How big is that cookie? The Integral Geometric approach to geometrical quantities

Abstract. Integral geometry studies the link between expectation of random variables and geometrical quantities like length, area or curvature. This thesis focuses on how expectation of random variables can be used to define reasonable notions of geometrical size. The simple idea of looking at the shadow when a compact convex body in $\mathbb{R}^n$ is orthogonally projected onto a random k-dimensional subspace will generate a collection of "continuous invariant valuations". A theorem of Hadwiger concludes that this collection spans the vector space of all continuous invariant valuations, and thus integral geometry elegantly provides a complete description about the size of a compact convex body. Furthermore, some of these valuations turn out to be extensions of familiar geometrical concepts like surface area, but escape the typically required continuity or smoothness conditions and offer an alternative interpretation of these concepts.

In choosing a k-dimensional subspace at random, one needs to specify a probability measure that is "invariant under the symmetries of Euclidean space" so that the size of an object is unchanged after it is displaced by a Euclidean motion. We explore the possibility of imposing measures without such restrictions to obtain new kinds of geometries, and perform this exploration on a generic smooth manifold. Finally, we look at how a variant of the integral geometric ideas explored permits the study of geometrical quantities of a different flavor: various forms of curvature.

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Contents

1. Introduction

There is no royal road to geometry.

– Euclid of Alexandria

In the 3^{rd} Century BC, Euclid wrote one of the most influential textbooks of all time – The Elements. In the treatise, Euclid revolutionized geometry by introducing an axiomatic approach. In the two millennia that have since passed, geometrical concepts like length, area and volume continue to be of relevance to the human mind but have been subjected to much higher levels of abstraction and complexity in order to meet the standards of modern mathematics. For example, the definition of surface area can only be made for certain subsets of $\mathbb{R}^3$, such as a smooth surface, and may involve technical machineries such as parametrization and working with integration of forms. Other geometrical concepts such as curvature are only given rigorous definitions on a relatively recent timescale using the abstract setting of manifold theory and differential geometry, and also involve highly technical formulations.

Following the wise words of Euclid, one might gain new insights by exploring an alternative road to understanding some of these geometrical quantities, especially in light of the high level of abstractions and technicalities involved in the definitions.

1.1. **Buffon and the Lens of Probability.** In 1733, the French mathematician Comte de Buffon first made the connection between probability theory and geometry with his famous Needle Problem (problem 1.1.1), opening the possibility of using a different approach to understanding geometry – using probability. Here is a quick recap of the problem and his solution published 4 years later, which will also make concrete what is meant by a possible probabilistic approach to understanding geometry.

Problem 1.1.1. (Buffon's Needle Problem) Let the xy-plane be ruled with parallel lines of 1 unit apart – for concreteness, take the system of lines of the form $y = m$ where $m \in \mathbb{Z}$. Drop a needle (i.e. line segment with endpoints) of length l randomly onto the plane. What is the probability that the needle will cross at least one of the lines? For the purpose of this discussion, we shall only consider the case $0 < l < 1$, so that the needle can only cross at most one of the lines.

In formulating the problem, one needs to be mathematically precise about what is meant by dropping the needle randomly onto the plane. To do this, note that it is sufficient to describe the position of the needle by a, the vertical distance between the centre of needle to the closest line, and θ, the acute angle formed by the needle and the vertical direction. We shall reasonably choose these random variables to independently follow a uniform distribution, specifically:

$$a \sim \text{Unif}\,[0, 1/2] \;\; \textit{and} \;\; \theta \sim \text{Unif}\,[0, \pi/2]$$

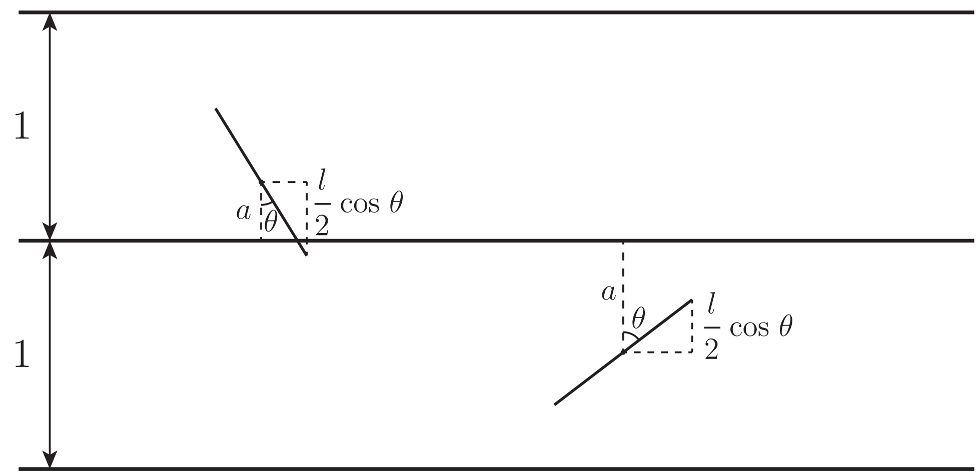


FIGURE 1.1.

This means that the joint probability density function of (a, θ) is

$$\begin{cases} 4/\pi & 0 \le a \le 1/2 \text{ and } 0 \le \theta \le \pi/2 \\ 0 & \text{otherwise} \end{cases}$$

Solution. A simple geometric analysis (figure 1.1) shows that the needle crosses a line if and only if

$$a \le \frac{l}{2}\cos\theta$$

The probability that our random variables satisfy the above inequality is given by

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{l}{2}\cos\theta} \frac{4}{\pi}\, da\, d\theta = \frac{2l}{\pi}$$

This solves the Buffon's Needle Problem (at least for $0 < l < 1$). □

There is an alternative interpretation of the above result. Let X be the random variable denoting the number of intersections between the needle and the system of lines. In our case of $0 < l < 1$, X can only take two possible values, 0 or 1. This means that $E[X]$ is the probability that the needle intersects some line. Therefore[1],

$$E[X] = \frac{2l}{\pi} \tag{1.1}$$

It is easy to show that equation (1.1) holds for all $l > 0$. Namely, mentally consider the needle to be composed to shorter needles of length $l_1, \dots, l_n$ pieced together[2], where $0 < l_i < 1$ and $l_1 + \dots + l_n = l$. Define X_i to be the number of intersections between the i^{th} piece and the system of lines, which means $X =$

[1]Note that this result holds even if our needle does not have one or both its endpoints included.

[2]The careful reader would worry about what happens at the gluing points between two short needle pieces, since they appear to be counted twice. However, by the previous footnote, we can ensure that each point of a long needle is only counted once by disregarding, where necessary, one or more endpoints of our short needle pieces.

$X_1 + \cdots + X_n$. Now, linearity of expectations holds regardless of the dependency between the X_i's, so[3]

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{2l_i}{\pi} = \frac{2l}{\pi}$$

This is a remarkable result. It states that the length of a needle can be computed (up to a scaling factor of $\pi/2$) by looking at the average number of intersections with the system of lines. This is a glimpse of how the expectation of a random variable can be used to calculate a geometrical quantity. We shall need this relation later, so let us record our discussion down as a proposition.

Proposition 1.1.2. Under the set-up of problem 1.1.1 (and the accompanying definition of "random"), the expected number of intersections between the needle and the system of lines is given by $2l/\pi$.

1.2. **Bertrand and Caution with Probability.** A key decision made in formulating the Buffon's Needle Problem was how to place the needle randomly. The infamous Bertrand's paradox tells us that this is not a decision that can be ignored. Although the paradox will not be discussed in detail here (see [Tis84]), it suffices to say that Bertrand exhibited the possibility of obtaining different answers to probability questions if the phrase "at random" can be interpreted in different ways. In our context, if we had chosen a different definition of placing our needle randomly, then $E[X]$ would turn out to be different, and may in fact be unrelated (or related in a complicated way) to l. We were fortunate that the most intuitive choice made above turns out to give not just a relation between $E[X]$ and l, but in fact the relation is linear (i.e. simple!). How might one choose the probability measure to ensure that there is a relation between $E[X]$ and l?

To get a sense of what additional properties our probability measure should have, consider the Buffon's Needle Problem generalized to 3 dimensions. Here, there are a system of parallel planes spaced 1 unit apart and a needle of length l. Previously, as a matter of respecting the convention of the original problem, we fixed the system of lines and placed the needle randomly. From this point on, it will be more useful to think that there is a needle fixed in space whose length we are interested in measuring, and we are doing this by randomly imposing a system of parallel planes and computing the expected number of intersections, which we will denote by $E[X]$. In other words, we get to choose the probability measure on systems of planes and want a relation between $E[X]$ and l that is true no matter how the needle has been fixed in space.

Suppose temporarily that we always fix our needle to be parallel to the z-axis of our xyz-space, with its middle at the origin. Here is a perfectly legitimate probability measure to impose on our system of planes that will turn out to give a linear relation between $E[X]$ and l in this temporary set-up, but will later be shown to actually be lacking. A system of planes is determined once we picked (θ, ϕ, a), where the polar angle θ and azimuth angle ϕ determines a direction in space for which the planes will be perpendicular to, and a is the closest distance of

[3]Before we can apply the result $E[X_i] = 2l_i/\pi$, we have to show that the position of the i^{th} piece has been placed in accordance to the (a, θ) distribution described above. Technically we only know this for the entire needle, but it is straightforward to see that this implies the same for the i^{th} piece.

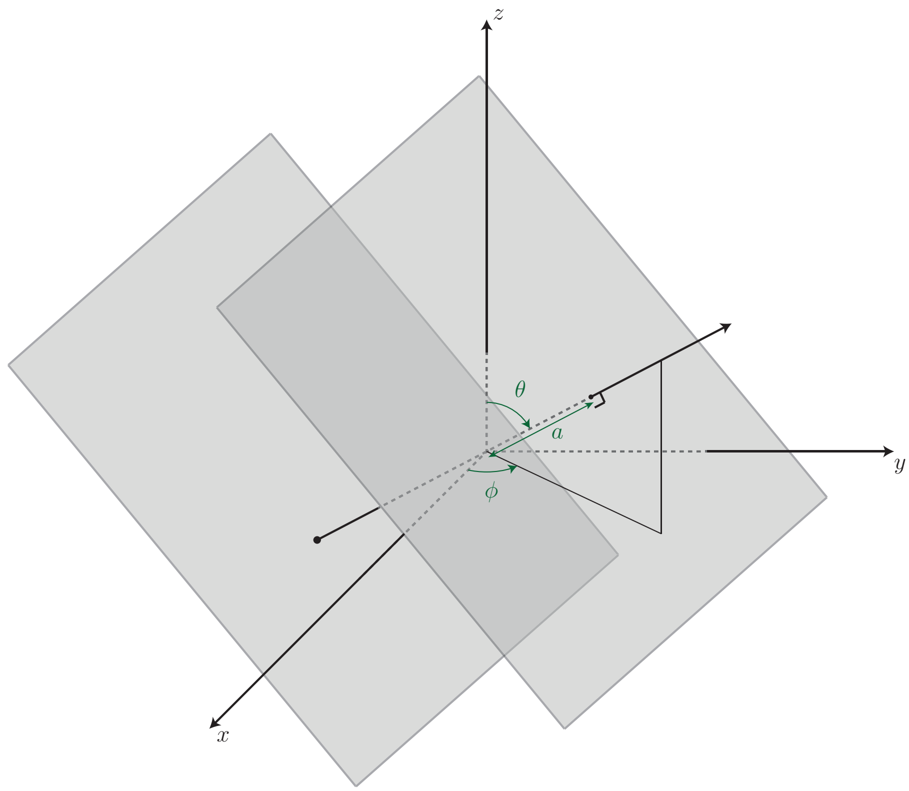


FIGURE 1.2.

the origin to a plane in the system (see figure 1.2). We set

$$\theta \sim \text{Unif}\,[0,\pi]\,,\phi \sim \text{Unif}\,[0,2\pi]\,,a \sim \text{Unif}\,[0,1/2]$$

with these three random variables being independent.

