# QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday September 2, 2025 (Day 1)

**1. (Algebra)** Prove that every group of size 45 is abelian.

**Solution.**

Let $G$ be a group of size 45. Let $n_3$ be the number of 3-Sylow subgroups of $G$. By the third Sylow theorem, we have

$$n_3|5 \quad \text{and} \quad n_3 \equiv 1 \bmod 3,$$

so that $n_3 = 1$. It follows that the unique 3-Sylow subgroup $P$ of $G$ is normal. Likewise, we find that $G$ has a unique 5-Sylow subgroup, denoted $Q$, which is again normal.

As $P$ is normal, $PQ$ is a subgroup of $G$. It follows from Lagrange's theorem that $PQ = G$. It is therefore enough to show that the elements of $P$ commute with those of $Q$. To see this, first note that $P \cap Q = \{e\}$ by Lagrange's theorem. Then, for every $p \in P$ and $q \in Q$, we have that

$$pqp^{-1}q^{-1} \in P \cap Q = \{e\},$$

thereby concluding the proof.

**2. (Algebraic Geometry)** Let $X \subset \mathbb{A}^3_{\mathbb{C}}$ be a subvariety defined by the equation $xy = z^2$.

(a) Show that $X$ is not smooth, compute the dimension of the Zariski tangent space at $(0,0,0) \in X$.

(b) Consider the blow up $Y := \mathrm{Bl}_{(0,0,0)} X$ at the point $(0,0,0)$. Show that $Y$ is smooth.

**Solution.**

(a) Let $S = \mathbb{C}[x,y,z]$ and $R = S/(f)$ with $f = xy - z^2$. Then:
- The Jacobian is $(\partial f/\partial x, \partial f/\partial y, \partial f/\partial z) = (y, x, -2z)$. All partial derivatives vanish simultaneously only at $(0,0,0)$, so $X$ has a (unique) singular point at the origin and is not smooth.
- We have $\Omega^1_{(0,0,0)}(X) = \mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m} \subset R$ is the maximal ideal generated by (the classes of) $x, y, z$. It is easy to see that classes of $x, y, z$ are linearly independent so must a basis in $\mathfrak{m}/\mathfrak{m}^2$, hence, $\dim \Omega^1_{(0,0,0)}(X) = 3$.

(b) Work on the standard affine charts of the blow-up of $\mathbb{A}^3$ at 0, and then intersect with the strict transform of $X$.

*Chart $U_x$:* coordinates $(x, y_1, z_1)$ with $y = xy_1$, $z = xz_1$. On this chart,

$$xy = z^2 \iff x(xy_1) = (xz_1)^2 \iff x^2 y_1 = x^2 z_1^2.$$

Removing the exceptional factor $x^2$ (i.e. passing to the strict transform) gives the equation

$$y_1 = z_1^2.$$

Thus the strict transform is isomorphic to $\operatorname{Spec} \mathbb{C}[x, z_1]$, an affine plane; in particular, it is smooth on $U_x$.

*Chart $U_y$:* coordinates $(y, x_1, z_1)$ with $x = yx_1$, $z = yz_1$. The strict transform is

$$y^2 x_1 = y^2 z_1^2 \implies x_1 = z_1^2,$$

again isomorphic to $\operatorname{Spec} \mathbb{C}[y, z_1]$, hence smooth.

*Chart $U_z$:* coordinates $(z, x_1, y_1)$ with $x = zx_1$, $y = zy_1$. Here

$$(zx_1)(zy_1) = z^2 \iff z^2 x_1 y_1 = z^2,$$

and the strict transform is given by

$$x_1 y_1 = 1,$$

which is isomorphic to $\operatorname{Spec} \mathbb{C}[z, x_1, x_1^{-1}] \simeq \mathbb{A}^1 \times \mathbb{G}_m$, hence smooth.

Since the strict transform is smooth on each affine chart, the blow-up $Y = \operatorname{Bl}_0 X$ is smooth.

**3. (Algebraic Topology)** Show that $S^2 \vee S^4$ and $\mathbb{CP}^2$ are not homotopy equivalent.

**Solution.**

Recall that as rings,

$$H^*(\mathbb{CP}^2; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^3),$$

where $\alpha$ has degree 2. In particular, there exists some $c \in H^2(\mathbb{CP}^2; \mathbb{Z})$ such that $c \smile c \neq 0 \in H^4(\mathbb{CP}^2; \mathbb{Z})$.

