QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday September 2, 2025 (Day 1)

1. (Algebra) Prove that every group of size 45 is abelian.

2. (Algebraic Geometry) Let $X \subset \mathbb{A}^3_{\mathbb{C}}$ be a subvariety defined by the equation $xy = z^2$.

(a) Show that X is not smooth, compute the dimension of the Zariski tangent space at $(0, 0, 0) \in X$.

(b) Consider the blow up $Y := \mathrm{Bl}_{(0,0,0)} X$ at the point $(0, 0, 0)$. Show that Y is smooth.

3. (Algebraic Topology) Show that $S^2 \vee S^4$ and $\mathbb{CP}^2$ are not homotopy equivalent.

4. (Complex Analysis) Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx.$$

5. (Differential Geometry) Let G be the Lie group $\mathrm{SU}(N)$.

(a) Show that a left-invariant one-form on G is never closed, unless it is zero.

(b) In the case $N = 2$, show that every left-invariant two-form on G is closed.

6. (Real Analysis) Let H and K be two Hilbert spaces. A set Q of bounded linear transformations $H \to K$ is *weakly bounded* if for every $f \in H$ and $g \in K$, there exists a scalar α such that $|\langle Af, g \rangle| \leq \alpha$ for all $A \in Q$.

Prove that every weakly bounded set of bounded linear transformations between Hilbert spaces is bounded.

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Wednesday September 3, 2025 (Day 2)

1. (Algebra) Let $G \cong S_4$ be the group of rotational symmetries of the cube in $\mathbb{R}^3$, and let V be its (complexified) geometric 3-dimensional irreducible representation. Let π be the complex representation of G arising from the permutation representation on the set of 4-element subsets of the 8 vertices. Write down the characters of the two representations π and V. What is the multiplicity of the irreducible representation V in π?

2. (Algebraic Geometry) By considering divisors in the canonical class, or otherwise, show that every smooth, complex projective curve C of genus 2 admits a regular map $C \to \mathbb{CP}^1$ of degree 2.

3. (Algebraic Topology) Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be a torus. For any homeomorphism $\varphi : T \to T$, consider the *mapping torus* M_φ, which is defined to be the quotient of $T \times [0,1]$ obtained by identifying each point $(x, 1)$ with $(\varphi(x), 0)$. Compute $\pi_n(M_\varphi)$ for all $n \geq 2$.

4. (Complex Analysis) Find a conformal map from the region

$$\Omega = \{z : |z-1| > 1 \text{ and } |z-2| < 2\} \subset \mathbb{C}$$

onto the upper half-plane $\mathbb{H} = \{z : \Im(z) > 0\}$.

5. (Differential Geometry) Let $V_k(\mathbb{R}^n) = \{A \in M_{n\times k}(\mathbb{R}) \mid A^\top A = I_k\}$.

(a) Show that $V_k(\mathbb{R}^n)$ is a smooth submanifold of $M_{n\times k}(\mathbb{R})$ and compute its dimension.

(b) Show that $T_A V_k(\mathbb{R}^n) = \{X \in M_{n\times k}(\mathbb{R}) \mid A^\top X + X^\top A = 0\}$.

(c) Using the inner product $\langle X, Y\rangle := \operatorname{tr}\left(X^\top Y\right)$ in $M_{n\times k}(\mathbb{R})$ or otherwise, construct a Riemannian metric on $V_k(\mathbb{R}^n)$ which is invariant under the natural (left) action of $\mathrm{O}(n)$ on $V_k(\mathbb{R}^n)$. Verify the invariance.

6. (Real Analysis) Let Ω be an open subset of $\mathbb{R}^d$ and $a < b$ be real numbers. For any positive integer n let $f_n(x, y)$ be a complex-valued measurable function on $\Omega \times (a, b)$. Let $a < c < b$. Assume that for each positive integer n the following three conditions are satisfied.

(i) For each n and for almost all $x \in \Omega$ the function $f_n(x, y)$ as a function of y is absolutely continuous in y for $y \in (a, b)$.

(ii) The function $\frac{\partial}{\partial y} f_n(x, y)$ is measurable on $\Omega \times (a, b)$ for each n and the function

$$\sum_{n=1}^{\infty} \left| \frac{\partial}{\partial y} f_n(x, y) \right|$$

is integrable on $\Omega \times (a, b)$.

(iii) The function $f_n(x, c)$ is measurable on Ω for each n and the function $\sum_{n=1}^{\infty} |f_n(x, c)|$ is integrable on Ω.