If $0 < l < 1$, then a similar geometric analysis tells us that the needle crosses a plane if and only if the needle, when projected onto the line normal to all the planes, crosses a plane. This happens if and only if

$$a \leq \frac{l}{2}|\cos\theta|$$

See figure 1.3. The probability that our random variables satisfy the above inequality is given by

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\frac{l}{2}|\cos\theta|} 2\cdot\frac{1}{\pi}\cdot\frac{1}{2\pi}\;da\;d\theta\;d\phi = \frac{2l}{\pi}$$

By a similar reasoning to those leading up to theorem 1.1.2, we see that for all $l > 0$, we have $E\,[X] = 2l/\pi$. There is indeed a linear relation between $E\,[X]$ and l.

However, in general, the needle is simply fixed in space, pointing in an arbitrary direction at an arbitrary position. We want to use a single probability measure on our system of planes to compute the length of any fixed needle. Unfortunately, the formula $E\,[X] = 2l/\pi$ is no longer always true. For example, if the fixed needle

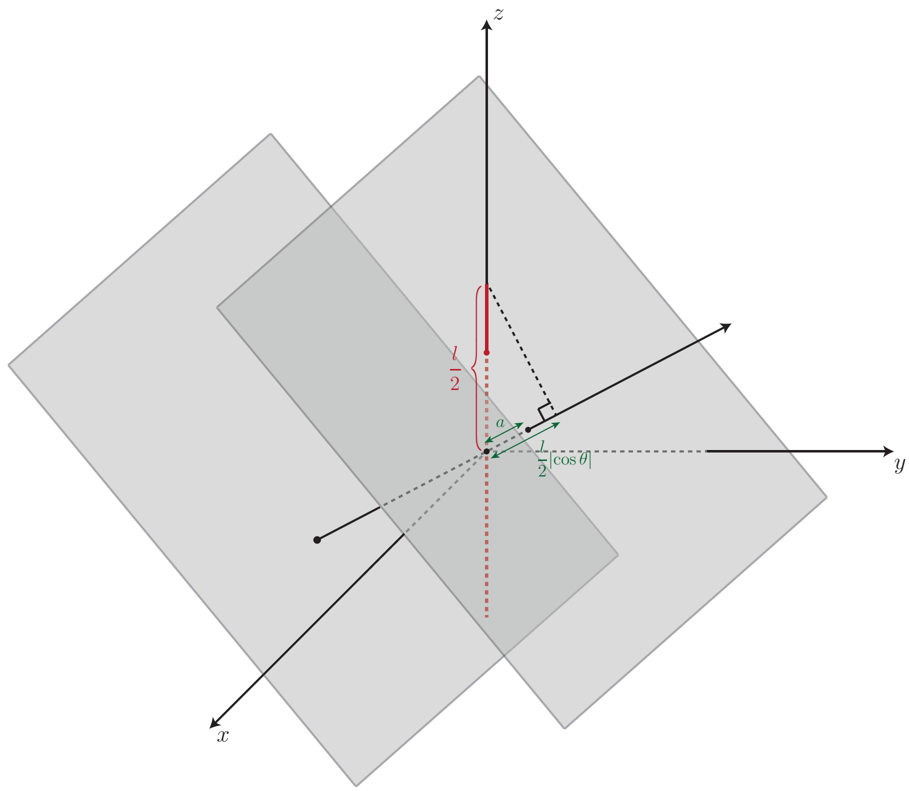


Figure 1.3.

is pointing along the x-direction with its middle at the origin, then by the same projection argument a needle of length $l < 1$ crosses a plane if and only if

$$a \leq \frac{l}{2} \sin\theta |\cos\phi|$$

The probability that our random variables satisfy the above inequality is given by

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{\frac{l}{2}\sin\theta|\cos\phi|} 2 \cdot \frac{1}{\pi} \cdot \frac{1}{2\pi} \ da \ d\theta \ d\phi = \frac{4l}{\pi^2}$$

This time, we have $E[X] = 4l/\pi^2 \neq 2l/\pi$.

In summary, the probability measure that was imposed failed to give any relation between $E[X]$ and l, beause $E[X]$ also depends on other parameters of the needle (e.g. its direction). There is actually a linear relation if we restrict the needles to always be lying along a certain direction in space, but this is a silly restriction. If only the probability measure is somehow compatible with rotation, then a single relation holding for needles along one direction will also hold for all needles in general. This motivates the following definition.

Definition 1.2.1. Suppose T is a set equipped with a group action by G. For a subset $S \subseteq T$ and an element $g \in G$, we write $gS := \{gt \mid t \in S\}$. The measure μ of a measure space (T, Σ, μ) is said to be G**-invariant** (or simply **invariant**) if for all $S \in \Sigma$, we have $gS \in \Sigma$ and $\mu(S) = \mu(gS)$.

The set T containing all possible systems of planes is equipped with the action by the Euclidean group $E(3)$. The preceding discussion motivates why any measure we consider imposing on T should preferably be $E(3)$-invariant (and Σ should be fine enough so that $X : T \to \mathbb{R} \cup \{\pm\infty\}$ is a random variable[4], i.e. measurable[5]). Specifically, if for a particular fixed needle we find $E[X]$ to be of a certain value, then we can subject the needle to any Euclidean motion and its length is not going to change under Euclidean motion, so $E[X]$ had better be unchanged if we want a relation between $E[X]$ and l. Choosing a $E(3)$-invariant measure on T will guarantee this. Of course, the relation may not be linear, but it is a relation nonetheless. We will finish up this discussion of the Buffon's Needle Problem once we have more tools (example 2.4.1).

1.3. **Integral Geometry.** In the late 1930s, Wilhelm Blaschke published a series of papers under the project "integral geometry". His idea was to investigate, as we have tried above, whether expectations of random variables could be used for calculating and understanding geometrical quantities like length, area or curvature. Morgan Crofton, in the late 1800s, actually preceded this effort with the discovery of many simple relations of this flavor. However, his work and even simply the idea of using probability to study geometrical quantities, which went by the umbrella title "geometric probability", came under great threat of being completely discredited after the paradox of Bertrand attacked Crofton's loose treatment of randomness. With no unified or consistent approach to value one definition of randomness over another, the theory appears highly arbitrary as to when a definition will lead to a relation. It was Poincaré who, in 1896, kept the idea alive by suggesting that the only definitions of randomness worth considering be limited to measures that are invariant under any symmetry group which the interested geometrical quantity is known to be invariant under. He coined the term "kinematic density" for what is more commonly called an invariant measure today.

Having been revived by the rebranding of Blaschke, the field has made noteworthy progress in the past half a century. The most important is probably establishing the use of homogeneous space theory as a unified approach to obtain the measure of interest, a work done by two of Blaschke's students, André Weil and Shiing-Shen Chern. Integral geometry offers itself as an interesting alternative tool to study geometrical problems (for example, Milnor [Mil50, 3.1] used this to study curvature of knots), but is also used today in stochastic geometry [SW08], computational modeling (for example, tomography) [KW03] and even made an appearance in statistical physics [Mec98].

The name "integral geometry" deserves a brief comment. The goal of the field is to link geometry with expectation of random variables, which is actually an integral of the random variable as a measurable function over the probability space, hence the name. In some cases, it is neater to not limit ourselves to probability spaces. Throughout, we will abuse terminology and also use the term "expectation" of a measurable function to denote its integral over the measure space.

[4]We considered the extended real numbers because the needle could very well lie in one of the planes, in which case the number of intersections is ∞. We might be tempted to disregard this situation by saying that it occurs with zero measure, but recall that we have yet to define our measure on T, so the phrase "zero measure" makes no sense.

[5]The Borel σ-algebra of the extended real numbers consists of all sets S for which $S \cap \mathbb{R}$ is a Borel set in the usual sense.

1.4. **Outline of Thesis.** Integral geometry is a rich field with its tentacles spanning into many areas of mathematics. This thesis will choose to focus on how expectations can be used to define reasonable notions of geometric sizes. For example, the "surface area" of a solid in $\mathbb{R}^3$ might be considered a reasonable notion of size. However, we remarked earlier that a formal definition by conventional methods only assigns surface area to certain subsets of space and involves certain technical machineries to set-up fully. Using the idea of expectations, we will – in a single breath – cleanly define an entire collection of notions of geometric size, one of which will be reminiscence of surface area (we will, for example, realize that there is agreement for convex polyhedrons), defined even for some subsets of $\mathbb{R}^3$ for which the conventional approach does not ascribe a surface area. A theorem due to Hadwiger will tell us why the single breath we took was enough to understand all reasonable notions of geometric sizes. This discussion will be done in chapter 3 and is the center point of the thesis.

Chapter 2 carries out the preparatory work of establishing certain invariant measures that we will need. In particular, for $0 \leq k \leq n$, the Grassmannian $\mathrm{Gr}(n,k) := \{k \text{ dimensional subspaces of } \mathbb{R}^n\}$ is acted upon by the orthogonal group $O(n)$ and the set of all k-planes $\mathrm{Aff}(n,k)$ is acted upon by the Euclidean group $E(n)$; we will need invariant measures for both. To do this, we will use the theory of homogeneous spaces and treat $O(n)$ and $E(n)$ as Lie groups.

Chapters 4 and 5 are bonus sections where we extend particular ideas presented in chapter 3. Treatment in these two chapters is designed to give a flavor of the power of integral geometry beyond this thesis.

In chapter 4, we present some original ideas and pursue the thread of using integral geometry to define the notion of length, but extended to curves on arbitrary smooth manifolds. The chapter will allow us to better appreciate how integral geometry can be used as the starting point for prescribing geometries, assigning distance functions to a smooth manifold with no presupposed geometry.

In chapter 5, we discuss how the integral geometric framework can also encompass the different geometrical notion of curvature.

2. **Invariant Measures**

In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics.

– Hermann Weyl

As we have seen, one of the key prerequisites to integral geometry is the study of invariant measures. In this chapter, we recap some of the main takeaways from the theory of homogeneous spaces that will be useful for integral geometry. In particular, we will establish the existence of invariant measures for a wide class of contexts that are frequently useful for practical applications in integral geometry. We will also pay particular attention to the examples of Grassmannians and the set of k-planes in $\mathbb{R}^n$.

For this purpose, we will be needing some standard material from smooth manifold theory, which we will be prepared to cite as long as it is discussed in [Lee12], a standard textbook on the subject.

2.1. **Theory of Lie Groups and Homogeneous Spaces.** In definition 1.2.1, we are under the most general context where there is simply a set T equipped with a group action by G. In most applications, G can have an additional structure of a Lie group, which will turn out to be a huge resource. Recall that a **Lie group** G is a group whose elements are also points of a smooth manifold, such that the group operation $(g, h) \mapsto gh$ and inversion map $g \mapsto g^{-1}$ are both smooth. There are two examples of Lie groups that will be relevant to our purpose. They are the orthogonal group $O(n)$, which is the set of $n \times n$ real matrices A satisfying $A^T A = AA^T = I$, and the Euclidean group $E(n)$, which is the symmetry group of n-dimensional Euclidean space (i.e. in particular including the translation group $T(n)$ and $O(n)$ as subgroups). For a proof that these are Lie groups, see [Lee12, 7.27] and [Lee12, 7.32].

Let us first study the situation of a Lie group G acting on by itself via left multiplication and see if we can obtain an invariant Borel measure on G (in the sense of definition 1.2.1). Note that here we chose the σ-algebra of the measure space to be $\Sigma = \mathcal{B}(G)$, where the notation $\mathcal{B}(X)$ will be used to denote the Borel σ-algebra of X whenever X is a topological space.

