \circ f(x) = x$. In other words, $f(x) = i(x)$.

(b) If $n = 2m - 1$ is odd then consider the unit sphere $S^{2m-1} \subseteq \mathbb{C}^m$ and the antipodal map $i(x) = -x$. Let $f : S^{2m-1} \to S^{2m-1}$ be the map

$$f(x_1, \ldots, x_m) = (\zeta x_1, \ldots, \zeta x_m)$$

for any $\zeta \in \mathbb{C}$ of unit length with $\zeta \neq \pm 1$. Then $f(x) \neq x$ and $f(x) \neq i(x)$ for any $x \in S^{2m-1}$.

4. (Complex Analysis)

(a) Determine the constant C in

$$u(x, y) = x^2 + Cy^2 - 2xy - 2x + 3y$$

so that u is a harmonic function.

(b) With the value C determined from (a), find a holomorphic function f with $f(0) = 0$ such that the real part of f is u.

Solution.

(a) From

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + 2C$$

it follows that C must be -1.

(b) For $f = u + iv$ to be holomorphic, the Cauchy-Riemann equations yield the condition for v as

$$v_x = -u_y = 2y + 2x - 3 \quad \text{and} \quad v_y = u_x = 2x - 2y - 2.$$

At this point we can integrate to determine v. Another way is to use $f_z = f_x = u_x + iv_x = 2x - 2y + i(2y + 2x - 3) = 2(1 + i)z - (2 + 3i)$ to get $f = (1 + i)z^2 - (2 + 3i)z$ with $f(0) = 0$.

5. (Differential Geometry) Consider $\mathbb{R}^3$ with coordinates (x_1, x_2, x_3) and define a 2-form β on the unit sphere $S^2 \subset \mathbb{R}^3$ by the formula

$$\beta = x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2.$$

Given an immersion $f : S^2 \to \mathbb{R}^3$ denote by $F : S^2 \to S^2$ the map assigning to a point $p \in S^2$ the unit normal vector to the immersion f at p. To fix orientation, choose F so that whenever (e_1, e_2) is a positively oriented basis of $T_p(S^2)$,

$$(df(e_1), df(e_2), F(p))$$

is positively oriented in $\mathbb{R}^3$. Show that

$$\int_{S^2} F^* \beta = 4\pi.$$

Solution. We will calculate that $F^*\beta = KdA$ where dA is the induced volume form and K the Gauss curvature of the immersion f. Given this, the claim follows from the Gauss-Bonnet theorem since the Euler characteristic of the 2-sphere is $\chi(S^2) = 2$.

Let u, v denote positively-oriented local coordinates in a coordinate neighborhood of a point $p \in S^2$ and write $f = (f_1(u,v), f_2(u,v), f_3(u,v))$. Since F is the (oriented) unit normal vector field, we have

$$F = \frac{f_u \times f_v}{|f_u \times f_v|}$$

where $f_u = (\partial_u f_1, \partial_u f_2, \partial_u f_3)$ and f_v is defined analogously. On the other hand, we have

$$F^*\beta = F \cdot (F_u \times F_v) du \wedge dv$$

where for $F = (F_1(u,v), F_2(u,v), F_3(u,v))$ we write $F_u = (\partial_u F_1, \partial_u F_2, \partial_u F_3)$ and F_v is defined in the analogous way. Since the Gauss curvature satisfies $K = \det(dF)$, we have $F_u \times F_v = K \cdot (f_u \times f_v)$ so that

$$F \cdot (F_u \times F_v) = KF \cdot (f_u \times f_v) = K|f_u \times f_v|.$$

Therefore, $F^*\beta = KdA$ as desired.

6. **(Real Analysis)** Show that if $f : \mathbb{R} \to \mathbb{C}$ is a $\mathcal{C}^\infty$ function satisfying $f(x+1) = f(x)$ for all $x \in \mathbb{R}$ then there exists a unique $\mathcal{C}^\infty$ function $g : \mathbb{R} \to \mathbb{C}$, satisfying the same periodicity $g(x+1) = g(x)$ for all $x \in \mathbb{R}$, that solves the differential equation $g'' + 2026g = f$.