By e.g. Mayer–Vietoris, the inclusion $i : S^2 \hookrightarrow S^2 \vee S^4$ induces an isomorphism $i^* : H^2(S^2 \vee S^4; \mathbb{Z}) \to H^2(S^2; \mathbb{Z})$. If $q : S^2 \vee S^4 \to S^2$ is the map that crushes $S^4$ to a point, then $q^* : H^2(S^2; \mathbb{Z}) \to H^2(S^4 \vee S^2; \mathbb{Z})$ is an inverse to $i^*$ and hence an isomorphism. For any $q^*(c) \in H^2(S^4 \vee S^2; \mathbb{Z})$,

$$q^*(c) \smile q^*(c) = q^*(0) = 0 \in H^4(S^2 \vee S^4; \mathbb{Z}).$$

Therefore the cohomology rings $H^*(\mathbb{CP}^2; \mathbb{Z})$ and $H^*(S^2 \vee S^4; \mathbb{Z})$ are not isomorphic as graded rings, and $\mathbb{CP}^2$ and $S^2 \vee S^4$ are not homotopy equivalent.

**4. (Complex Analysis)** Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx.$$

**Solution.** Let $C$ (resp., $C'$) be a contour on the complex plane from $-\infty$ to $+\infty$ which goes along the real axis for most part but goes above (resp., below) the origin. Then,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx &= \int_{C'} \frac{\sin^2 z}{z^2} dz \\
&= \int_{C'} \frac{2 - e^{2iz} - e^{-2iz}}{4z^2} dz \\
&= \int_{C'} \frac{1 - e^{2iz}}{4z^2} dz + \int_{C'} \frac{1 - e^{-2iz}}{4z^2} dz \\
&= 2\pi i \operatorname{Res}_{z=0} \left( \frac{1 - e^{2iz}}{4z^2} \right) + \int_{C} \frac{1 - e^{2iz}}{4z^2} dz + \int_{C'} \frac{1 - e^{-2iz}}{4z^2} dz \\
&= 2\pi i \left( -\frac{i}{2} \right) \ + \ 0 \ + \ 0 \\
&= \pi,
\end{aligned}$$

where, in the second last equality, we have pushed the contour $C$ (resp., $C'$) up (resp., down) to infinity, using the fact that $1 - e^{2iz}$ (resp., $1 - e^{-2iz}$) is bounded in the upper (resp., lower) half-plane.

**5. (Differential Geometry)** Let $G$ be the Lie group $\mathrm{SU}(N)$.

(a) Show that a left-invariant one-form on $G$ is never closed, unless it is zero.

(b) In the case $N = 2$, show that every left-invariant two-form on $G$ is closed.

**Solution.**

(a) Let $\alpha$ be a closed, left-invariant 1-form. For a vector field $X$ on $G$, we have the Cartan formula for the Lie derivative:

$$\begin{aligned}
\mathcal{L}_X \alpha &= d\,\imath_X \alpha + \imath_X \, d\alpha, \\
&= d\,\imath_X \alpha
\end{aligned}$$

because $d\alpha = 0$. If $X$ is also left-invariant, then $\imath_X \alpha$ is a constant function, so the Cartan formula gives $\mathcal{L}_X \alpha = 0$. If $A$ is the vector field that is dual to $\alpha$ using the bi-invariant Riemannian metric on $G$, then this becomes

$$0 = \mathcal{L}_X A = [X, A].$$

This means that $A$ is a central element of the Lie algebra, or in other words a trace-zero, skew-adjoint matrix that commutes with all others.

The only such $A$ is zero, so $\alpha = 0$. (Almost the same argument works for any semisimple Lie group.)

*Alternative for this part.* Use the fact that the de Rham cohomology group $H^1(\mathrm{SU}(N);\mathbb{R})$ is zero. So if $\alpha$ is a closed, then $\alpha = df$ for some real function $f$. This means that $\alpha$ must vanish somewhere (at the maximum of $f$, for example), and since $\alpha$ is left-invariant it must vanish everywhere.

(b) If $\beta$ is a left-invariant 2-form that is not closed, then $d\beta$ is a non-zero, exact 3-form which is also left-invariant. In the case $N = 2$, the group $G$ is the sphere $S^3$, and a non-zero left-invariant form must either be everywhere positive or everywhere negative. In any case, $d\beta$ would have non-zero integral on $S^3$, which contradicts Stokes' theorem.

*Alternative for this part.* The left-invariant 1-forms are a 3-dimensional vector space. The exterior derivative is an injective map from left-invariant 1-forms to left-invariant 2-forms, by the first part. The left-invariant 2-forms are also a 3-dimensional vector space, so the exterior derivative is surjective. In other words, every left-invariant 2-form is exact.

**6. (Real Analysis)** Let $H$ and $K$ be two Hilbert spaces. A set $Q$ of bounded linear transformations $H \to K$ is *weakly bounded* if for every $f \in H$ and $g \in K$, there exists a scalar $\alpha$ such that $|\langle Af, g\rangle| \leq \alpha$ for all $A \in Q$.

Prove that every weakly bounded set of bounded linear transformations between Hilbert spaces is bounded.