Indeed, we have the following useful starting point from smooth manifold theory.

Theorem 2.1.1. Every Lie group, which can always be endowed with a left-invariant orientation, has a nowhere-vanishing positively-oriented left-invariant (continuous) n-form (where $n = \dim G$). Moreover, all left-invariant n-forms are equal, up to constant multiples.

Proof. See [Lee12, 16.10]. □

Theorem 2.1.2. Let G be a Lie group acting on itself by left multiplication. Then there is a measure μ on $(G, \mathcal{B}(G))$ such that μ is invariant.

Before proceeding with the proof, we need a few pieces of notation. We let $C_c(G)$ denote the space of continuous functions on G with compact support. If U is an open set in G and $f \in C_c(G)$, we use the notation

$$f \prec U$$

to mean $0 \le f \le 1$ and $\operatorname{supp}(f) \subseteq U$. Also observe that if $f \in C_c(G)$ and ω is a (continuous) n-form, then $f\omega$ is a compactly supported (continuous) n-form, so $\int_G f\omega$ makes sense (after we chose an orientation for G).

Proof. Endow G with a left-invariant orientation and let ω denote a nowhere-vanishing positively-oriented left-invariant (continuous) n-form.

Step 1: Let us first construct a candidate Borel measure. We define, for $U \subseteq G$ open,

$$\mu_0(U) := \sup \left\{ \int_G f\omega \mid f \in C_c(G),\ f \prec U \right\}$$

and for an arbitrary $E \subseteq G$, we define

$$\mu^*(E) := \inf \{\mu_0(U) \mid U \supseteq E,\ U \text{ open}\}$$

Observe that $\mu_0(U) \le \mu_0(V)$ if $U \subseteq V$ and so $\mu^*(U) = \mu_0(U)$ for U open. Let us show that μ^* is an outer measure. For that, we need this lemma.

Lemma: If $U_1, U_2, \dots$ is a sequence of open sets and $U := \bigcup_{j=1}^{\infty} U_j$, then $\mu_0(U) \le \sum_{j=1}^{\infty} \mu_0(U_j)$.

Proof of Lemma: Let $f \in C_c(G)$ be such that $f \prec U$. By compactness of $\operatorname{supp}(f)$, we can pick N such that $\operatorname{supp}(f) \subseteq \bigcup_{j=1}^{N} U_j$. Since $\bigcup_{j=1}^{N} U_j$ is a smooth manifold, by the Existence of Partitions of Unity theorem [Lee12, 2.23], we can find smooth functions $g_1, \dots, g_N : \bigcup_{j=1}^{N} U_j \to \mathrm{R}$ such that $g_j \prec U_j$ and $\sum_{j=1}^{N} g_j \equiv 1$ on $\bigcup_{j=1}^{N} U_j$. Then $f = \sum_{j=1}^{N} f g_j$ and $f g_j \prec U_j$, so

$$\int_G f\omega = \sum_{j=1}^{n} \int_G f g_j \omega \le \sum_{j=1}^{n} \mu_0(U_j) \le \sum_{j=1}^{\infty} \mu_0(U_j)$$

and therefore by definition of μ_0, we conclude that $\mu_0(U) \le \sum_{j=1}^{\infty} \mu_0(U_j)$.

As a consequence of the lemma and basic measure theory (e.g. [Fol99, 1.10]), we learn that μ^* is an outer measure. Now, we show that every open set is μ^*-measurable, that is, let U be open and we want to show that whenever $E \subseteq X$ is such that $\mu^*(E) < \infty$, we will have $\mu^*(E) \ge \mu^*(E \cap U) + \mu^*(E \cap U^c)$. Fix $\epsilon > 0$.

If E happens to be open, then we can find $f_1 \in C_c(G)$ with $f_1 \prec E \cap U$ such that $\int_G f_1\omega > \mu_0(E \cap U) - \epsilon$, and similarly find $f_2 \in C_c(G)$ with $f_2 \prec E - \operatorname{supp}(f_1)$ such that $\int_G f_2\omega > \mu_0(E - \operatorname{supp}(f_1)) - \epsilon$. Note $f_1 + f_2 \prec E$ and so

$$\begin{aligned} \mu^*(E) = \mu_0(E) \ge \int_G (f_1 + f_2)\,\omega &> \mu_0(E \cap U) + \mu_0(E - \operatorname{supp}(f_1)) - 2\epsilon \\ &= \mu^*(E \cap U) + \mu^*(E - \operatorname{supp}(f_1)) - 2\epsilon \ge \mu^*(E \cap U) + \mu^*(E \cap U^c) - 2\epsilon \end{aligned}$$

For E not necessarily open, we find open V such that $V \supseteq E$ and $\mu^*(V) = \mu_0(V) < \mu^*(E) + \epsilon$. Then

$$\mu^*(E) + \epsilon > \mu^*(V) \ge \mu^*(V \cap U) + \mu^*(V \cap U^c) \ge \mu^*(E \cap U) + \mu^*(E \cap U^c)$$

and so the desired inequality also holds.

Therefore, the collection of μ^*-measurable sets will contain $\mathcal{B}(G)$, and we can define $\mu := \mu^*|_{\mathcal{B}(G)}$ as a Borel measure.

Step 2: We check that μ is invariant. Let $E \in \mathcal{B}(G)$ and $g \in G$. Note that left multiplication by g is a homeomorphism, so $gE \in \mathcal{B}(G)$. By definition,

$$\mu\left(gE\right) = \inf\left\{\mu_0(U) \mid U \supseteq gE, U \text{ open}\right\} = \inf\left\{\mu_0(gU) \mid U \supseteq E, U \text{ open}\right\}$$

But

$$\begin{aligned}\mu_0(gU) = \sup\left\{\int_G f\omega \mid f \in C_c(G), f \prec gU\right\} \\ = \sup\left\{\int_G f(L^*_{g^{-1}}\omega) \mid f \in C_c(G), f \prec U\right\} = \mu_0(U)\end{aligned}$$

where $L^*_{g^{-1}}$ denotes the pullback by the diffeomorphism $L_{g^{-1}}$ that is left multiplication by g^{-1}, and where we used the left-invariance of ω in the last equality. Therefore $\mu(gE) = \mu(E)$ as desired. $\square$

As a step towards increasing generality, let G be a Lie group and H a closed subgroup of G. The left coset space G/H can be equipped with a smooth manifold structure such that the quotient map $\pi : G \to G/H$ is smooth [Lee12, 21.17]. Under this structure, G/H can be acted on by G via

$$g_1\overline{g_2} = \overline{g_1 g_2}$$

and this action is smooth.

Proposition 2.1.3. There is an invariant Borel measure for G/H equipped with the above action by G.

Proof. Equip G with the invariant Borel measure μ. G/H here is equipped with the Borel σ-algebra. The map $\pi : G \to G/H$ is smooth, thus continuous, and hence measurable. Therefore, the pushforward measure $\pi_*(\mu)$ is a Borel measure on G/H. It is invariant because for $S \in \mathcal{B}(G/H)$, we have $gS \in \mathcal{B}(G/H)$ because the action of g on G/H is a homeomorphism, and

$$(\pi_*(\mu))(gS) = \mu(\pi^{-1}(gS)) = \mu(g(\pi^{-1}(S))) = \mu(\pi^{-1}(S)) = (\pi_*(\mu))(S)$$

as desired. $\square$

The above consideration involving G/H is actually a lot more general than it might first appear. A **homogeneous G-space** (or simply **homogeneous space**) is a smooth manifold endowed with a transitive smooth action by the Lie group G. Clearly, G/H is an example of a homogeneous G-space. The following theorem tells us that in fact all homogeneous spaces look like this.

Theorem 2.1.4. Let G be a Lie group and M a homogeneous G-space. Pick any point $p \in M$. Then the stabilizer of p, denoted $\mathrm{Stab}(p)$ is a closed subgroup of G and the map $F : G/\mathrm{Stab}(p) \to M$ defined by $F(\overline{g}) = gp$ is a diffeomorphism.

Proof. See [Lee12, 21.28]. $\square$

Corollary 2.1.5. A homogeneous G-space can be equipped with an invariant Borel measure.

Proof. Pick any $p \in M$ and consider the pushforward of the invariant Borel measure of $G/\mathrm{Stab}(p)$ using the map F as defined in theorem 2.1.4. $\square$

Finally, the following theorem is useful if we start with a set that might not yet be equipped with a smooth manifold structure.

Theorem 2.1.6. Let T be a set, endowed with a transitive action by a Lie group G such that for some $p \in T$, we know that the stabilizer subgroup $\mathrm{Stab}(p)$ is closed in G. Then T can be made into a smooth manifold such that it is a homogeneous G-space.

Proof. See [Lee12, 21.20]. □

2.2. **Grassmannians.** The above discussion can be applied to two examples of particular interest to integral geometry. The first is $\mathrm{Gr}(n, k)$, the set of all k-dimensional subspaces of $\mathbb{R}^n$, acted transitively upon by $O(n)$ (k here is some fixed integer satisfying $0 \leq k \leq n$). If we arbitrarily pick an element $L_0 \in \mathrm{Gr}(n, k)$, then $\mathrm{Stab}(L_0)$ is some copy of $O(k) \times O(n-k)$ inside $O(n)$ which can be shown to be a closed subgroup. For example, if we view $O(n)$ as the set of $n \times n$ orthogonal matrices with respect to the standard basis $(\epsilon_1, \ldots, \epsilon_n)$ and select $L_0 = \mathrm{span}(\epsilon_1, \ldots, \epsilon_k)$, then $\mathrm{Stab}(L_0)$ is the set of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

where A and D are orthogonal matrices of sizes $k \times k$ and $(n-k) \times (n-k)$ respectively. We shall abuse notation and understand $O(k) \times O(n-k)$ as a specific subgroup of $O(n)$. The smooth manifold structure (which includes the topology) of $\mathrm{Gr}(n, k)$ is then better understood as

$$\mathrm{Gr}(n, k) \simeq \frac{O(n)}{O(k) \times O(n-k)}$$

and has an $O(n)$-invariant Borel measure.

Let θ_n denote a $O(n)$-invariant Borel measure on $O(n)$ itself, which exists by theorem 2.1.2. The corresponding $O(n)$-invariant Borel measure on $\mathrm{Gr}(n, k)$, denoted[6] by $\gamma_{n,k}$, is obtained by (two) pushforwards and concretely is given by

$$\gamma_{n,k}(A) = \theta_n(\{g \in O(n) \mid gL_0 \in A\})$$

for any $A \in \mathcal{B}(\mathrm{Gr}(n, k))$.

In turn, here is one way of defining some θ_n that is practical for our applications. Observe that there is a bijection between elements of $O(n)$ and an ordered orthonormal basis under the map $g \mapsto (e_1, \ldots, e_n) = (g(\epsilon_1), \ldots, g(\epsilon_n))$. We begin by picking $e_1 \in S^{n-1}$ in accordance to the normalized spherical measure σ_{n-1} (normalized means $\sigma_{n-1}(S^{n-1}) = 1$). Now, e_2 must be picked from the copy of S^{n-2} that lies in the hyperplane orthogonal to e_1, and we pick e_2 in accordance to the normalized spherical measure σ_{n-2}. In general, we pick e_i uniformly from the copy of S^{n-i} that lies in the $(n-i+1)$-dimensional plane orthogonal to $e_1, \ldots, e_{i-1}$. In

[6] We chose the symbol γ because "gamma" and "Grassmannians" both start with "g". Likewise, hopefully the symbol θ reminds one of angles and thus serves as a mnemonic for the measure on $O(n)$.

other words, we can define $\theta_n$via

$$\int_{O(n)} f(g)\ \theta_n(dg)$$
$$= \int_{S^{n-1}} \int_{S^{n-2}\perp\{e_1\}} \cdots \int_{S^0\perp\{e_1,\dots,e_{n-1}\}} f(\epsilon_i \mapsto e_i)\ \sigma_0(de_n)\dots\sigma_{n-2}(de_2)\ \sigma_{n-1}(de_1)$$

for any measurable function f (and $\epsilon_i \mapsto e_i$ denotes the unique element of $O(n)$ that maps ϵ_i to e_i).