Solution. A $\mathbb{Z}$-periodic function $\phi : \mathbb{R} \to \mathbb{C}$ is $\mathcal{C}^\infty$ if and only if it has a Fourier series $\phi(x) = \sum_{n\in\mathbb{Z}} a_n(\phi)e^{2\pi inx}$ whose coefficients a_n decay faster than any inverse power of n, i.e. such that $n^M a_n(\phi) \to 0$ as $n \to \pm\infty$ for every $M = 1, 2, 3, \ldots$. For $\phi = g$ we compute

$$g''(x) + 2026g(x) = \sum_{n\in\mathbb{Z}} (2026 - (2n\pi)^2)a_n(g)e^{2\pi inx}.$$

Thus $a_n(f) = (2026 - (2n\pi)^2)\, a_n(g)$. Since π^2 is irrational, $2026 - (2n\pi)^2$ is never zero, so for each n there is a unique solution

$$a_n(g) = \frac{a_n(f)}{2026 - (2n\pi)^2},$$

which are the Fourier coefficients of a smooth function g, because the $a_n(g)$ decay faster than any inverse power of n if and only if the $a_n(f)$ do.

QUALIFYING EXAMINATION

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Department of Mathematics

Wednesday September 2, 2026 (Day 2)

1. (Algebra) Consider the polynomial $f(x) = x^{12} - 2$.

(a) If K denotes the splitting field of f over $\mathbb{F}_7$, compute the degree of the extension $K/\mathbb{F}_7$.

(b) Describe a generator for $\mathrm{Gal}(K/\mathbb{F}_7)$.

Solution.

(a) Noting that 2 is a primitive cube root of 1 in $\mathbb{F}_7$, we see that the splitting field of f is the minimal extension of $\mathbb{F}_7$ containing a primitive 36^{th} root of 1. Any finite extension of $\mathbb{F}_7$ is of the form $\mathbb{F}_{7^k}$ and so we seek the minimal k solving the equation

$$7^k - 1 \equiv 0 \mod 36.$$

Calculation shows $k = 6$; thus the splitting field of f is $K = \mathbb{F}_{7^6}$. In particular, the extension $K/\mathbb{F}_7$ has degree 6.

(b) The Galois group $\mathrm{Gal}(K/\mathbb{F}_7)$ is the cyclic group of order 6 generated by the Frobenius automorphism of K given by $x \mapsto x^7$.

2. (Algebraic Geometry) Let $X \subset \mathbb{P}^n$ be an irreducible projective variety of dimension d over an algebraically closed field. Let $\mathbb{G}(\ell, n)$ be the Grassmannian of ℓ-planes in $\mathbb{P}^n$ with $1 \leq \ell < n-d$, and let $C(X) \subset \mathbb{G}(\ell, n)$ be the locus of ℓ-planes meeting X. Prove that $C(X)$ is irreducible, and find its dimension.

Solution. Consider the incidence correspondence,

$$I = \{\, (x, L) \mid x \in L \,\} \subset X \times \mathbb{G}(\ell, n).$$

The map $p : I \to X$ is a bundle whose fiber $p^{-1}(x)$ is a Grassmannian $\mathbb{G}(\ell - 1, n - \ell)$ of dimension $\ell(n-\ell)$. Since X and the fiber are irreducible, the projective variety I is also irreducible and its dimension is the sum: $d + \ell(n-\ell)$.

Consider next the map $q : I \to \mathbb{G}(\ell, n)$. Its image is $C(X)$. As the image of an irreducible projective variety under a regular map, the locus $C(X)$ is irreducible. Its dimension can be computed as the sum of the dimension of $C(X)$ and the generic dimension of a fiber $q^{-1}(L)$ for $L \in C(X)$. We claim that the generic fiber $q^{-1}(L)$ is zero-dimensional. For this, it is enough to show there is an $L \in \mathbb{G}(\ell, n)$ meeting X in a given point x and no other point. When $\ell \geq 2$, for distinct x and y in X, the dimension of the locus of $L \in \mathbb{G}(\ell, n)$ containing x and y is the dimension of a Grassmannian $\mathbb{G}(\ell - 2, n - 2)$, which is $(\ell - 1)(n - \ell)$. Also when $\ell = 1$, two distinct points determine a unique line, so the same formula $(\ell - 1)(n - \ell)$ applies. So the locus of those $L \in \mathbb{G}(\ell, n)$ meeting X in x and at least one other point has dimension at most

$$\dim(X) + (\ell - 1)(n - \ell)$$
$$< \ell(n - \ell)$$

by the hypothesis. Thus a general ℓ-plane through x contains no other point of X.