**Solution.**

For any fixed $g \in K$, the subset $\{\, A^*g \mid A \in Q\} \subseteq H$ is weakly bounded. Namely, for every $f \in H$, there exists a scalar $\alpha$ such that

$$|\langle f, A^*g\rangle| = |\langle Af, g\rangle| \leq \alpha.$$

By the principle of uniform boundedness for Hilbert spaces, it follows that there exists a scalar $\beta$, depending on $g$ only, such that $||A^*g|| \leq \beta$ for all $A \in Q$.

We now consider the subset $\{\, Af \mid A \in Q,\ f \in H \text{ with } ||f|| = 1\,\} \subseteq K$. We claim that it is also bounded. Namely, for every $g \in K$, we have

$$|\langle Af, g\rangle| = |\langle f, A^*g\rangle| \leq ||f|| \cdot ||A^*g|| \leq \beta.$$

It follows again from uniform boundedness for Hilbert spaces that there exists a scalar $\gamma$ such that $||Af|| \leq \gamma$ for all $A \in Q$ and unit vector $f \in H$. This shows that $||A|| \leq \gamma$ for all $A \in Q$, i.e. we have shown that $Q$ is bounded as desired.

**QUALIFYING EXAMINATION**

Harvard University

Department of Mathematics

Wednesday September 3, 2025 (Day 2)

**1. (Algebra)** Let $G \cong S_4$ be the group of rotational symmetries of the cube in $\mathbb{R}^3$, and let $V$ be its (complexified) geometric 3-dimensional irreducible representation. Let $\pi$ be the complex representation of $G$ arising from the permutation representation on the set of 4-element subsets of the 8 vertices. Write down the characters of the two representations $\pi$ and $V$. What is the multiplicity of the irreducible representation $V$ in $\pi$?

**Solution.**

The answer is 7. The multiplicity of $V$ in $\pi$ is the inner product

$$m = \langle \chi_V, \chi_\pi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)\, \chi_\pi(g).$$

Here, for the permutation representation $\pi$, the character $\chi_\pi(g)$ is the number of 4-element subsets fixed by $g$.

This sum can be written as a sum over conjugacy classes. The conjugacy classes are:

1. $C_1$, the conjugacy class of the identity element, size 1.
2. $C_2$, the conjugacy class of elements of order 2 arising as rotations about axes through faces, size 3.
3. $C_2'$, the conjugacy class of elements of order 2 arising as rotations about axes through midpoints of edges, size 6.
4. $C_3$, the conjugacy class of rotations through angle $2\pi/3$ about axes through vertices, size 8.
5. $C_4$, the conjugacy class of rotations through angle $\pi/2$ about face axes, size 6.

For each class, we find the number of fixed 4-element subsets (the character of $\pi$):

1. for $C_1$, $\chi_\pi = \binom{8}{4} = 70$.
2. for $C_2$, $\chi_\pi = \binom{4}{2} = 6$.
3. for $C_2'$, $\chi_\pi = \binom{4}{2} = 6$.

4. for $C_3$, $\chi_\pi = 2 \times 2 = 4$ (because we choose one fixed vertex and one 3-cycle to make up the 4).
5. for $C_4$, $\chi_\pi = 2$.

Meanwhile for $\chi_V$, the character of a rotation through angle $\theta$ is $1 + 2\cos(\theta)$, so we have:

1. for $C_1$, $\chi_V = 3$.
2. for $C_2$, $\chi_V = -1$.
3. for $C_2'$, $\chi_V = -1$.
4. for $C_3$, $\chi_V = 0$.
5. for $C_4$, $\chi_V = 1$.

We compute the sum:

$$\frac{1}{24}\left(1 \cdot 3 \cdot 70 + 3 \cdot (-1) \cdot 6 + 6 \cdot (-1) \cdot 6 + 8 \cdot 0 \cdot 4 + 6 \cdot 1 \cdot 2\right).$$

This is,

$$(210 - 18 - 36 + 12)/24 = 7.$$

**2. (Algebraic Geometry)** By considering divisors in the canonical class, or otherwise, show that every smooth, complex projective curve $C$ of genus 2 admits a regular map $C \to \mathbb{CP}^1$ of degree 2.

**Solution.**

Let $K_C$ denote the canonical class of $C$. Then $\ell(0) = 1$ and $\ell(K_C) = 2$ is the genus of $C$. Let $D$ denote a divisor in the class $K_C$; because $\ell(K_C) > 0$, we may assume that $D$ is effective after possibly replacing $D$ with a linearly equivalent divisor. Moreover, there exists a nonconstant rational function $f$ of $C$ so that $(f) + D \geq 0$ because $\ell(K_C) > 1$. The Riemann–Roch theorem shows that $\deg(D) = 2$.

The rational function $f$ defines a nonconstant map $f : C \to \mathbb{CP}^1$. Because $D$ is effective and $\deg(D) = 2$, $f$ has at most two poles (counted with multiplicity). Because $f$ is nonconstant and not an isomorphism, the number of poles (with multiplicity) cannot be zero or one. Therefore, $f$ has exactly two poles, and $\deg(f) = 2$.