From the above description of θ_n, we have $\gamma_{n,k}(\mathrm{Gr}(n,k)) = \theta_n(O(n)) = 1$, so for each fixed n and k, $\mathrm{Gr}(n,k)$ is already a probability space. No normalization is needed.

We will need the following lemma in the future.

Lemma 2.2.1. Let f be a measurable function on $\mathrm{Gr}(n,k)$. Then

$$\int_{\mathrm{Gr}(n,k)} f(L)\ \gamma_{n,k}(dL) = \int_{\mathrm{Gr}(n,n-k)} f(L^\perp)\ \gamma_{n,n-k}(dL)$$

Proof. Pick any $L_0 \in \mathrm{Gr}(n,k)$. For a subset $A \subseteq \mathrm{Gr}(n,k)$, we let $A^\perp \subseteq \mathrm{Gr}(n,n-k)$ denote

$$A^\perp := \left\{L^\perp \mid L \in A\right\}$$

Observe that

$$\gamma_{n,k}(A) = \theta_n(\{g \in O(n) \mid gL_0 \in A\}) = \theta_n(\left\{g \in O(n) \mid gL_0^\perp \in A^\perp\right\}) = \gamma_{n,n-k}(A^\perp)$$

from which the desired conclusion follows. ☐

2.3. ***k*-planes.** A similar discussion can be made for $\mathrm{Aff}(n,k)$, the set of all k-dimensional planes in $\mathbb{R}^n$, acted upon transitively by $E(n)$. Recall that $E(n)$ can be written as the semi-direct product $T(n) \rtimes O(n)$, where $T(n)$ is the translation group and $O(n)$ acts on $T(n) \simeq \mathbb{R}^n$ in the usual way (for a review of Lie group structure under semi-direct products, see [Lee12, 7.32]). As a matrix group, $E(n)$ can be realized as matrices of the form

$$\begin{pmatrix} A_{11} & \dots & A_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ A_{n1} & \dots & A_{nn} & b_n \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

where A is an orthogonal matrix (standard basis chosen here) and b is a vector (again, standard basis). Pick an element $P_0 \in \mathrm{Aff}(n,k)$. Then $\mathrm{Stab}(P_0)$ is some copy of $(T(k)\times\{e\})\rtimes(O(k)\times O(n-k))$ inside $E(n)$ which can be shown to be a closed subgroup. For example, without loss of generality, suppose $P_0 = \mathrm{span}(\epsilon_1,\dots,\epsilon_k)$. Then $\mathrm{Stab}(P_0)$ consists of matrices of the form

$$\begin{pmatrix} A' & 0 & b \\ 0 & A'' & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where A' is orthogonal of size $k \times k$, A'' is orthogonal of size $(n-k)\times(n-k)$ and b is a vector of length k. We will abuse notation and understand $(T(k)\times\{e\})\rtimes(O(k)\times O(n-k))$ as a subgroup of $E(n)$. Then, as smooth manifolds,

$$\mathrm{Aff}(n,k) \simeq \frac{E(n)}{(T(k)\times\{e\})\rtimes(O(k)\times O(n-k))}$$

and has an invariant Borel measure.

There is a natural candidate for an invariant Borel measure on $E(n)$. First, we perform a verification that $\mathcal{B}(E(n)) = \mathcal{B}(T(n)) \otimes \mathcal{B}(O(n))$. Indeed, the smooth manifold structure of semi-direct product is taken to be that of the Cartesian product, in which case the topology is the product topology. Now $T(n)$ and $O(n)$ are Lie groups, hence second-countable and thus separable; they are also metrizable (they inherit the metric from $\mathbb{R}^n$ and $\mathbb{R}^{n^2}$ respectively). The result follows from this standard lemma from measure theory.

Lemma 2.3.1. If $X_1, \ldots, X_N$ are metric spaces and X is $\prod_{j=1}^N X_j$ equipped with the product topology, then $\bigotimes_{j=1}^N \mathcal{B}(X_j) \subseteq \mathcal{B}(X)$. If furthermore $X_1, \ldots, X_N$ are separable, then $\bigotimes_{j=1}^N \mathcal{B}(X_j) = \mathcal{B}(X)$.

Proof. See any text in measure theory, for example [Fol99, 1.5]. □

Let λ_n denote the n-dimensional Lebesgue measure, which we restrict to Borel sets but continue to use the same symbol as a mild abuse of notation, and recall θ_n from section 2.2. Since we have $\mathcal{B}(E(n)) = \mathcal{B}(T(n)) \otimes \mathcal{B}(O(n))$, it makes sense for us to talk about the product measure $\lambda_n \times \theta_n$ as a Borel measure on $E(n)$. One can check that this is an $E(n)$-invariant measure. Under the pushforwards, we obtain an $E(n)$-invariant measure on $\mathrm{Aff}(n,k)$, which we denote[7] by $\alpha_{n,k}$.

One can work through the details of the pushforwards and obtain the following practical interpretation of $\alpha_{n,k}$: choosing a k-plane can be seen as choosing an element of $\mathrm{Gr}(n,k)$ in accordance to $\gamma_{n,k}$ and then translating the subspace along a direction in its orthogonal complement chosen in accordance to λ_k. In other words, we can understand $\alpha_{n,k}$ as

$$\int_{\mathrm{Aff}(n,k)} f(P)\ \alpha_{n,k}(dP) = \int_{\mathrm{Gr}(n,k)} \int_{L^\perp} f(L+y)\ \lambda_k(dy)\ \gamma_{n,k}(dL)$$

for any measurable function f.

2.4. **Example - Buffon's Needle Problem in $\mathbb{R}^n$.**

As a nice opportunity to illustrate the ideas of this chapter, let us finish off the Buffon's Needle Problem that was unsettled in the introduction.

Example 2.4.1 (Buffon's Needle Problem in $\mathbb{R}^n$)**.** Let there be a needle of length l fixed in $\mathbb{R}^n$, $n \geq 2$. We have to choose a probability measure on systems of parallel hyperplanes spaced a unit apart, and hopefully obtain a relation between the expected number of intersections and l.

We have already discussed that because the needle length is invariant under $E(n)$, we had better choose a probability measure that is $E(n)$-invariant. The set T containing all systems of hyperplanes is equipped with the action by $E(n)$, and very similar to the discussion of k-planes, we can obtain an $E(n)$-invariant Borel measure $\widetilde{\alpha}_{n,n-1}$ where

$$\int_T f(P)\ \widetilde{\alpha}_{n,n-1}(dP) = \int_{\mathrm{Gr}(n,n-1)} \int_0^1 f(L+y)\ \lambda_1(dy)\ \gamma_{n,n-1}(dL)$$

for any measurable function f. The inner integral ranges from 0 to 1 because translating the system of hyperplanes along the normal direction will lead to the

[7] Again, "lambda" and "Lebesgue" both start with the same letter, as do "alpha" and "affine".

same configuration every integer distance. Note that $\widetilde{\alpha}_{n,n-1}$ is already normalized to be a probability measure because take $f(P) \equiv 1$ and we get

$$\widetilde{\alpha}_{n,n-1}(T) = \lambda_1([0,1]) \cdot \gamma_{n,n-1}(\mathrm{Gr}(n, n-1)) = 1$$

Let $X : T \to \mathrm{R} \cup \{\pm\infty\}$ denote the number of intersections, which can be shown to be a random variable, i.e. measurable. First assume $0 < l < 1$. Then

$$\begin{aligned} E[X] &= \int_T X(P)\ \widetilde{\alpha}_{n,n-1}(dP) \\ &= \int_{\mathrm{Gr}(n,n-1)} \int_0^1 X(L+y)\ \lambda_1(dy)\ \gamma_{n,n-1}(dL) \end{aligned}$$

Now, the inner integral is simply the length of the needle after projected onto $L^\perp$. Introducing the notation $S|L$ to denote the orthogonal projection of a subset $S \subseteq \mathbb{R}^n$ onto a linear subspace L, we want to compute

$$\begin{aligned} &\int_{\mathrm{Gr}(n,n-1)} \mathrm{length}(\mathrm{needle}|L^\perp)\ \gamma_{n,n-1}(dL) \\ =&\int_{\mathrm{Gr}(n,1)} \mathrm{length}(\mathrm{needle}|L)\ \gamma_{n,1}(dL) \end{aligned}$$

where we have applied lemma 2.2.1.

Now we have to do actual computations. Pick $L_0 \in \mathrm{Gr}(n, 1)$ to be the line spanned by the first standard basis vector ϵ_1. For a subset $A \subseteq \mathrm{Gr}(n, 1)$, note that

$$\begin{aligned} \gamma_{n,1}(A) &= \theta_n(\{g \in O(n) \mid gL_0 \in A\}) \\ &= \theta_n\left(\left\{g \in O(n) \mid g\epsilon_1 \in \bigcup_{L\in A} (L \cap S^{n-1})\right\}\right) \end{aligned}$$

So, by our definition of θ_n, the integral becomes

$$\begin{aligned} &\int_{S^{n-1}} \mathrm{length}(\mathrm{needle}|\mathrm{span}(u))\ \sigma_{n-1}(du) \\ =&\ l\int_{S^{n-1}} |v \cdot u|\ \sigma_{n-1}(du) \\ =&\frac{2 \cdot \mathrm{vol}(B^{n-1})}{\mathrm{area}(S^{n-1})} l \end{aligned}$$

where v is a unit vector pointing along the direction of the needle and B^{n-1} is the closed unit sphere in $\mathbb{R}^{n-1}$. The last equality is by lemma 2.4.2 below, with a sketched proof for those who have not seen this fact from calculus. Note that σ_{n-1} here is the *normalized* spherical measure, hence the differing factor with lemma 2.4.2.

Substituting general expressions for volume and surface area of spheres, we conclude that

$$E[X] = \frac{2}{n\sqrt{\pi}} \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{2}+\frac{1}{2})} l$$

For example, for $n = 2, 3, 4$, we respectively get the proportionality constant to be $2/\pi$, $1/2$ and $4/(3\pi)$. $\square$

Lemma 2.4.2. Let v be a fixed unit vector. Then

$$\int_{S^{n-1}} |v \cdot n| \ dS = 2 \cdot \mathrm{vol}(B^{n-1})$$

where dS is the usual surface element and n is the outward unit normal vector.

Proof. (Sketch) A full proof can be found in [KR97, 5.5.1]. Chop up the surface into many small regions A_i with surface area $S(A_i)$ and we have the approximation

$$\int_{S^{n-1}} |v \cdot n| \ dS \approx \sum_i |v \cdot n_i| \ S(A_i)$$

Observe that $|v \cdot n_i| \ S(A_i)$ is approximately the area of the shadow when A_i is orthogonally projected onto the hyperplane $v^\perp$. On orthogonal projection, each hemisphere as separated by $v^\perp$ covers the solid unit ball in $v^\perp$ once. Thus the above sum is approximately equal to $2 \cdot \mathrm{vol}(B^{n-1})$. The approximations become equality in the limit. $\square$

3. Notions of Size

Mathematics is the art of giving the same name to different things.
– Henri Poincaré

We are ready to see how expectations can be used to define reasonable notions of size. Actually the Buffon's Needle Problem of example 2.4.1 gave us a glimpse of this idea – the length of a needle can be computed (up to scaling factor) as the expectation of a random variable. However, the example is not very satisfactory on two accounts: the length of a line segment is not a difficult concept to grapple with, and we want to avoid having to search from scratch a random variable for each reasonable notion of size that one can think of. Fortunately, this chapter will explain how integral geometry provides an elegant and unified way to think about all reasonable notions of size, many of which are difficult to grasp by classical approaches.