Since the generic fiber of the surjective map $q : I \to C(X)$ is 0-dimensional, the dimension of $C(X)$ is equal to the dimension of the incidence correspondence I, which is $d + \ell(n - \ell)$.

3. **(Algebraic Topology)** Let X be the space obtained from the union of $S^1 \times S^2$ and the closed 3-disk D^3 by gluing $\partial D^3 = S^2$ to $\{p\} \times S^2$ for some $p \in S^1$. Compute $H_n(X; \mathbb{Z})$ for $n = 1$ and $n = 2$.

Solution. There exist open neighborhoods $U \subseteq X$ of $S^1 \times S^2$ and $V \subseteq X$ of D^3 that deformation retract onto $S^1 \times S^2$ and D^3 respectively, and such that $U \cap V$ deformation retracts onto $(S^1 \times S^2) \cap (D^3) = \partial D^3$, which is homeomorphic to S^2. We apply Mayer–Vietoris to the union $X = U \cup V$ using the homotopy equivalences above:

$$\cdots \to H_n(S^2) \to H_n(S^1 \times S^2) \oplus H_n(D^3) \to H_n(X) \to H_{n-1}(S^2) \to \ldots$$

For $n = 1$, we have the exact sequence

$$H_1(S^2) \to H_1(S^1 \times S^2) \oplus H_1(D^3) \to H_1(X) \to H_0(S^2) \to H_0(S^1 \times S^2) \oplus H_0(D^3)$$

The map $H_0(S^2) \to H_0(S^1 \times S^2) \oplus H_0(D^3)$ is injective, because the induced maps $H_0(S^2) \to H_0(S^1 \times S^2)$ and $H_0(S^2) \to H_0(D^3)$ are isomorphisms. Moreover, $H_1(S^2) = 0$ and $H_1(D^3) = 0$, and so

$$0 \to H_1(S^1 \times S^2) \to H_1(X) \to 0.$$

In particular, $H_1(X) \cong H_1(S^1 \times S^2) \cong \mathbb{Z}$.

For $n = 2$, we have the exact sequence

$$H_2(S^2) \to H_2(S^1 \times S^2) \oplus H_2(D^3) \to H_2(X) \to H_1(S^2)$$

The induced map $H_2(S^2) \to H_2(S^1 \times S^2)$ induced by inclusion is an isomorphism. Using the fact that $H_1(S^2) = 0$ and $H_2(D^3) = 0$, we see that $H_2(X) = 0$.

4. (Complex Analysis) Let $\mathbb{D}$ denote the open unit disk in $\mathbb{C}$. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. Show for all $z \in \mathbb{D}$ that

$$\left| \frac{f(0) - f(z)}{1 - \overline{f(0)}f(z)} \right| \leq |z|.$$

Solution. Let ψ denote the automorphism of $\mathbb{D}$ given by

$$\psi(z) = \frac{f(0) - z}{1 - \overline{f(0)}z}.$$

The composition

$$\psi \circ f(z) = \frac{f(0) - f(z)}{1 - \overline{f(0)}f(z)}$$

satisfies $\psi \circ f(0) = 0$. By the Schwarz lemma,

$$|\psi \circ f(z)| \leq |z|$$

for all $z \in \mathbb{D}$.

5. (Differential Geometry) Consider the following smooth real manifolds.

(a) S^n

(b) $\mathbb{RP}^n$.

Do these admit a non-exact non-zero closed differential k-form for $k \geq 1$? If yes construct an explicit example. If not explain why.

Solution. We recall the de Rham cohomology groups

$$H^k_{\mathrm{dR}}(X, \mathbb{R}) := \mathrm{Ker}(d : \Omega^k(X) \to \Omega^{k+1}(X))/\mathrm{Im}(d : \Omega^{k-1}(X) \to \Omega^k(X)).$$