**3. (Algebraic Topology)** Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be a torus. For any homeomorphism $\varphi : T \to T$, consider the *mapping torus* $M_\varphi$, which is defined to be the quotient of $T \times [0,1]$ obtained by identifying each point $(x,1)$ with $(\varphi(x), 0)$. Compute $\pi_n(M_\varphi)$ for all $n \geq 2$.

**Solution.**

The projection $T \times [0,1] \to [0,1]$ descends to a well-defined map $M_\varphi \to S^1$, turning $M_\varphi$ into an $T$-bundle over $S^1$. For each $n \geq 2$,

$$\pi_n(T) = 0, \qquad \pi_n(S^1) = 0$$

because the universal covers of $T$ and $S^1$ are both contractible. So the long exact sequence of the fiber bundle $M_\varphi \to S^1$ shows that $\pi_n(M_\varphi) = 0$ for all $n \geq 2$ as well.

Alternatively, one can explicitly show that the universal cover of $M_\varphi$ is homeomorphic to $\mathbb{R}^3$, and then conclude that $\pi_n(M_\varphi) = \pi_n(\mathbb{R}^3) = 0$ for all $n \geq 2$.

**4. (Complex Analysis)** Find a conformal map from the region

$$\Omega = \{z : |z-1| > 1 \;\text{ and }\; |z-2| < 2\} \subset \mathbb{C}$$

onto the upper half-plane $\mathbb{H} = \{z : \Im(z) > 0\}$.

**Solution.**

Let $S = \{z : \frac{1}{4} < \Re(z) < \frac{1}{2}\}$. Then, we have a conformal equivalence $\Omega \cong S$ given by the map $z \mapsto \frac{1}{z}$, and $S \cong \mathbb{H}$ by $z \mapsto e^{2\pi i (z - \frac{1}{4})}$. That is,

$$\begin{aligned} \Omega &\cong \mathbb{H} \\ z &\mapsto e^{2\pi i (\frac{1}{z} - \frac{1}{4})}. \end{aligned}$$

**5. (Differential Geometry)** Let $V_k(\mathbb{R}^n) = \{A \in M_{n\times k}(\mathbb{R}) \mid A^\top A = I_k\}$.

(a) Show that $V_k(\mathbb{R}^n)$ is a smooth submanifold of $M_{n\times k}(\mathbb{R})$ and compute its dimension.

(b) Show that $T_A V_k(\mathbb{R}^n) = \{X \in M_{n\times k}(\mathbb{R}) \mid A^\top X + X^\top A = 0\}$.

(c) Using the inner product $\langle X, Y\rangle := \operatorname{tr}\left(X^\top Y\right)$ in $M_{n\times k}(\mathbb{R})$ or otherwise, construct a Riemannian metric on $V_k(\mathbb{R}^n)$ which is invariant under the natural (left) action of $\mathrm{O}(n)$ on $V_k(\mathbb{R}^n)$. Verify the invariance.

**Solution.**

(a) Consider $F\colon M_{n\times k}(\mathbb{R}) \to \mathrm{Sym}_k(\mathbb{R})$ given by $F(A) = A^\top A$. Then $V_k(\mathbb{R}^n) = F^{-1}(I_k)$. The differential at $A$ in direction $X$ is

$$d_A F(X) = A^\top X + X^\top A \in \mathrm{Sym}_k(\mathbb{R}).$$

If $A \in V_k(\mathbb{R}^n)$, we claim $d_A F$ is surjective onto $\mathrm{Sym}_k(\mathbb{R})$. Indeed, for any $S \in \mathrm{Sym}_k(\mathbb{R})$, choose $X = \frac{1}{2}AS$, then

$$d_A F(X) = \tfrac{1}{2}(A^\top A S + S^\top A^\top A) = \tfrac{1}{2}(S+S) = S.$$

Hence $I_k$ is a regular value and $V_k(\mathbb{R}^n)$ is a smooth submanifold.
The ambient space $M_{n\times k}(\mathbb{R})$ has dimension $nk$, while the target $\mathrm{Sym}_k(\mathbb{R})$ has dimension $\frac{k(k+1)}{2}$. By the regular value theorem,

$$\dim V_k(\mathbb{R}^n) = nk - \frac{k(k+1)}{2}.$$

(b) The tangent space at $A$ is the kernel of $d_A F$:

$$T_A V_k(\mathbb{R}^n) = \ker(d_A F) = \{X \in M_{n\times k}(\mathbb{R}) \mid A^\top X + X^\top A = 0\}.$$