As the Banach-Tarski paradox shows, attempting to assign sizes to every subset of $\mathbb{R}^n$ is a tricky business. Our theory will focus only on $\mathcal{K}^n$, the collection of all compact convex subsets of $\mathbb{R}^n$. This collection is wide enough to include some examples of smooth manifold (or rather, the boundary of the subset is a smooth manifold), polyhedrons, but also other less well-behaved examples. Yet, it is narrow enough to produce an elegant theory. We shall also remark that once we completed a discussion for $\mathcal{K}^n$, it is possible to extend our size functions to the collection of polyconvex sets (i.e. the collection consisting of finite unions of compact convex sets) by a direct application of an extension theorem by Groemer. However, we will not pursue this discussion here – a good reference on Groemer's extension theorem will be [KR97, Chapter2]. The collection of polyconvex sets practically covers all geometrical objects one would deal with in real life[8]; even this 'o' is really a finite union of convex pixels[9]. We merely wish to point out that the generality of this theory, though limited to $\mathcal{K}^n$, is not something to be underestimated.

3.1. "Reasonable" notion of size.

A notion of size will be a map $\phi : \mathcal{K}^n \to \mathbb{R}$, but we should not allow every possible such mapping. There will be some properties we hope ϕ should satisfy before it will be convincing to call ϕ a notion of size. Below, three such properties are described.

Definition 3.1.1. The map $\phi : \mathcal{K}^n \to \mathbb{R}$ is said to be a **valuation** if $\phi(\emptyset) = 0$ and

$$\phi(K \cup L) = \phi(K) + \phi(L) - \phi(K \cap L)$$

whenever $K, L \in \mathcal{K}^n$ are such that $K \cup L \in \mathcal{K}^n$.

Definition 3.1.2. The map $\phi : \mathcal{K}^n \to \mathbb{R}$ is said to be **invariant** if $\phi(gK) = \phi(K)$ for all $K \in \mathcal{K}^n$ and $g \in E(n)$, where $gK := \{g(x)|x \in K\}$.

[8] That said, I am not completely sure what mathematicians will regard as "real life".

[9] In some sense, everything we encounter in real life is a finite union of atoms, but I will leave it to the physicists to decide if we are permitted to talk about atoms as "convex".

We will want ϕ to be an invariant valuation. Before describing the third property, it will be useful to recall what is known as the Hausdorff metric.

Definition 3.1.3. For $K, L \in \mathcal{K}^n$, define the **Hausdorff distance** between them to be

$$\delta(K,L) := \max\left\{\sup_{k\in K} d(k,L), \sup_{l\in L} d(K,l)\right\} = \max\left\{\sup_{k\in K}\inf_{l\in L} d(k,l), \sup_{l\in L}\inf_{k\in K} d(k,l)\right\}$$

where d refers to the Euclidean metric in $\mathbb{R}^n$.

Note that δ is not yet shown to be a metric. Before doing that, here is a lemma that provides a more intuitive way of understanding δ. The statement of the lemma requires us to first understand the ϵ-fattening of an object in $\mathcal{K}^n$, where $\epsilon \geq 0$. We define the ϵ-fattening of $\emptyset$ to be $\emptyset$, and for non-empty $K \in \mathcal{K}^n$,

$$K_{+\epsilon} := \{x \in \mathbb{R}^n \mid d(K,x) \leq \epsilon\}$$

Recall that for $K, L \in \mathcal{K}^n$, their Minkowski sum is given by

$$K + L := \{k + l \mid k \in K,\ l \in L\}$$

Thus, we can also write $K_{+\epsilon} = K + \epsilon B^n$, where ϵB^n is the solid unit sphere of radius ϵ. Note that $K_{+\epsilon}$ is bounded, closed (from continuity of $d(K,\cdot)$) and convex (given $x, y \in K_{+\epsilon}$, decompose x, y using the Minkowski sum and then use the fact that both K and ϵB^n are convex), whence $K_{+\epsilon} \in \mathcal{K}^n$.

Lemma 3.1.4. Let $K, L \in \mathcal{K}^n$. Then $\delta(K,L) \leq \epsilon$ if and only if $K \subseteq L_{+\epsilon}$ and $L \subseteq K_{+\epsilon}$. Hence, we may equivalently define

$$\delta(K,L) = \inf\{\epsilon \geq 0 \mid K \subseteq L_{+\epsilon} \textit{ and } L \subseteq K_{+\epsilon}\}$$

and in fact, the above infimum is achieved.

Proof. If $L \subseteq K_{+\epsilon}$, then every $l \in L$ obeys $d(K,l) \leq \epsilon$. Similarly if $K \subseteq L_{+\epsilon}$, then every $k \in K$ obeys $d(k,L) \leq \epsilon$. Therefore,

$$\delta(K,L) = \max\left\{\sup_{k\in K} d(k,L), \sup_{l\in L} d(K,l)\right\} \leq \epsilon$$

Conversely, we prove the contrapositive. WLOG suppose $L \subseteq K_{+\epsilon}$ is violated, so there exists $\tilde{l} \in L$ such that $d(K,\tilde{l}) > \epsilon$. Then $\delta(K,L) \geq d(K,\tilde{l}) > \epsilon$.

Note that if $K \subseteq L_{+\epsilon}$ and $L \subseteq K_{+\epsilon}$ hold, then $K \subseteq L_{+\eta}$ and $L \subseteq K_{+\eta}$ also hold for any $\eta > \epsilon$. Thus

$$\mathrm{E} := \{\epsilon \geq 0 \mid K \subseteq L_{+\epsilon} \textit{ and } L \subseteq K_{+\epsilon}\}$$

is either of the form (c,∞) or $[c,\infty)$ for some $c \geq 0$. By the first half of the lemma, $\delta(K,L)$ cannot be bigger or smaller than c, so $\delta(K,L) = c$. Then again by the first half of the lemma, $c \in \mathrm{E}$ so $\mathrm{E} = [c,\infty)$ and the infimum is achieved. $\square$

Proposition 3.1.5. The Hausdorff distance δ is a metric and turns $\mathcal{K}^n$ into a metric space.

Proof. First, δ maps into $\mathbb{R}$ because our sets are compact (in particular bounded). δ is clearly symmetric, non-negative, and $\delta(K,K) = 0$ for $K \in \mathcal{K}^n$. If $K, L \in \mathcal{K}^n$ are distinct, at least one of $K - L$ or $L - K$ will be non-empty; WLOG say we can pick some $\tilde{l} \in L - K$. Since K is compact (in particular closed), $d(K,\tilde{l}) > 0$. Thus $\delta(K,L) \geq d(K,\tilde{l}) > 0$.

It remains to prove the triangle inequality. Let $K, L, M \in \mathcal{K}^n$ and $\epsilon = \delta(K, M)$, $\eta = \delta(L, M)$. By lemma 3.1.4,

$$K \subseteq M_{+\epsilon} \textit{ and } M \subseteq L_{+\eta}$$

and so $K \subseteq L_{+(\epsilon+\eta)}$ and similarly $L \subseteq K_{+(\epsilon+\eta)}$. Again by lemma 3.1.4, $\delta(K, L) \leq \epsilon + \eta$. □

Now we may state the final property that ϕ should have.

Definition 3.1.6. The map $\phi : \mathcal{K}^n \to \mathbb{R}$ is simply said to be **continuous** if it is a continuous map with the Hausdorff metric on $\mathcal{K}^n$ and the usual Euclidean metric on $\mathbb{R}$.

In summary, we shall be interested in studying the collection of all continuous invariant valuations on $\mathcal{K}^n$, which we shall denote Val_n. Here is an easy property of Val_n that already provides a lot of information about its structure.

Proposition 3.1.7. Val_n is a real vector space under the usual definition of how to add functions and perform scalar multiplication.

Proof. It is clear that for $\phi, \psi \in \mathrm{Val}_n$ and $c \in \mathbb{R}$,

- ϕ, ψ are valuations $\Rightarrow \phi + \psi$ and $c\phi$ are valuations
- ϕ, ψ are invariant $\Rightarrow \phi + \psi$ and $c\phi$ are invariant
- ϕ, ψ are continuous $\Rightarrow \phi + \psi$ and $c\phi$ are continuous

Therefore we have $\phi + \psi \in \mathrm{Val}_n$ and $c\phi \in \mathrm{Val}_n$. □

Example 3.1.8. One natural candidate element of Val_n would be the n-dimensional Lebesgue measure (restricted to $\mathcal{K}^n$), which we shall still denote by λ_n. It follows from any standard exposition on Lebesgue measure that this is an invariant valuation. What is believable intuitively but not so clear rigorously is continuity. This has been established in [Bee74]. □

Example 3.1.9. It is not obvious how we might define the perimeter of a compact convex subset in $\mathbb{R}^2$ (the boundary may behave rather wildly!). However, if we temporarily restrict ourselves to $\mathcal{P}^2 \subseteq \mathcal{K}^2$ defined to be the collection of all compact convex polygons (including degenerate ones which are simply line segments), then we have a clear way of defining what we mean by perimeter[10]. We leave it as an exercise for the reader to show that the perimeter is a continuous invariant valuation on $\mathcal{P}^2$. Later, we will see how integral geometry allows us to easily produce a continuous invariant valuation on $\mathcal{K}^2$ that is essentially the idea of perimeter (i.e. the two valuations coincide when restricted to $\mathcal{P}^2$). A similar discussion can be made for surface area of compact convex subsets in $\mathbb{R}^3$. □

3.2. **Intrinsic Volumes.** Here is one natural way to measure the size of an object K in $\mathcal{K}^n$ which will indeed turn out to be a continuous invariant valuation. The idea is to take a random k-subspace L of $\mathbb{R}^n$ ($1 \leq k \leq n$), look at the orthogonal projection $K|L$ of K onto that plane, and compute the expected k-dimensional Lebesgue measure of the resulting shadow. We want our size function to be invariant, which means the probability measure we impose on $\mathrm{Gr}(n, k)$ should be the $O(n)$-invariant measure $\gamma_{n,k}$.

[10] For line segments, we have to "double-count" the length.

Definition 3.2.1. For $n \geq 1$, the intrinsic volume functions $V_{n,0}, \ldots, V_{n,n} : \mathcal{K}^n \to \mathbb{R}$ are defined as follows. For $1 \leq k \leq n$,

$$V_{n,k}(K) := \int_{\mathrm{Gr}(n,k)} \lambda_k(K|L)\ \gamma_{n,k}(dL)$$

and for $V_{n,0}$, we define $V_{n,0}(K)$ to be 1 whenever K non-empty and 0 otherwise.

A couple of remarks about the definition needs to be made. First, we handled the $V_{n,0}$ case separately but if we wish, we could have thought about "orthogonal projection" onto the origin and regard λ_0 as the counting measure. Second, one can see that $K|L$ is indeed a compact subset of L and hence $\lambda_k(K|L)$ makes sense and is finite; moreover, one should technically check that $\lambda_k(K|\cdot)$ is a measurable function, but here we will be content in saying that the Borel σ-algebra is fine enough to handle the behavior of compact convex K in this aspect. Third, $V_{n,n}$ agrees with λ_n.

Proposition 3.2.2. The intrinsic volume functions are continuous invariant valuations.

Proof. The claim is clearly true for $V_{n,0}$, so let us ignore that case.

Invariance under $E(n)$ follows from the $O(n)$-invariance of $\gamma_{n,k}$, the translational invariance of λ_k in the subspace L, and the fact that translating K in the direction orthogonal to L does not change $K|L$.