In particular, a differential form in $\mathrm{Ker}(d : \Omega^k(X) \to \Omega^{k+1}(X))$ is called closed, and a differential form in $\mathrm{Im}(d : \Omega^{k-1}(X) \to \Omega^k(X))$ is called exact. Therefore, we are only interested in whether these spaces have non-vanishing higher de Rham cohomology. For the sphere S^n, we recall that $H^k_{\mathrm{dR}}(S^n, \mathbb{R})$ is equal to 0 unless $k = 0$ or $k = n$ by induction and Mayer-Vietoris applied to the standard charts coming from stereographic projection. It admits a generator given by the volume form with respect to the standard euclidean metric inherited from the natural embedding $S^n \subset \mathbb{R}^{n+1}$. In standard coordinates $(x_0, x_1, \ldots, x_{n-1}, x_n)$, this is given as

$$\omega_{S^n} = \sum_{i=0}^{n} (-1)^i x_i \, dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

Since $\dim(S^n) = n$ the form ω_{S^n} is automatically closed. For non-exactness, we recall that this has the property that

$$\int_{S^n} \omega_{S^n} = \mathrm{Vol}(S^n),$$

and this implies that ω_{S^n} cannot be exact. Indeed, if $\omega_{S^n} = d\eta$ then Stokes' theorem would imply that

$$\int_{S^n} \omega_{S^n} = 0.$$

For the real projective space, we recall that $\mathbb{RP}^n$ is the quotient of S^n by the antipodal map $A : S^n \to S^n$ sending $x \in S^n$ to $-x \in S^n$. Any differential form on $\mathbb{RP}^n$ pulls back to one on S^n via the quotient map $\pi : S^n \to \mathbb{RP}^n$. Conversely, any differential form ω on S^n gives rise to a differential form which descends to $\mathbb{RP}^n$ by the operation of averaging

$$\omega \mapsto \frac{1}{2}(\omega + A^*\omega),$$

which commutes with cohomology. This gives an isomorphism

$$H^k_{\mathrm{dR}}(\mathbb{RP}^n, \mathbb{R}) \simeq H^k(S^n, \mathbb{R})^{\mathbb{Z}/2\mathbb{Z}}.$$

We note that ω_{S^n} is only invariant under $\mathbb{Z}/2\mathbb{Z}$ if and only if n is odd since $A^*\omega_{S^n} = (-1)^{n+1}\omega_{S^n}$. Therefore, there only exists a non-exact non-zero closed differential form if n is odd in which case it is represented by a differential form pulling back to ω_{S^n} under π.

6. (Real Analysis) Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of real functions ($n \in \mathbb{N}$).

(a) Give a clear statement of the hypotheses and conclusions of the monotone convergence theorem as it applies to such a sequence.

(b) Give a clear statement of the hypotheses and conclusions of the dominated convergence theorem as it applies to such a sequence.

(c) Let $g_n : \mathbb{R} \to \mathbb{R}$ be Lebesgue integrable, for all n. Using the monotone and dominated convergence theorems, show that the series

$$\sum_{n=1}^{\infty} g_n(x)$$

converges for almost all $x \in \mathbb{R}$ provided that the series

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}} |g_n|$$

converges. Further, writing $G(x)$ for the limit of the series (defined almost everywhere), show that G is integrable and

$$\int G = \sum \int g_n.$$

Solution.

(a) Suppose each f_n is measurable and non-negative, suppose $f_n \leq f_{n+1}$ for all n and let $I \in [0, \infty]$ be the limit

$$I = \lim_n \int f_n.$$

Let $f : \mathbb{R} \to [0, \infty]$ be defined as $f(x) = \lim f_n(x)$. Then f is measurable and

$$I = \int f,$$

which is to be interpreted as including the statement that f is integrable if and only if I is finite.

(b) Suppose that each f_n is measurable and

$$f_n(x) \to f(x)$$

almost everywhere. Suppose that there is a function $h \in L^1(\mathbb{R})$ with

$$|f_n(x)| \leq h(x)$$

almost everywhere for all n. Then $f \in L^1(\mathbb{R})$ and

$$\int f = \lim_n \int f_n.$$

(c) Let

$$H_N(x) = \sum_{n=1}^{N} |g_n(x)|.$$

Then H_N is an increasing sequence of nonnegative measurable functions. By MCT,

$$\begin{aligned}\int \sum_n |g_n(x)|\, dx &= \lim_{N \to \infty} \int H_N(x)\, dx \\ &= \sum_n \int_{\mathbb{R}} |g_n(x)|\, dx\end{aligned}$$

which is finite by hypothesis. It follows that

$$\sum_{n=1}^{\infty} |g_n(x)| < \infty$$

for almost every $x \in \mathbb{R}$, otherwise its integral would be infinite. Hence the series

$$\sum_{n=1}^{\infty} g_n(x)$$

converges absolutely for almost every x. Let

$$G(x) = \sum_{n=1}^{\infty} g_n(x)$$

where the series converges, defining G arbitrarily on the remaining null set.