(c) Restrict the inner product on $M_{n\times k}(\mathbb{R})$ to each tangent space $T_A V_k(\mathbb{R}^n)$ to obtain a Riemannian metric

$$g_A(X,Y) = \mathrm{tr}(X^\top Y), \qquad X, Y \in T_A V_k(\mathbb{R}^n).$$

The natural left action of $\mathrm{O}(n)$ on $V_k = V_k(\mathbb{R}^n)$ is $h \cdot A := hA$ for $h \in \mathrm{O}(n)$.
The differential sends $X \in T_A V_k$ to $hX \in T_{hA} V_k$. Since $h^\top h = \mathrm{Id}_n$,

$$g_{hA}(hX, hY) = \mathrm{tr}\left((hX)^\top (hY)\right) = \mathrm{tr}\left(X^\top h^\top h Y\right) = \mathrm{tr}(X^\top Y) = g_A(X,Y).$$

Thus $g$ is $\mathrm{O}(n)$-invariant.

**6. (Real Analysis)** Let $\Omega$ be an open subset of $\mathbb{R}^d$ and $a < b$ be real numbers. For any positive integer $n$ let $f_n(x,y)$ be a complex-valued measurable function on $\Omega \times (a,b)$. Let $a < c < b$. Assume that for each positive integer $n$ the following three conditions are satisfied.

(i) For each $n$ and for almost all $x \in \Omega$ the function $f_n(x,y)$ as a function of $y$ is absolutely continuous in $y$ for $y \in (a,b)$.

(ii) The function $\frac{\partial}{\partial y} f_n(x,y)$ is measurable on $\Omega \times (a,b)$ for each $n$ and the function

$$\sum_{n=1}^{\infty} \left| \frac{\partial}{\partial y} f_n(x,y) \right|$$

is integrable on $\Omega \times (a,b)$.

(iii) The function $f_n(x,c)$ is measurable on $\Omega$ for each $n$ and the function $\sum_{n=1}^{\infty} |f_n(x,c)|$ is integrable on $\Omega$.

Prove that the function

$$y \mapsto \int_{x\in\Omega} \sum_{n=1}^{\infty} f_n(x,y)dx$$

is a well-defined function for almost all points $y$ of $(a,b)$ and that

$$\frac{d}{dy} \int_{x\in\Omega} \sum_{n=1}^{\infty} f_n(x,y)dx = \sum_{n=1}^{\infty} \int_{x\in\Omega} \left( \frac{\partial}{\partial y} f_n(x,y) \right) dx$$

for almost all $y \in (a,b)$.

*Hint:* Use Fubini's theorem to exchange the order of integration and use convergence theorems for integrals of sequences of functions to exchange the order of summation and integration.

**Solution.**

The theorem of Fubini which we will use states that if $F(x,y)$ on $\Omega_1 \times \Omega_2$ (with $\Omega_j$ open in $\mathbb{R}^{d_j}$ for $j = 1, 2$) and if

$$\int_{(x,y)\in\Omega_1\times\Omega_2} |F(x,y)| < \infty,$$

then

$$\int_{x\in\Omega_1} \left( \int_{y\in\Omega_2} F(x,y)dy \right) dx = \int_{y\in\Omega_2} \left( \int_{x\in\Omega_1} F(x,y)dx \right) dy.$$

One consequence of the theorem of dominated convergence which we will use is the following exchange of integration and summation. If $F_n(x)$ is a sequence of measurable functions on an open subset $\tilde{\Omega}$ of $\mathbb{R}^{\tilde{d}}$ such that

$$\int_{x\in\tilde{\Omega}} \sum_{n=1}^{\infty} |F_n(x)| < \infty,$$

then

$$\int_{x\in\tilde{\Omega}} \sum_{n=1}^{\infty} F_n(x) = \sum_{n=1}^{\infty} \int_{x\in\tilde{\Omega}} F_n(x).$$

These two results make it possible for us to both exchange the order of integration and the order of summation and integration in the following equation for $a < \eta < b$,

(†)

$$\int_{y=c}^{\eta} \left( \sum_{n=1}^{\infty} \int_{x\in\Omega} \left( \frac{\partial}{\partial y} f_n(x,y) \right) dx \right) dy = \int_{x\in\Omega} \left( \sum_{n=1}^{\infty} \int_{y=c}^{\eta} \left( \frac{\partial}{\partial y} f_n(x,y) \right) dy \right) dx,$$

because the function

$$\sum_{n=1}^{\infty} \left| \frac{\partial}{\partial y} f_n(x,y) \right|$$

is integrable on $\Omega \times (a,b)$. Since for almost all $x \in \Omega$ the function $f_n(x,y)$ as a function of $y$ is absolutely continuous in $y$, it follows that

$$\int_{y=c}^{\eta} \left( \frac{\partial}{\partial y} f_n(x,y) \right) dy = f_n(x,\eta) - f_n(x,c)$$