Next, we show continuity of $V_{n,k}$. Suppose $K_1, K_2, \ldots, K \in \mathcal{K}^n$ are such that $K_i \to K$. Note that there is a sufficiently giant closed ball B that contains all of $K_1, K_2, \ldots, K$. For fixed L, it is easy to see from lemma 3.1.4 that $(K_i|L) \to (K|L)$. Since λ_k is continuous, $\lambda_k(K_i|L) \to \lambda_k(K|L)$. Finally, observe that $\lambda_k(K_i|L) \leq \lambda_k(B|L)$ for all i and so we may apply the dominated convergence theorem to conclude that

$$V_{n,k}(K_i) = \int_{\mathrm{Gr}(n,k)} \lambda_k(K_i|L)\ \gamma_{n,k}(dL) \to \int_{\mathrm{Gr}(n,k)} \lambda_k(K|L)\ \gamma_{n,k}(dL) = V_{n,k}(K)$$

Finally, we show that $V_{n,k}$ is a valuation. Suppose $K_1, K_2 \in \mathcal{K}^n$ are such that $K_1 \cup K_2$ is also convex. Note that for any L, we have

$$(K_1 \cup K_2)|L = (K_1|L) \cup (K_2|L)$$

and that this is convex. A priori, we also have

$$(K_1 \cap K_2)|L \subseteq (K_1|L) \cap (K_2|L)$$

but we must have equality. Indeed, pick $x \in (K_1|L) \cap (K_2|L)$. Define $x + L^\perp$ to be the translated copy of $L^\perp$ that passes through x. Let

$$X_i := \{k \in K_i \mid k \text{ projects to } x\} = K_i \cap (x + L^\perp)$$

Note that X_1, X_2 and $X_1 \cup X_2$ are non-empty compact convex. By the lemma following this proposition, $X_1 \cap X_2$ is non-empty so $x \in (K_1 \cap K_2)|L$. Having proven the identities

$$(K_1 \cup K_2)|L = (K_1|L) \cup (K_2|L)$$
$$(K_1 \cap K_2)|L = (K_1|L) \cap (K_2|L)$$

We conclude using the fact that λ_k is a valuation and the linearity of integration. $\square$

Lemma 3.2.3. Suppose $K, L \in \mathcal{K}^n$ non-empty are such that $K \cup L \in \mathcal{K}^n$. Then $K \cap L \in \mathcal{K}^n$ is non-empty.

Proof. If $L \subseteq K$, we are done. Otherwise, pick any $l \in L - K$. Since K is closed and convex, it follows from theory of Hilbert spaces that there exists a unique $k \in K$ that is the closest point of K to l. Consider the line segment joining k and l, which must lie in $K \cup L$ by convexity of $K \cup L$. By construction, k must be the only point on the line segment that lies in K, so all other points of the line segment must lie in L. Since L is closed, we conclude $k \in L$ and so $k \in K \cap L$. □

There is an alternative idea that is first explored by Crofton, which is to pick a random $(n-k)$-dimensional hyperplane and ask whether it will intersect the convex body. Intuitively, the larger the body, the more "likely" an intersection will occur. It turns out that this idea leads back to the same intrinsic volume functions. In the proposition, note that $V_{n,0}$ has be used to detect whether an intersection has occurred or not.

Proposition 3.2.4. (Crofton formula) For $1 \leq k \leq n$, we have

$$V_{n,k}(K) = \int_{\mathrm{Aff}(n,n-k)} V_{n,0}(K \cap P)\ \alpha_{n,n-k}(dP)$$

Before proceeding with the proof, here is an obligatory remark that one should technically check that $V_{n,0}(K \cap \cdot)$ is a measurable function for the integral to make sense, but we will again leave the issue aside.

Proof. We have

$$\begin{aligned} &\int_{\mathrm{Aff}(n,n-k)} \chi(K \cap P)\ \alpha_{n,n-k}(dP) \\ = &\int_{\mathrm{Gr}(n,n-k)} \int_{L^\perp} \chi(K \cap (L+y))\ \lambda_k(dy)\ \gamma_{n,n-k}(dL) \end{aligned}$$

Now, for a fixed $L \in \mathrm{Gr}(n, n-k)$, as y varies in $L^\perp$, $K \cap (L+y)$ is non-empty if and only if $y \in K|L^\perp$. Therefore, the above expression simplifies to

$$\int_{\mathrm{Gr}(n,n-k)} \lambda_k(K|L^\perp)\ \gamma_{n,n-k}(dL)$$

Finally, we recall proposition 2.2.1 and conclude that this expression is equal to

$$\int_{\mathrm{Gr}(n,k)} \lambda_k(K|L)\ \gamma_{n,k}(dL)$$

which is $V_{n,k}(K)$. □

3.3. **Hadwiger's Theorem.** Recall from proposition 3.1.7 that Val_n is a real vector space. The key importance of intrinsic volume functions lies in the following theorem, which tells us that the intrinsic volume functions that we have defined through integral geometric language are really all that is needed to understand the collection of continuous invariant valuations.

Theorem 3.3.1. (Hadwiger) The intrinsic volume functions $V_{n,0}, \ldots, V_{n,n}$ are a basis for Val_n. In particular, $\dim(\mathrm{Val}_n) = n + 1$.

Proof. A typical proof is long and will derail us form the discussion. We refer interested readers to [Kla95]. □

Actually, it is not difficult to see that $V_{n,0}, \ldots, V_{n,n}$ are linearly independent. A function $\phi : \mathcal{K}^n \to \mathbb{R}$ is said to be homogeneous of degree i if $\phi(tK) = t^i \phi(K)$ where $tK := \{tx \mid x \in K\}$. Since the k-dimensional Lebesgue measure is homogeneous of degree k, it follows from the definition of the intrinsic volume functions that $V_{n,k}$ is homogeneous of degree k. If $a_0, \ldots, a_n$ are scalars such that

$$a_0 V_{n,0} + \cdots + a_n V_{n,n} \equiv 0$$

then we simply evaluate the above expression on tB_n and obtain

$$a_0 V_{n,0}(B_n) t^0 + \cdots + a_n V_{n,n}(B_n) t^n = 0 \text{ for all } t \geq 0$$

which means the above polynomial in t is the zero polynomial. Since $V_{n,i}(B_n) \neq 0$ for all i, we conclude that $a_0 = \cdots = a_n = 0$. Using a similar idea, we can deduce the following corollary from theorem 3.3.1.

Corollary 3.3.2. If ϕ is a continuous invariant valuation that is homogeneous of degree i, then it must be that $\phi = cV_{n,i}$ for some real constant c.

Proof. By theorem 3.3.1, we can write $\phi = a_0 V_{n,0} + \cdots + a_n V_{n,n}$ for some scalars $a_0, \ldots, a_n$. Evaluate both sides of the expression on tB_n and conclude as above that all the scalars are zero except possibly $a_i = \phi(B_n)/V_{n,i}(B_n)$. □

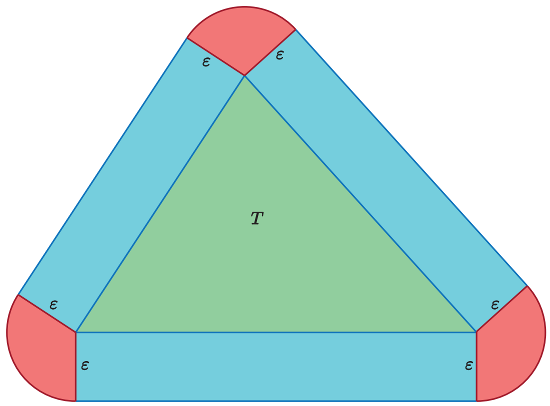
Now, we are in a position to perform a normalization procedure that captures the full meaning of the word "intrinsic" in the name "intrinsic volume". Suppose $K \in \mathcal{K}^m$. If $n \geq m$, we may also consider $\mathbb{R}^m$ as a subset of $\mathbb{R}^n$ in the most natural way and view K as an element of $\mathcal{K}^n$. There is no guarantee that $V_{m,k}(K) = V_{n,k}(K)$. However, we can easily rescale our intrinsic volume functions to achieve this desirable property, without affecting any of the theory that we have developed thus far.

We proceed inductively. To prevent confusion, V will continue to denote the intrinsic volumes as we have defined previously, and W will be used to denote the rescaled versions of V. We want to define $W_{n,k}$ for all $n \geq 1,\ 0 \leq k \leq n$.

(1) For all $n \geq 1$, we let $W_{n,0} = V_{n,0}$. Also set $W_{1,1} = \lambda_1$ which is also $V_{1,1}$.
(2) Suppose $W_{n-1,k}$ has been set for all $0 \leq k \leq n-1$. First set $W_{n,n} = \lambda_n$ which is also $V_{n,n}$. Next, consider a fixed $1 \leq k \leq n-1$ and we want to define $W_{n,k}$. If $K \in \mathcal{K}^{n-1}$, then we can consider it as an element of $\mathcal{K}^n$ and compute $V_{n,k}(K)$. Thus $V_{n,k}$ induces a k-homogeneous continuous invariant valuation on $\mathcal{K}^{n-1}$. By corollary 3.3.2, we can write this induced map as $cV_{n-1,k}$ for some $c \neq 0$, which is also of the form $c' W_{n-1,k}$ for some $c' \neq 0$. Set $W_{n,k} = \frac{1}{c'} V_{n,k}$.

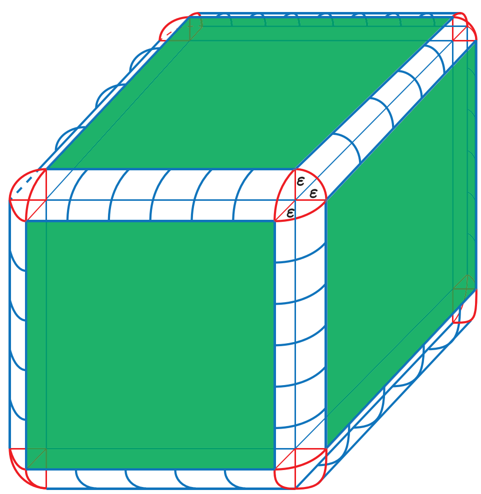
By construction, $W_{n,k}$ has the desired property that $W_{m,k}(K) = W_{n,k}(K)$ whenever $n \geq m$ and $K \in \mathcal{K}^m$ (and the bonus property that $W_{n,n}$ coincides with Lebesgue measure). We will call the $W_{n,k}$ the **rescaled intrinsic volume functions**. Now the intrinsic volumes of K are indeed "intrinsic" to K itself, in the sense that it does not depend on the dimension of the ambient space.

3.4. **Steiner's formula.** We now consider a formula due to Steiner that will give an alternative interpretation of the intrinsic volume functions. For example, we will see (up to scaling factors) how $W_{2,1}$ can be thought of as the perimeter function, of how $W_{3,2}$ can be thought of as the surface area function, generalized to all compact convex subsets. The correct scaling factor can be easily obtained by performing computation on say B^n and is an unimportant detail theoretically.



$$\lambda_2(T_{+\varepsilon}) = \text{area}(T) + \varepsilon \cdot \text{perimeter}(T) + \varepsilon^2 \cdot \pi$$

FIGURE 3.1.



$$\lambda_3(C_{+\varepsilon}) = \text{volume}(C) + \varepsilon \cdot \text{surface area}(C) + \varepsilon^2 \cdot \pi \cdot \frac{\text{total edge length}(C)}{4} + \varepsilon^3 \cdot \frac{4\pi}{3}$$

FIGURE 3.2.