Set

$$H(x) = \sum_{n=1}^{\infty} |g_n(x)|.$$

The calculation above shows that $H \in L^1(\mathbb{R})$. Moreover,

$$|G(x)| \leq H(x)$$

almost everywhere, so $G \in L^1(\mathbb{R})$.

Now let

$$S_N(x) = \sum_{n=1}^{N} g_n(x).$$

Then $S_N(x) \to G(x)$ almost everywhere, and

$$|S_N(x)| \leq \sum_{n=1}^{N} |g_n(x)| \leq H(x).$$

Since $H \in L^1(\mathbb{R})$, the DCT gives

$$\int_{\mathbb{R}} G(x)\,dx = \lim_{N\to\infty} \int_{\mathbb{R}} S_N(x)\,dx.$$

So,

$$\int_{\mathbb{R}} S_N(x)\,dx = \sum_{n=1}^{N} \int_{\mathbb{R}} g_n(x)\,dx.$$

Therefore

$$\int_{\mathbb{R}} G(x)\,dx = \sum_{n=1}^{\infty} \int_{\mathbb{R}} g_n(x)\,dx.$$

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday September 3, 2026 (Day 3)

1. (Algebra) Let G be the symmetric group S_5.

(a) Show that G has 7 conjugacy classes, and determine the order of their elements.

(b) It is known that G has two non-isomorphic irreducible complex representations of dimension 4, and two non-isomorphic irreducible representations of dimension 5. Using this, determine the dimensions of the remaining isomorphism classes of irreducible representations of G.

Solution.

(a) The cycle structures give a bijection between conjugacy classes of S_n and partitions of n. For $n = 5$ there are seven, with cycle structures 11111, 1112, 122, 113, 23, 14, 5. The order is the least common multiple of the cycle lengths, so here 1, 2, 2, 3, 6, 4, 5 respectively.

(b) The total number of irreducible representations equals the number of conjugacy classes, which we just checked equals 7. We are given the dimensions of 4 of them, so there are $7 - 4 = 3$ others, including the trivial representation which has dimension 1. We use the sum-of-squares formula $|G| = \sum_\rho \rho(1)^2$. Here $|G| = 5! = 120$, and the four known dimensions are $\rho(1) = 4, 4, 5, 5$, so we need three positive integers that include 1 and whose squares sum to $120 - 82 = 38$. These must be $1, 1, 6$. (Alternatively, G has a second one-dimensional representation, the sign character; this leaves only one $\rho(1)$ unaccounted for, which must be $\sqrt{38 - 1^2 - 1^2} = 6$.)

Remark: One 4-dimensional representation, call it V, is the trace-zero subspace of the 5-dimensional permutation representation; the other is $V \otimes \epsilon$ where ϵ is the sign character. The 6-dimensional representation is then the exterior square of V.

2. (Algebraic Geometry) Let $\mathbb{A}^n$ denote the affine space of dimension n over an algebraically closed field k.

(a) Describe an open covering of $\mathbb{A}^n \setminus \{0\}$ by affine open subsets.

(b) For $n \geq 2$, describe the ring of regular functions on $\mathbb{A}^n \setminus \{0\}$.

(c) Show that $\mathbb{A}^n \setminus \{0\}$ is not affine when $n \geq 2$.

Solution.

(a) Write $U = \mathbb{A}^n \setminus \{0\}$. In the standard coordinates $x_1, \dots, x_n$, let $U_j \subset \mathbb{A}^n \setminus \{0\}$ be the subset $\{x_j \neq 0\}$. This is isomorphic to a product,

$$U_j \cong (\mathbb{A} \setminus \{0\}) \times \mathbb{A}^{n-1},$$

and is affine because $\mathbb{A} \setminus \{0\}$ is isomorphic to the locus $xy = 1$ in $\mathbb{A}^2$ for example. The union of the U_j is U.