for almost all $x \in \Omega$, which implies that

$$\begin{aligned} \int_{x\in\Omega} \left( \sum_{n=1}^{\infty} \int_{y=c}^{\eta} \left( \frac{\partial}{\partial y} f_n(x,y) \right) dy \right) dx &= \int_{x\in\Omega} \left( \sum_{n=1}^{\infty} (f_n(x,\eta) - f_n(x,c)) \right) dx \\ &= \int_{x\in\Omega} \left( \sum_{n=1}^{\infty} f_n(x,\eta) \right) dx - \int_{x\in\Omega} \left( \sum_{n=1}^{\infty} f_n(x,c) \right) dx, \end{aligned}$$

because $\sum_{n=1}^{\infty} |f_n(x,c)|$ is integrable on $\Omega$. Putting this together with (†) yields
(‡)
$$\int_{y=c}^{\eta} \left( \sum_{n=1}^{\infty} \int_{x\in\Omega} \left( \frac{\partial}{\partial y} f_n(x,y) \right) dx \right) dy = \int_{x\in\Omega} \left( \sum_{n=1}^{\infty} f_n(x,\eta) \right) dx - \int_{x\in\Omega} \left( \sum_{n=1}^{\infty} f_n(x,c) \right) dx.$$

Differentiating both sides of (‡) with respect to $\eta$ and applying the fundamental theorem of calculus in the theory of Lebesgue and then replacing $\eta$ by $y$, we obtain

$$\sum_{n=1}^{\infty} \int_{x\in\Omega} \left( \frac{\partial}{\partial y} f_n(x,y) \right) dx = \frac{\partial}{\partial y} \int_{x\in\Omega} \left( \sum_{n=1}^{\infty} f_n(x,y) \right) dx$$

for almost all $y \in (a,b)$.

**QUALIFYING EXAMINATION**

HARVARD UNIVERSITY

Department of Mathematics

Thursday September 4, 2025 (Day 3)

**1. (Algebra)** Let $K \subset \mathbb{C}$ be the field generated over $\mathbb{Q}$ by the 12th root of unity $\alpha = e^{2\pi i/12}$.

(a) Describe the structure of the Galois group of this extension and its action on $K$.

(b) Find the minimal polynomial of $\alpha$ over $\mathbb{Q}$.

(c) Describe the intermediate fields, contained strictly between $\mathbb{Q}$ and $K$. Express each one as $\mathbb{Q}(\sqrt{d})$ for an explicit $d \in \mathbb{Z}$.

**Solution.**

(a) The Galois conjugates of $\alpha$ are the other primitive 12th roots of 1, namely (in addition to $\alpha$) the elements $\alpha^r$ for $r = 5, 7$ and 11 (the residues coprime to 12). A Galois automorphism $g$ is determined by $g(\alpha)$, and there are four such: the automorphisms $g(\alpha) = \alpha^r$, for $r = 1, 5, 7, 11$. Each of these automorphisms has square 1, because $r^2 = 1 \bmod 12$ for $r = 1, 5, 7$ and 11. The Galois group is therefore isomorphic to $C_2 \times C_2$, this being the only group of order 4 that does not contain an element of order 4.

(b) We seek the 12th cyclotomic polynomial. Its roots are obtained from the set of all 12th roots by striking out the 6th roots and the two primitive 4th roots ($\pm i$). So the polynomial is

$$(x^{12} - 1)/((x^6 - 1)(x^2 + 1))$$

which is $x^4 - x^2 + 1$, by straightforward division.

(c) There are three proper, non-trivial subgroups of $C_2 \times C_2$, each of index 2. So by the Galois correspondence we seek 3 intermediate fields of degree 2 of $\mathbb{Q}$. The field $K$ contains the primitive 3rd roots of unity, $(1 \pm \sqrt{-3})/2$, so $K$ contains $\mathbb{Q}(\sqrt{-3})$. The field contains the 4th roots of unity, so contains $\mathbb{Q}(\sqrt{-1})$. The product of $\sqrt{-3}$ and $\sqrt{-1}$ is $\sqrt{3}$. So $K$ contains $\mathbb{Q}(\sqrt{3})$. Indeed, $K$ is the field $\mathbb{Q}(\sqrt{3}, i)$.

In summary, $K$ contains the three fields $\mathbb{Q}(\sqrt{d})$ for $d = -1, -3, 3$. These three subfields are distinct: the last because it is the only one contained in $\mathbb{R}$ and the first two are different because $\sqrt{3}$ is irrational.

**2.** **(Algebraic Geometry)** Let $x, y$ denote coordinates of the affine plane $\mathbb{A}^2$ over $\mathbb{C}$. Consider the following affine plane curves $C_i$ over $\mathbb{C}$:

$$\begin{aligned}C_1 &= V(xy-1)\\ C_2 &= V(xy)\\ C_3 &= V(y-x^2)\\ C_4 &= V(x^2+y^2)\\ C_5 &= V(x^2-x)\end{aligned}$$

(a) For each $1 \le i, j \le 5$, determine whether the curves $C_i$ and $C_j$ are isomorphic.

