QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday September 1, 2026 (Day 1)

1. (Algebra) Let A be a commutative ring with unit, and M a module over A.

(a) Define what it means for M to be a *Noetherian* A-module, and what it means for A to be a *Noetherian ring*.

(b) Now suppose A is the polynomial ring $\mathbb{C}[z]$, and M its fraction field $\mathbb{C}(z)$ considered as an A-module. Show that A is a Noetherian ring but M is not a Noetherian A-module.

2. (Algebraic Geometry) Consider the cubic curve

$$C := V(y^2 z - x^3) \subset \mathbb{P}^2_{\mathbb{C}}.$$

Show that C is birational to $\mathbb{P}^1_{\mathbb{C}}$ and exhibit an explicit birational equivalence. Prove that C is not isomorphic to $\mathbb{P}^1_{\mathbb{C}}$.

3. (Algebraic Topology) Let $n \geq 1$ and let $i : S^n \to S^n$ denote the antipodal map.

(a) Suppose n is even. For any continuous map $f : S^n \to S^n$, show that there exists a point $x \in S^n$ such that $f(x) = x$ or $f(x) = i(x)$.

(b) For every odd integer n, find a continuous map $f : S^n \to S^n$ such that $f(x) \neq x$ and $f(x) \neq i(x)$ for all $x \in S^n$.

4. (Complex Analysis)

(a) Determine the constant C in

$$u(x, y) = x^2 + Cy^2 - 2xy - 2x + 3y$$

so that u is a harmonic function.

(b) With the value C determined from (a), find a holomorphic function f with $f(0) = 0$ such that the real part of f is u.

5. **(Differential Geometry)** Consider $\mathbb{R}^3$ with coordinates (x_1, x_2, x_3) and define a 2-form β on the unit sphere $S^2 \subset \mathbb{R}^3$ by the formula

$$\beta = x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2.$$

Given an immersion $f : S^2 \to \mathbb{R}^3$ denote by $F : S^2 \to S^2$ the map assigning to a point $p \in S^2$ the unit normal vector to the immersion f at p. To fix orientation, choose F so that whenever (e_1, e_2) is a positively oriented basis of $T_p(S^2)$,

$$(df(e_1), df(e_2), F(p))$$

is positively oriented in $\mathbb{R}^3$. Show that

$$\int_{S^2} F^*\beta = 4\pi.$$

6. **(Real Analysis)** Show that if $f : \mathbb{R} \to \mathbb{C}$ is a $\mathcal{C}^\infty$ function satisfying $f(x+1) = f(x)$ for all $x \in \mathbb{R}$ then there exists a unique $\mathcal{C}^\infty$ function $g : \mathbb{R} \to \mathbb{C}$, satisfying the same periodicity $g(x+1) = g(x)$ for all $x \in \mathbb{R}$, that solves the differential equation $g'' + 2026g = f$.

QUALIFYING EXAMINATION

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Wednesday September 2, 2026 (Day 2)

1. (Algebra) Consider the polynomial $f(x) = x^{12} - 2$.

(a) If K denotes the splitting field of f over $\mathbb{F}_7$, compute the degree of the extension $K/\mathbb{F}_7$.

(b) Describe a generator for $\mathrm{Gal}(K/\mathbb{F}_7)$.

2. (Algebraic Geometry) Let $X \subset \mathbb{P}^n$ be an irreducible projective variety of dimension d over an algebraically closed field. Let $\mathbb{G}(\ell, n)$ be the Grassmannian of ℓ-planes in $\mathbb{P}^n$ with $1 \leq \ell < n-d$, and let $C(X) \subset \mathbb{G}(\ell, n)$ be the locus of ℓ-planes meeting X. Prove that $C(X)$ is irreducible, and find its dimension.

3. (Algebraic Topology) Let X be the space obtained from the union of $S^1 \times S^2$ and the closed 3-disk D^3 by gluing $\partial D^3 = S^2$ to $\{p\} \times S^2$ for some $p \in S^1$. Compute $H_n(X; \mathbb{Z})$ for $n = 1$ and $n = 2$.