Steiner's formula aims to describe how the n-dimensional Lebesgue measure of a fixed compact convex subset of $\mathbb{R}^n$ changes as it undergoes ϵ-fattening. It turns out that $\lambda_n(K_{+\epsilon})$, as a function of ϵ, will be a $\leq n$-degree polynomial with coefficients involving the intrinsic volumes $W_{n,0}(K), \ldots, W_{n,n}(K)$. Figures 3.1 and 3.2 illustrate that we indeed obtain a polynomial in ϵ for the fixed triangle $T \subseteq \mathbb{R}^2$ and the fixed cube $C \subseteq \mathbb{R}^3$, and will be helpful for understanding the various definitions made in the proof below. (Note: the polynomial would be different if we had considered T to be in $\mathbb{R}^3$ - we would get a degree 3 polynomial with the constant term being zero.)

To begin the investigation, we will first obtain the formula for $\lambda_n(P_{+\epsilon})$ for P a polytope.

Definition 3.4.1. A polytope in $\mathbb{R}^n$ is a bounded non-empty subset which can be represented as the intersection of finitely many closed halfspaces.

Note that a polytope is necessarily in $\mathcal{K}^n$ (and it can very well be a "lower-dimensional" object - because two closed halfspaces can intersect to give a hyperplane and further intersections after that will give a "lower-dimensional" polytope). In fact, the following proposition suggests why it might be useful to first study the collection $\mathcal{P}^n$ of polytopes.

Proposition 3.4.2. Under the Hausdorff metric, $\mathcal{P}^n$ is a dense subset of $\mathcal{K}^n$.

Proof. Fix $K \in \mathcal{K}^n$ and $\epsilon > 0$. We wish to show the existence of $P \in \mathcal{P}^n$ for which $\delta(P, K) \leq \epsilon$. For each point $k \in K$, we consider the open ball $\mathrm{ball}_k(\epsilon)$ centered at k of radius ϵ. Then

$$\bigcup_{k \in K} \mathrm{ball}_k(\epsilon)$$

is an open cover of K. By compactness of K, there exists $k_1, \ldots, k_w$ such that $K \subseteq \bigcup_{i=1}^{w} \mathrm{ball}_{k_i}(\epsilon)$. Set P to be the convex hull of $k_1, \ldots, k_w$. Then $P \in \mathcal{P}^n$ and because K is convex, we have $P \subseteq K \subseteq K_{+\epsilon}$. By construction of P, we have $K \subseteq P_{+\epsilon}$. Therefore by lemma 3.1.4, we have $\delta(P, K) \leq \epsilon$. $\square$

Recall that a **supporting hyperplane** of $P \in \mathcal{P}^n$ is a hyperplane H such that $H \cap P \neq \emptyset$ and P lies entirely in one of the two closed halfspaces defined by H. In fact, $H \cap P$ is again a polytope and is called an **k-face** if its dimension is k $(0 \leq k \leq n-1)$. We also consider P to be a n-face of itself. For $0 \leq k \leq n$, we let $\mathcal{F}_k$ be the collection of all k-faces of P and denote the collection of all faces of P by $\mathcal{F} := \bigcup_{k=0}^{n} \mathcal{F}_k$. For a face $F \in \mathcal{F}$, we define its **relative interior** $\mathrm{relint}(F)$ to be all elements of F that do not belong to any face of strictly lower dimension. Observe that P can be written as the disjoint union

$$P = \coprod_{F \in \mathcal{F}} \mathrm{relint}(F)$$

In fact, we can decompose the fattened $P_{+\epsilon}$ into a disjoint union as well. From theory of Hilbert space, to each point $x \in \mathbb{R}^n$, we can associate a unique closest point in P; let $\mathrm{proj} : \mathbb{R}^n \to \mathbb{R}^n$ denote this map (the image will be in P). By the partition above, we end up with a partition of $\mathbb{R}^n$ depending on where $\mathrm{proj}(x)$ lies. Thus we may write

$$P_{+\epsilon} = \coprod_{F \in \mathcal{F}} (P_{+\epsilon} \cap \mathrm{proj}^{-1}(\mathrm{relint}(F)))$$

Now we have to find a way to compute the size of each piece. For $0 \leq k \leq n-1$ and $F \in \mathcal{F}_k$ we define the normal cone of F, denoted $N(F)$, as follows: pick any $x \in \mathrm{relint}(F)$, and consider the closed convex cone consisting of the zero vector and all (not necessarily unit) outer normal vectors of supporting hyperplanes at x. It is easy to check that this definition does not depend on the choice of x. Finally, we observe that

$$\begin{aligned} &\lambda_n(P_{+\epsilon} \cap \mathrm{proj}^{-1}(\mathrm{relint}(F))) \\ &= \lambda_{n-k}(N(F) \cap \epsilon B^n) \cdot \lambda_k(F) \\ &= \epsilon^{n-k} \cdot \lambda_{n-k}(N(F) \cap B^n) \cdot \lambda_k(F) \end{aligned}$$

and of course for the separate case $k = n$, we have $\lambda_n(P_{+\epsilon} \cap \mathrm{proj}^{-1}(\mathrm{relint}(P))) = \lambda_n(P)$.

Therefore, if we set, for $0 \leq k \leq n-1$,

$$\tilde{W}_{n,k}(P) := \sum_{F \in \mathcal{F}_k} \lambda_{n-k}\left(N\left(F\right) \cap B_n\right) \cdot \lambda_k\left(F\right) \tag{3.1}$$

and $\tilde{W}_{n,n}(P) := \lambda_n(P)$, then we have the following theorem.

Theorem 3.4.3. (Steiner's formula for polytopes) For a polytope $P \in \mathcal{P}^n$, $\lambda_n(P_{+\epsilon})$ as a function of ϵ is a $\leq n$-degree polynomial. In fact, with $\tilde{W}_{n,0}, \ldots, \tilde{W}_{n,n} : \mathcal{P}^n \to \mathbb{R}$ defined as above,

$$\lambda_n(P_{+\epsilon}) = \sum_{k=0}^{n} \epsilon^{n-k} \cdot \tilde{W}_{n,k}(P)$$

Proof. The proof is given by the discussion above. □

Now, we extend the discussion from $\mathcal{P}^n$ to $\mathcal{K}^n$.

Theorem 3.4.4. (Steiner's formula) There are functions $\tilde{W}_{n,0}, \ldots, \tilde{W}_{n,n} : \mathcal{K}^n \to \mathbb{R}$ such that for any $K \in \mathcal{K}^n$, we have

$$\lambda_n(K_{+\epsilon}) = \sum_{k=0}^{n} \epsilon^{n-k} \cdot \tilde{W}_{n,k}(K)$$

Proof. First, observe that for any polytope $P \in \mathcal{P}^n$, we have the following system of linear expressions from theorem 3.4.3:

$$\begin{bmatrix} \lambda_n(P_{+1}) \\ \lambda_n(P_{+2}) \\ \vdots \\ \lambda_n(P_{+n}) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 2^n & 2^{n-1} & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ n^n & n^{n-1} & \ldots & 1 \end{bmatrix} \begin{bmatrix} \tilde{W}_{n,0}(P) \\ \tilde{W}_{n,1}(P) \\ \vdots \\ \tilde{W}_{n,n}(P) \end{bmatrix}$$

Inverting the Vandermonde matrix, we can express

$$\tilde{W}_{n,k}(P) = \sum_{j=0}^{n} c_{kj} \cdot \lambda_n(P_{+j})$$

where c_{kj} are entries of the inverted matrix. Now for any $K \in \mathcal{K}^n$, we simply define

$$\tilde{W}_{n,k}\left(K\right) := \sum_{j=0}^{n} c_{kj} \cdot \lambda_n\left(K_{+j}\right) \tag{3.2}$$

We already noted that λ_n is continuous on $\mathcal{K}^n$ (example 3.1.8), but in fact for a fixed $\epsilon \geq 0$, the mapping $K \mapsto \lambda_n(K_{+\epsilon})$ is continuous as well, since it is the composition of continuous maps[11] $K \mapsto K_{+\epsilon} \mapsto \lambda_n(K_{+\epsilon})$. Therefore, the $\tilde{W}_{n,k}$, defined by equation (3.2), are continuous. Since we have Steiner's formula for polytopes and P is dense in K (proposition 3.4.3), we conclude that Steiner's formula holds for each $K \in \mathcal{K}^n$ by choosing a sequence of polytopes converging to K. □

[11]Continuity of the first map follows from the fact that $K \subseteq L_{+\eta}$ implies $K_{+\epsilon} \subseteq (L_{+\eta})_{+\epsilon} = L_{+(\eta+\epsilon)} = (L_{+\epsilon})_{+\eta}$ and an application of lemma 3.1.4.

From the definition by equation (3.2), we see that $\tilde{W}_{n,k}$ is invariant under $E(n)$, and we already saw that it is continuous. In fact, $\tilde{W}_{n,k}$ is a valuation. This follows directly from our previous observation that λ_n is a valuation (example 3.1.8) and the following lemma (lemma 3.4.5), because from the lemma we know

$$\begin{aligned}
\tilde{W}_{n,k}(K \cup L) &= \sum_{j=0}^{n} c_{kj} \cdot \lambda_n((K \cup L)_{+j}) \\
&= \sum_{j=0}^{n} c_{kj} \cdot \lambda_n(K_{+j} \cup L_{+j}) \\
&= \sum_{j=0}^{n} c_{kj} \cdot \lambda_n(K_{+j}) + \sum_{j=0}^{n} c_{kj} \cdot \lambda_n(L_{+j}) - \sum_{j=0}^{n} c_{kj} \cdot \lambda_n(K_{+j} \cap L_{+j}) \\
&= \sum_{j=0}^{n} c_{kj} \cdot \lambda_n(K_{+j}) + \sum_{j=0}^{n} c_{kj} \cdot \lambda_n(L_{+j}) - \sum_{j=0}^{n} c_{kj} \cdot \lambda_n((K \cap L)_{+j}) \\
&= \tilde{W}_{n,k}(K) + \tilde{W}_{n,k}(L) - \tilde{W}_{n,k}(K \cap L)
\end{aligned}$$

whenever $K, L \in \mathcal{K}^n$ are such that $K \cup L \in \mathcal{K}^n$.

Lemma 3.4.5. Suppose $K, L \in K$ are such that $K \cup L \in K$. Then

$$\begin{aligned}
(K \cup L)_{+\epsilon} &= K_{+\epsilon} \cup L_{+\epsilon} \\
(K \cap L)_{+\epsilon} &= K_{+\epsilon} \cap L_{+\epsilon}
\end{aligned}$$

Proof. $x \in (K \cup L)_{+\epsilon}$ if and only if there exists some $y \in K \cup L$ such that $d(x, y) \leq \epsilon$, which is equivalent to either there being a $y \in K$ or there being a $y \in L$ such that $d(x, y) \leq \epsilon$, which is equivalent to $x \in K_{+\epsilon} \cup L_{+\epsilon}$.

One direction of the second equality is equally clear. Namely, if $x \in (K \cap L)_{+\epsilon}$, then there exists some $y \in K \cap L$ such that $d(x, y) \leq \epsilon$, so in particular $d(K, x)$, $d(L, x) \leq \epsilon$, whence $x \in K_{+\epsilon} \cap L_{+\epsilon}$.To show the reverse inclusion, suppose $x \in K_{+\epsilon} \cap L_{+\epsilon}$. Let $k \in K$ and $l \in L$ such that $d(k, x) \leq \epsilon$ and $d(l, x) \leq \epsilon$. Let $\mathfrak{L}$ be the line segment joining k and l. Observe that $K \cup L$ is convex implies $\mathfrak{L} \subseteq K \cup L$. Also observe that k, l lie in the closed ball of radius ϵ centered at x, so every point of $\mathfrak{L}$ is of distance $\leq \epsilon$ from x. Therefore, it remains to show that some point on $\mathfrak{L}$ lies in $K \cap L$. Indeed, $\mathfrak{L}$ is connected and $K \cap \mathfrak{L}, L \cap \mathfrak{L}$ are non-empty closed subsets of $\mathfrak{L}$ with $(K \cap \mathfrak{L}) \cup (L \cap \mathfrak{L}) = \mathfrak{L}$, so $K \cap \mathfrak{L}$ and $L \cap \mathfrak{L}$ cannot be disjoint. □

From this discussion, we conclude that $\tilde{W}_{n,k} \in \mathrm{Val}_n$. In fact, from the definition by equation (3.1), we see that for any $P \in \mathcal{P}^n$, we have

$$\tilde{W}_{n,k}(tP) = t^k \tilde{W}_{n,k}(P)$$

and by continuity with denseness of P, we conclude that $\tilde{W}_{n,k}$ is homogeneous of degree k. Therefore, we have the following proposition.