(b) Consider the curve
$$C_6 = V(y^2 - x^3).$$
Show that there exists a regular map $C_3 \to C_6$ which is bijective on points but that the curves $C_3$ and $C_6$ are not isomorphic.

**Solution.**

The curves $C_2$ and $C_4$ are isomorphic via the map $\varphi : C_4 \to C_2$

$$(x, y) \mapsto (x + iy, x - iy).$$

The curves $C_1$ and $C_3$ are irreducible, while $C_2$ and $C_5$ are reducible. The curves $C_2$ and $C_5$ are not isomorphic because $C_2$ is singular at $(0, 0)$ while $C_5$ is nonsingular.

It remains to distinguish $C_1$ and $C_3$. Their coordinate rings are

$$\mathbb{C}[C_1] = \mathbb{C}[x, y]/(xy - 1) \cong \mathbb{C}[s, s^{-1}], \qquad \mathbb{C}[C_3] = \mathbb{C}[x, y]/(y - x^2) \cong \mathbb{C}[t]$$

where the isomorphism $\mathbb{C}[C_1] \to \mathbb{C}[s, s^{-1}]$ is given by $x \mapsto s$, $y \mapsto s^{-1}$ and the isomorphism $\mathbb{C}[C_3] \to \mathbb{C}[t]$ is given by $x \mapsto t$, $y \mapsto t^2$. Finally, note that there is no isomorphism $\mathbb{C}[s, s^{-1}] \to \mathbb{C}[t]$; if so, $s \in \mathbb{C}[s, s^{-1}]$ must be mapped to a unit of $\mathbb{C}[t]$, i.e. a nonzero constant.

The regular map
$$C_3 \to C_6, \qquad (x, y) \mapsto (x^2, x^3)$$
is bijective on points. The curves $C_3$ and $C_6$ are not isomorphic because $C_6$ is singular at $(0, 0)$ but $C_3$ is nonsingular.

**3.** **(Algebraic Topology)** Let $\Sigma_g$ denote a closed, oriented surface of genus $g$. Prove that there is a covering map $\Sigma_g \to \Sigma_h$ if and only if $g - 1$ is a positive integer multiple of $h - 1$.

**Solution.**

If there is a covering map $\Sigma_g \to \Sigma_h$, say, of degree $d \geq 1$, then

$$2 - 2g = \chi(\Sigma_g) = d \cdot \chi(\Sigma_h) = d(2 - 2h),$$

and it follows that $g - 1$ must be a positive integer multiple of $h - 1$.

On the other hand, if $(g-1) = d(h-1)$ for some positive integer $d$, then there is an explicit covering map of degree $d$ constructed as follows. First, cut $\Sigma_h$ along an essential $S^1$—playing the role of a branch cut in this construction—to obtain a surface of genus $h - 1$ with 2 boundary components. Take $d$ copies of such surfaces, and glue the boundaries in a cyclic manner. The resulting surface has genus $d(h-1) + 1 = g$. This gives a degree $d$ covering map $\Sigma_g \to \Sigma_h$. Alternatively here, use the classification of covering spaces by subgroups of $\pi_1$: a subgroup of index $d$ exists, for example as the kernel of a homomorphism $\pi_1(\Sigma_h) \to H^1(\Sigma_h; \mathbb{Z}) = \mathbb{Z}^{2h} \to \mathbb{Z}/d$.

**4. (Complex Analysis)** Let

$$f(z) = z^8 - 2z^2 + 18z - 3 + e^z.$$

Use Rouché's theorem to find, with multiplicities counted,

(a) the number of roots of $f(z)$ in $|z| < 1$,

(b) the number of roots of $f(z)$ in $|z| < 2$.

*Hint:* Use $|e^z| \leq 9$ on $|z| \leq 2$. In both parts, write $f(z)$ as the sum of a monomial-term and the rest of its terms.

**Solution.**

(a) Use

$$|18z| = 18 > 1 + 2 + 3 + 3 \geq |z^8 - 2z^2 - 3 + e^z|$$

on $|z| = 1$ to conclude by Rouché's theorem that the number of roots of $f(z) = 0$ in $|z| < 1$ is the same as the number of roots of $18z = 0$ in $|x| < 1$ which is 1.

(b) Use

$$|z^8| = 2^8 = 256 > 8 + 36 + 3 + 9 \geq \left|-2z^2 + 18z - 3 + e^z\right|$$

on $|z| = 2$ to conclude by Rouché's theorem that the number of roots of $f(z) = 0$ in $|z| < 2$ is the same as the number of roots of $z^8 = 0$ in $|x| < 2$ (with multiplicities counted) which is 8.