4. (Complex Analysis) Let $\mathbb{D}$ denote the open unit disk in $\mathbb{C}$. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. Show for all $z \in \mathbb{D}$ that

$$\left| \frac{f(0) - f(z)}{1 - \overline{f(0)} f(z)} \right| \leq |z|.$$

5. (Differential Geometry) Consider the following smooth real manifolds.

(a) S^n

(b) $\mathbb{RP}^n$.

Do these admit a non-exact non-zero closed differential k-form for $k \geq 1$? If yes construct an explicit example. If not explain why.

6. (Real Analysis) Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of real functions ($n \in \mathbb{N}$).

(a) Give a clear statement of the hypotheses and conclusions of the monotone convergence theorem as it applies to such a sequence.

(b) Give a clear statement of the hypotheses and conclusions of the dominated convergence theorem as it applies to such a sequence.

(c) Let $g_n : \mathbb{R} \to \mathbb{R}$ be Lebesgue integrable, for all n. Using the monotone and dominated convergence theorems, show that the series

$$\sum_{n=1}^{\infty} g_n(x)$$

converges for almost all $x \in \mathbb{R}$ provided that the series

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}} |g_n|$$

converges. Further, writing $G(x)$ for the limit of the series (defined almost everywhere), show that G is integrable and

$$\int G = \sum \int g_n.$$

QUALIFYING EXAMINATION

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Thursday September 3, 2026 (Day 3)

1. (Algebra) Let G be the symmetric group S_5.

(a) Show that G has 7 conjugacy classes, and determine the order of their elements.

(b) It is known that G has two non-isomorphic irreducible complex representations of dimension 4, and two non-isomorphic irreducible representations of dimension 5. Using this, determine the dimensions of the remaining isomorphism classes of irreducible representations of G.

2. (Algebraic Geometry) Let $\mathbb{A}^n$ denote the affine space of dimension n over an algebraically closed field k.

(a) Describe an open covering of $\mathbb{A}^n \setminus \{0\}$ by affine open subsets.

(b) For $n \geq 2$, describe the ring of regular functions on $\mathbb{A}^n \setminus \{0\}$.

(c) Show that $\mathbb{A}^n \setminus \{0\}$ is not affine when $n \geq 2$.

3. (Algebraic Topology) Let $A : S^2 \times S^2 \to S^2 \times S^2$ denote the map

$$A(x, y) = (-x, -y)$$

where $x \mapsto -x$ denotes the antipodal map of S^2. Consider the quotient space

$$X := (S^2 \times S^2)/\langle A \rangle.$$

(a) Show that $\pi_n(\mathbb{RP}^2 \times S^2) \cong \pi_n(X)$ for all $n \geq 1$.

(b) Show that $\mathbb{RP}^2 \times S^2$ and X are not homotopy equivalent. (*Hint.* For this part you may want to consider H^4 of each space.)

4. (Complex Analysis) Use the integral

$$\int_{|z|=1} \left(z + \frac{1}{z} \right)^{2n} \frac{dz}{z}$$

to compute

$$\int_{\theta=0}^{2\pi} \cos^{2n}\theta d\theta$$

to yield an answer of the form

$$C\pi\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$$

where C is a positive integer. Determine C.

5. (Differential Geometry) Consider the subset $\Sigma \subset \mathbb{C}^2$ given by

$$\Sigma = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 = |w|^2 = 1/2\}.$$

(a) Prove that Σ is a smooth submanifold of $\mathbb{C}^2$ and compute its codimension.

(b) Show that there is a smoothly embedded 3-torus in $\mathbb{C}^2$ disjoint from Σ.

6. (Real Analysis) Let V be a real inner-product space.

(a) Define what it means for V to be a *Hilbert space*.

(b) Let V be a Hilbert space and suppose $W \subseteq V$ is a closed subspace (in the metric topology associated to the inner product). Prove that for all $v \in V$ there exists a unique $w \in W$ that minimizes the distance from v to w. (You must prove both existence and uniqueness for full credit.)