(b) A regular function on U arises from a rational function h on $\mathbb{A}^n$ whose restriction to each U_j is regular. This means that there is an expression

$$h = a/b$$

where a and b are polynomials with no common factor and, for each j, every irreducible factor of b must divide x_j. Since x_1 and x_2 are relatively prime polynomials, b must be a unit. So h is a polynomial in the variables. So the ring of regular functions on U is the polynomial ring

$$k[x_1, x_2, \dots, x_n].$$

(c) The coordinate functions $x_1, \dots, x_n$ define a map $x : U \to \mathbb{A}^n$ and the above calculation shows that the resulting map (in the other direction) on the rings of regular functions is an isomorphism. If U were affine, it would follow that x itself is an isomorphism. But it is not surjective.

3. (Algebraic Topology) Let $A : S^2 \times S^2 \to S^2 \times S^2$ denote the map

$$A(x, y) = (-x, -y)$$

where $x \mapsto -x$ denotes the antipodal map of S^2. Consider the quotient space

$$X := (S^2 \times S^2)/\langle A \rangle.$$

(a) Show that $\pi_n(\mathbb{RP}^2 \times S^2) \cong \pi_n(X)$ for all $n \geq 1$.

(b) Show that $\mathbb{RP}^2 \times S^2$ and X are not homotopy equivalent. (*Hint.* For this part you may want to consider H^4 of each space.)

Solution.

(a) Both $\mathbb{RP}^2 \times S^2$ and X have universal cover homeomorphic to $S^2 \times S^2$, and so

$$\pi_n(\mathbb{RP}^2 \times S^2) \cong \pi_n(S^2 \times S^2) \cong \pi_n(X)$$

for all $n \geq 2$. For $n = 1$, first compute that

$$\pi_1(\mathbb{RP}^2 \times S^2) \cong \pi_1(\mathbb{RP}^2) \times \pi_1(S^2) \cong \mathbb{Z}/2\mathbb{Z}.$$

Moreover, $\pi_1(X) \cong \mathbb{Z}/2\mathbb{Z}$ as well because $\langle A \rangle \cong \mathbb{Z}/2\mathbb{Z}$ acts on the simply-connected space $S^2 \times S^2$ freely and properly discontinuously.

(b) By, e.g. the Künneth theorem,

$$H^4(\mathbb{RP}^2 \times S^2; \mathbb{Z}) \cong H^2(\mathbb{RP}^2; \mathbb{Z}) \otimes H^2(S^2; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$

On the other hand, X is a closed 4-manifold because it has a closed 4-manifold $S^2 \times S^2$ as a covering space. Moreover, X is orientable because the deck transformation A is orientation-preserving. One way to see this is to compute the induced action of A on a generator

$$\alpha \smile \beta \in H^2(S^2; \mathbb{Z}) \otimes H^2(S^2; \mathbb{Z}) \cong H^4(S^2 \times S^2) \cong \mathbb{Z}$$

where α, β are generators of the two copies of $H^2(S^2; \mathbb{Z})$ respectively:

$$A^*(\alpha \smile \beta) = A^*(\alpha) \smile A^*(\beta) = (-\alpha) \smile (-\beta) = \alpha \smile \beta.$$

This shows that A^* acts as the identity on $H^4(S^2 \times S^2; \mathbb{Z})$, and so A is orientation-preserving. Hence $H^4(X; \mathbb{Z}) \cong \mathbb{Z}$ because X is a closed, orientable 4-manifold.

4. (Complex Analysis) Use the integral

$$\int_{|z|=1} \left(z + \frac{1}{z} \right)^{2n} \frac{dz}{z}$$

to compute

$$\int_{\theta=0}^{2\pi} \cos^{2n} \theta d\theta$$

to yield an answer of the form

$$C\pi \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$$

where C is a positive integer. Determine C.

Solution.

$$\begin{aligned}
\int_{\theta=0}^{2\pi} \cos^{2n} \theta d\theta &= \int_{|z|=1} \left[\frac{1}{2} \left(z + \frac{1}{z} \right) \right]^{2n} \frac{dz}{iz} \\
&= \frac{1}{2^{2n} i} \int_{|z|=1} \left(z^{2n-1} + \binom{2n}{1} z^{2n-3} + \binom{2n}{2} z^{2n-5} \right. \\
&\qquad \left. + \cdots + z^{-2n-1} \right) dz \\
&= \frac{1}{2^{2n} i} 2\pi i \binom{2n}{n} = 2\pi \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}
\end{aligned}$$

with $C = 2$.