**5. (Differential Geometry)** Let $S^2$ be the unit sphere in $\mathbb{R}^3$, so that $TS^2$ is regarded as a subbundle of the trivial bundle $\underline{\mathbb{R}}^3$ on $S^2$. The rotation of $S^2$ about the $z$-axis is generated by the vector field $V$ on $S^2$ given by

$$V(x, y, z) = (-y, x, 0).$$

(a) Compute the covariant derivative $\nabla_V V$ on $S^2$.

(b) From the calculation in the previous part, which non-trivial integral curves of $V$ are geodesics on $S^2$? Give a geometric interpretation of your answer.

**Solution.**

(a) Extend $V$ to a vector field $\tilde{V}$ on all of $\mathbb{R}^3$ by the same formula,

$$\tilde{V}(x, y, z) = (-y, x, 0).$$

Compute the derivative $\tilde{\nabla}_{\tilde{V}}\tilde{V}$ on $\mathbb{R}^3$ using the trivial connection:

$$\begin{aligned}\tilde{\nabla}_{\tilde{V}}\tilde{V}(x, y, z) &= -y(\partial/\partial x)(-y, x, 0) + x(\partial/\partial y)(-y, x, 0)\\ &= (-x, -y, 0).\end{aligned}$$

The Levi-Civita connection on the submanifold is obtained from the (trivial) Levi-Civita connection on the ambient space by projection. So we must compute the projection of $W = \tilde{\nabla}_{\tilde{V}}\tilde{V}(x, y, z)$ to $TS^2$ at points on the sphere. This means subtracting the component in the direction of the unit vector $(x, y, z)$:

$$\begin{aligned}\nabla_V V(x, y, z) &= W - \big(W \cdot (x, y, z)\big)(x, y, z)\\ &= (-x, -y, 0) + (x^2 + y^2)(x, y, z)\\ &= (-x, -y, 0) + (1 - z^2)(x, y, z)\\ &= (-z^2 x, -z^2 y, z(1 - z^2)).\end{aligned}$$

(b) The vector field $V$ has constant length along trajectories of $V$, so for geodesic curves we want $\nabla_V V = 0$. From the expression above, this requires at least $z = 0$ or $z = \pm 1$. The latter happens only at the zeros of $V$ at the points $(0, 0, \pm 1)$. So the only non-trivial geodesic is at $z = 0$. Of the many possible geometric interpretations, one may say that the trajectories are all circles and only the great circles (such as $z = 0$) are geodesics, these being the ones that maximize length, so are critical points of the energy functional.

**6. (Real Analysis)** Denote by $\mathcal{S}(\mathbb{R})$ the *Schwarz space* on $\mathbb{R}$ consisting of all complex-valued $C^\infty$ functions $f(x)$ on $\mathbb{R}$ such that

$$\sup_{x\in\mathbb{R}} |x|^k \left|\frac{d^\ell f}{dx^\ell}(x)\right| < \infty \quad \text{for all } k, \ell \in \mathbb{N} \cup \{0\}.$$

Suppose $\psi(x)$ is a function in $\mathcal{S}(\mathbb{R})$ with

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1. \tag{1}$$

Denote by $\hat{\psi}(\xi)$ the *Fourier transform* of $\psi(x)$ defined by

$$\hat{\psi}(\xi) = \int_{-\infty}^{\infty} \psi(x) e^{-2\pi i x\xi} dx.$$

Prove the following Fourier-transform version of the *Heisenberg uncertainty principle*

$$\left(\int_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx\right)\left(\int_{-\infty}^{\infty} \xi^2 |\hat{\psi}(\xi)|^2 d\xi\right) \geq \frac{1}{16\pi^2}.$$

*H*int: Write the integrand in equation (1) as $1 \cdot |\psi(x)|^2$ and integrate by parts. Use the Plancherel formula which equates the $L^2$ norm of an element of $\mathcal{S}(\mathbb{R})$ to the $L^2$ norm of its Fourier transform. Apply it to the derivative of an element of $\mathcal{S}(\mathbb{R})$.

**Solution.**

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\psi(x)|^2 dx \\
&= -\int_{-\infty}^{\infty} x \frac{d}{dx} |\psi(x)|^2 dx \\
&= -\int_{-\infty}^{\infty} \left(x\psi'(x)\overline{\psi(x)} + x\overline{\psi'(x)}\psi(x)\right) dx \\
&\leq 2\int_{-\infty}^{\infty} |x||\psi'(x)||\psi(x)| dx \\
&\leq 2\left(\int_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |\psi'(x)|^2 dx\right)^{\frac{1}{2}} \\
&\qquad\qquad \text{(by Cauchy-Schwarz inequality)} \\
&= 2\left(\int_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx\right)^{\frac{1}{2}} \left(4\pi^2 \int_{-\infty}^{\infty} \xi^2 |\hat{\psi}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \\
&\qquad\qquad \text{(by applying Plancherel formula to } \psi'(x)\text{)},
\end{aligned}$$

which, upon squaring both sides, implies the inequality sought.