Prove that the function

$$y \mapsto \int_{x \in \Omega} \sum_{n=1}^{\infty} f_n(x, y) dx$$

is a well-defined function for almost all points y of (a, b) and that

$$\frac{d}{dy} \int_{x \in \Omega} \sum_{n=1}^{\infty} f_n(x, y) dx = \sum_{n=1}^{\infty} \int_{x \in \Omega} \left(\frac{\partial}{\partial y} f_n(x, y) \right) dx$$

for almost all $y \in (a, b)$.

Hint: Use Fubini's theorem to exchange the order of integration and use convergence theorems for integrals of sequences of functions to exchange the order of summation and integration.

QUALIFYING EXAMINATION

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Thursday September 4, 2025 (Day 3)

1. (Algebra) Let $K \subset \mathbb{C}$ be the field generated over $\mathbb{Q}$ by the 12th root of unity $\alpha = e^{2\pi i/12}$.

(a) Describe the structure of the Galois group of this extension and its action on K.

(b) Find the minimal polynomial of α over $\mathbb{Q}$.

(c) Describe the intermediate fields, contained strictly between $\mathbb{Q}$ and K. Express each one as $\mathbb{Q}(\sqrt{d})$ for an explicit $d \in \mathbb{Z}$.

2. (Algebraic Geometry) Let x, y denote coordinates of the affine plane $\mathbb{A}^2$ over $\mathbb{C}$. Consider the following affine plane curves C_i over $\mathbb{C}$:

$$C_1 = V(xy - 1)$$
$$C_2 = V(xy)$$
$$C_3 = V(y - x^2)$$
$$C_4 = V(x^2 + y^2)$$
$$C_5 = V(x^2 - x)$$

(a) For each $1 \leq i, j \leq 5$, determine whether the curves C_i and C_j are isomorphic.

(b) Consider the curve

$$C_6 = V(y^2 - x^3).$$

Show that there exists a regular map $C_3 \to C_6$ which is bijective on points but that the curves C_3 and C_6 are not isomorphic.

3. (Algebraic Topology) Let Σ_g denote a closed, oriented surface of genus g. Prove that there is a covering map $\Sigma_g \to \Sigma_h$ if and only if $g - 1$ is a positive integer multiple of $h - 1$.

4. (Complex Analysis) Let

$$f(z) = z^8 - 2z^2 + 18z - 3 + e^z.$$

Use Rouché's theorem to find, with multiplicities counted,

(a) the number of roots of $f(z)$ in $|z| < 1$,

(b) the number of roots of $f(z)$ in $|z| < 2$.

Hint: Use $|e^z| \leq 9$ on $|z| \leq 2$. In both parts, write $f(z)$ as the sum of a monomial-term and the rest of its terms.

5. (Differential Geometry) Let S^2 be the unit sphere in $\mathbb{R}^3$, so that TS^2 is regarded as a subbundle of the trivial bundle $\underline{\mathbb{R}}^3$ on S^2. The rotation of S^2 about the z-axis is generated by the vector field V on S^2 given by

$$V(x, y, z) = (-y, x, 0).$$

(a) Compute the covariant derivative $\nabla_V V$ on S^2.

(b) From the calculation in the previous part, which non-trivial integral curves of V are geodesics on S^2? Give a geometric interpretation of your answer.

6. (Real Analysis) Denote by $\mathcal{S}(\mathbb{R})$ the *Schwarz space* on $\mathbb{R}$ consisting of all complex-valued C^∞ functions $f(x)$ on $\mathbb{R}$ such that

$$\sup_{x \in \mathbb{R}} |x|^k \left| \frac{d^\ell f}{dx^\ell}(x) \right| < \infty \quad \text{for all } k, \ell \in \mathbb{N} \cup \{0\}.$$

Suppose $\psi(x)$ is a function in $\mathcal{S}(\mathbb{R})$ with

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1. \tag{1}$$

Denote by $\hat{\psi}(\xi)$ the *Fourier transform* of $\psi(x)$ defined by

$$\hat{\psi}(\xi) = \int_{-\infty}^{\infty} \psi(x) e^{-2\pi i x \xi} dx.$$

Prove the following Fourier-transform version of the *Heisenberg uncertainty principle*

$$\left(\int_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx \right) \left(\int_{-\infty}^{\infty} \xi^2 |\hat{\psi}(\xi)|^2 d\xi \right) \geq \frac{1}{16\pi^2}.$$

*H*int: Write the integrand in equation (1) as $1 \cdot |\psi(x)|^2$ and integrate by parts. Use the Plancherel formula which equates the L^2 norm of an element of $\mathcal{S}(\mathbb{R})$ to the L^2 norm of its Fourier transform. Apply it to the derivative of an element of $\mathcal{S}(\mathbb{R})$.