5. **(Differential Geometry)** Consider the subset $\Sigma \subset \mathbb{C}^2$ given by

$$\Sigma = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 = |w|^2 = 1/2\}.$$

(a) Prove that Σ is a smooth submanifold of $\mathbb{C}^2$ and compute its codimension.

(b) Show that there is a smoothly embedded 3-torus in $\mathbb{C}^2$ disjoint from Σ.

Solution.

(a) Define a smooth map $F : \mathbb{C}^2 = \mathbb{R}^4 \to \mathbb{R}^2$ by the formula

$$F(x_1, y_1, x_2, y_2) = (x_1^2 + y_1^2 - 1/2, x_2^2 + y_2^2 - 1/2)$$

so that $F^{-1}(0, 0) = \Sigma$. The differential dF is given by

$$dF = \begin{pmatrix} 2x_1 & 2y_1 & 0 & 0 \\ 0 & 0 & 2x_2 & 2y_2 \end{pmatrix},$$

which has rank 2 at points of Σ (where it cannot be that x_1 and y_1 are simultaneously zero, and likewise for x_2 and y_2) and so in particular $(0, 0) \in \mathbb{R}^2$ is a regular value of the map F. By the regular value theorem, Σ is a smooth submanifold of $\mathbb{C}^2$ of (real) codimension 2.

(b) Observe that the normal bundle to Σ in $\mathbb{C}^2$ is a trivial real vector bundle of rank 2 (one can exhibit an explicit global frame, for instance by observing that Σ is the product of two circles of radius $1/\sqrt{2}$). Considering the sphere bundle in the normal bundle of radius some $\epsilon > 0$ sufficiently small, its image under the exponential map will be a submanifold of $\mathbb{C}^2$, diffeomorphic to $S^1 \times \Sigma = S^1 \times S^1 \times S^1$, which is disjoint from Σ.

6. **(Real Analysis)** Let V be a real inner-product space.

(a) Define what it means for V to be a *Hilbert space.*

(b) Let V be a Hilbert space and suppose $W \subseteq V$ is a closed subspace (in the metric topology associated to the inner product). Prove that for all $v \in V$ there exists a unique $w \in W$ that minimizes the distance from v to w. (You must prove both existence and uniqueness for full credit.)

Solution.

(a) V is a Hilbert space if and only if it is complete with respect to the metric topology associated to the inner product.

(b) For $w, w' \in W$ we apply the parallelogram identity $\|a+b\|^2 + \|a-b\|^2 = 2(\|a\|^2 + \|b\|^2)$ to $(a, b) = (v - w, v - w')$, then divide both sides by 4 to find

$$\left\| v - \frac{w + w'}{2} \right\|^2 + \left\| \frac{w - w'}{2} \right\|^2 = \frac{1}{2} \left(\|v - w\|^2 + \|v - w'\|^2 \right).$$

Uniqueness soon follows (and without need for the hypothesis that V is Hilbert): If w and w' both minimize the distance, say at d, the RHS is $\frac{1}{2}(d^2+d^2) = d^2$, and the LHS is at least $d^2 + \frac{1}{4}\|w - w'\|^2$, so $\|w - w'\| = 0$ and $w = w'$.

For existence: the subset $\{\|v-w\| : w \in W\}$ of $\mathbb{R}$ is nonempty (let $w = 0$) and bounded below (because distances are nonnegative). Thus it has an infimum, call it d. Let $w_1, w_2, w_3, \ldots$ be a sequence of vectors $w_n \in W$ such that $\|v - w_n\| \to d$. We claim that this sequence $\{w_n\}$ is Cauchy. Since V is complete, the sequence then converges to some $w_\infty \in V$; since W is closed, this limit must be in W; and since the distance function is continuous, $\|v - w_\infty\| = d$ as desired.

To prove that the sequence is Cauchy: given $\epsilon > 0$, there exists N such that $n > N \Rightarrow \|v - w_n\|^2 < d^2 + \epsilon^2$. Suppose now that $n, n' > N$, and take $w = w_n, w' = w_{n'}$ in our identity. The RHS is less than $d^2 + \epsilon^2$. On the LHS, the first term is at least d^2, so the second is less than ϵ^2. It follows that $\|w_n - w_{n'}\| < 2\epsilon$. Since ϵ is an arbitrary positive number, we are done.