Proposition 3.4.6. $\tilde{W}_{n,k}$ is equal to the intrinsic volume functions $W_{n,k}$ up to a scale factor.

Proof. This follows from corollary 3.3.2. □

For polytopes in $\mathbb{R}^2$ and $\mathbb{R}^3$, it is easy to see that $\tilde{W}_{2,1}$ and $\tilde{W}_{3,2}$ are the perimeter and surface area function respectively. Thus, we see how $W_{2,1}$ and $W_{3,2}$ are ways to generalize these concepts to compact convex subsets of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively (up to scaling factors). These generalizations are also natural ideas in the sense that they are obtain by approximating with polytopes. The other $\tilde{W}_{n,k}$'s are also not too difficult to understand concretely for polytopes, and provide a different perspective of understanding our intrinsic volume functions $W_{n,k}$.

To sum up the rewards of this chapter, it is worth emphasizing that through the lens of probability, we were able to generate the entire class of size functions on $\mathcal{K}^n$ by the application of one single idea (be it looking at shadow size or intersection counts). These size functions are not only generalizations of conventional notions such as surface area and volume, but also apply to sets whose poor boundary behavior typically exclude them from classical definitions.

4. Lengths on Manifolds

Everyone knows what a curve is, until he has studied enough mathematics to become confused through the countless number of possible exceptions,
– Felix Klein

The average shadow length when $K \in \mathcal{K}^n$ is projected onto a random element of $\mathrm{Gr}(n, 1)$, which we denoted by $V_{n,1}$, is also called the **mean width**. Proposition 3.2.4 due to Crofton tells us that $V_{n,1}$ can be equivalently understood as the measure of hyperplanes that intersects K. For a 1-dimensional object in $\mathcal{K}^n$, i.e. a line segment, the rescaled version $W_{n,1}$ gives its length (because $W_{1,1}$ agrees with λ_1). However, studying the length of line segments alone is not very interesting, and in fact, studying the length of the wider class of piecewise smooth curves is a lot more important, in the following sense.

Suppose M is a (smooth) Riemannian manifold where g is its Riemannian metric (i.e. a smooth symmetric covariant 2-tensor field on M that is positive definite at each point). Recall that M can be turned into a metric space as follows. If $\gamma : [a, b] \to M$ is a piecewise smooth curve (from now all curves without any adjectives are assumed to be piecewise smooth), define the length of γ to be

$$L_g(\gamma) := \int_a^b |\gamma'(t)|_g dt$$

where $\gamma'(t)$ lies in the tangent space at each point and so we may compute its norm using g. We then define the distance between two points $p, q \in M$ to be

$$d_g(p, q) := \inf_{\gamma} L_g(\gamma)$$

where γ ranges over all curves with endpoints p and q. It is a theorem that d_g turns M into a metric space whose metric topology coincides with the manifold topology (for example, see [Lee12, 13.29]).

Although every smooth manifold M admits a Riemannian metric (see [Lee12, 13.3]), the idea is we want to start with just M alone and see if integral geometry can be used to define length of curves on M. For example, in $\mathbb{R}^n$, integral geometry allowed us to reproduce the length of line segments and we have good intuitive reason to believe that Crofton's idea of looking at intersection with hyperplanes will allow us to reproduce length of curves (proposition 4.1.1) - and indeed Crofton himself proved this in $\mathbb{R}^2$. Once we have length of curves on M, we can talk about distances between points of M. This is a powerful alternative way to introduce geometry (in a loose sense of the word) on M, rather than seeking for a Riemannian metric. Note that this distance function might have nothing to do with the smooth manifold topology of M.

The goal of this chapter is to lay the beginnings of this idea that is original to this thesis. We will see how definitions of curve length permissible by our application

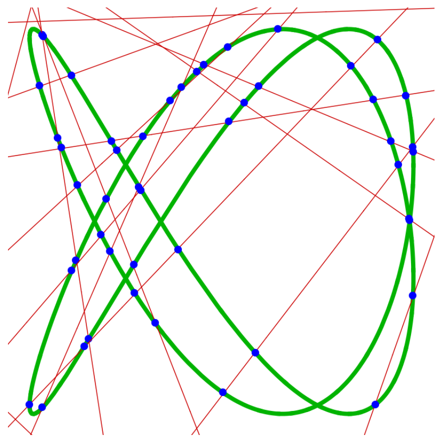


FIGURE 4.1.

of integral geometry include those obtainable by some Riemannian metric, but also much more.

4.1. **From Needles to Noodles.** Before plunging into a general smooth manifold M, let us first convince ourselves that a direct extension of Crofton's formula (proposition 3.2.4) from line segments to curves can be done. Intuitively, it says that the length of a curve is proportional to the expected number of intersections with randomly chosen hyperplanes (figure 4.1).

Proposition 4.1.1. (Crofton's formula for curves) Fix $n \geq 1$. For any curve γ, we have

$$\mathrm{length}(\gamma) = c_n \int_{\mathrm{Aff}(n,n-1)} N_\gamma(P)\ \alpha_{n,n-1}(dP)$$

where c_n is a constant and $N_\gamma : P \to \mathbb{R} \cup \{\pm\infty\}$ maps P to the number of intersections between P and $\mathrm{Im}(\gamma)$.

Of course one should check that $N_\gamma(\cdot)$ is a measurable function. We will also present a sketched proof rather than derail ourselves by fiddling with the details of taking the limit.

Proof. (Sketch) We already have this result for γ a line segment. A key property of N_γ that will give us the general result is additivity, namely, if $\gamma : [a, b] \to \mathbb{R}^n$ is a curve and $a < z < b$, then we can define $\gamma_1 := \gamma|_{[a,z]}$ and $\gamma_2 := \gamma|_{[z,b]}$, and we have

$$N_\gamma = N_{\gamma_1} + N_{\gamma_2}$$

almost everywhere (there is some complication of double counting at the point $\gamma_1(z) = \gamma_2(z)$, hence the "almost everywhere"). Using additivity, we thus have the result for γ a piecewise linear curve. Finally, a curve can be well-approximated by a sequence of piecewise linear curves, from which the result follows. $\square$

4.2. **Extension to Manifolds.** Let M be a smooth manifold and $\gamma : [a, b] \to M$ be a curve. Our first goal is to find a way to define the length of γ using inspiration from integral geometry. We would want to define the length as the expected number of intersections with random choices of "planes", and one way we can do this is to use level sets of smooth functions on M.

Definition 4.2.1. Let M be a smooth manifold and γ a curve. Whenever we have a smooth function ω on M, denote by $N_\gamma(\omega)$ the number of intersections between γ and the level set $\{x \in M | \omega(x) = 0\}$. Now, if Ω is a set of smooth functions on M and (Ω, Σ, μ) is a measure space such that $N_\gamma : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ is measurable (i.e. Σ needs to be sufficiently fine), then we may define

$$L_\mu(\gamma) := \int_\Omega N_\gamma(\omega)\ \mu(d\omega)$$

We should remark that $N_\gamma(\omega)$, as the cardinality of the intersection set, may well be infinite, and we have no reason to believe that this will occur with measure zero. As such, $L_\mu(\gamma)$ may well be infinite.

For reasons that will be clear later, we will make a small change to what we mean by $N_\gamma(\omega)$. We have the smooth function $\omega \circ \gamma : [a, b] \to M \to \mathbb{R}$. Let us instead define $N_\gamma(\omega)$ to be the number of transitions between $\omega \circ \gamma \le 0$ and $\omega \circ \gamma > 0$. This notion is well-defined because $\omega \circ \gamma$ is continuous and so the set of points satisfying $\omega \circ \gamma > 0$ is an open subset of $[a, b]$. By the property of open subsets of $\mathbb{R}$ being a countable disjoint union of open intervals, the same can be said for open subset of $[a, b]$ (except "open interval" here means relatively open, so it could be something that includes the endpoints). Separating into the cases where the disjoint union is countably infinite or finite, we respectively are able to compute an answer for $N_\gamma(\omega)$.

Definition 4.2.2. Let M be a smooth manifold, γ a curve and $N_\gamma(\cdot)$ be defined as above. If Ω is a set of smooth functions on M and (Ω, Σ, μ) is a measure space such that $N_\gamma : \Omega \to [-\infty, \infty]$ is measurable, then we may define

$$L_\mu(\gamma) := \int_\Omega N_\gamma(\omega)\ \mu(d\omega)$$

We will use definition 4.2.2 from now on. For a fixed M, a choice of (Ω, Σ, μ) for which all N_γ are measurable (from now on implicitly assumed or verified) gives us a definition of "length of curves". In turn, a choice of (Ω, Σ, μ) gives us a distance function

$$d_\mu(x, y) := \inf_\gamma L_\mu(\gamma)$$

where γ ranges over all curves with endpoints x and y, and the infimum is taken to be ∞ if there is no such γ or if all values of $L_\mu(\gamma)$ happen to be infinite.

Although we allow (Ω, Σ, μ) to be incredibly general, the induced distance function has decent properties.

Proposition 4.2.3. Suppose (Ω, Σ, μ) is such that all pairs of points on M have finite distance. Then d_μ is a pseudometric (i.e. a metric except that one could have $d_\mu(x, y) = 0$ for distinct $x \neq y$).

Proof. Let x, y, z be points of M. $d_\mu(x, x) = 0$ because we simply take γ to be the curve $\gamma([a, b]) = \{x\}$ for which $L_\mu(\gamma) = 0$ since $N_\gamma(\omega) = 0$ for all $\omega \in \Omega$. Next, it is clear from the definition that $d_\mu(x, y) = d_\mu(y, x)$. Lastly, the triangle inequality holds because a path with endpoints x and y followed by a path with endpoints y and z is in particular a path with endpoints x and z. $\square$

The finiteness hypothesis of proposition 4.2.3 is not a serious obstacle. We can define equivalence classes on M where $x \sim y$ if and only if $d_\mu(x, y) < \infty$. A similar reasoning to the above proof shows that $\sim$ is indeed an equivalence relation. We then have broken M into a disjoint union of so-called galaxies, in which d_μ is a pseudometric on each galaxy. Of course each galaxy may no longer be a manifold, but we have already achieved some form of geometrical structure on each galaxy, which is our real goal. One may further perform the metric identification on each galaxy (i.e. $x \sim y$ if and only if $d_\mu(x, y) = 0$) and turn each galaxy into a metric space.

Remark 4.2.4. The new definition for $N_\gamma(\omega)$ is crucial for us to obtain $d_\mu(x, x) = 0$. If we had stuck to definition 4.2.1, then $N_\gamma(\omega)$ need not be zero for the curve $\gamma([a, b]) = \{x\}$, because the zero level curve of ω may very well contain x (so γ intersects the level curve). In particular, consider the example where Ω contains only the zero function. Then $d_\mu(x, x) = 1$ for all $x \in M$. □

Example 4.2.5. We can recover (up to a scaling factor) the Euclidean metric as follows. Here $M = \mathbb{R}^n$,

$$\Omega = \left\{x \mapsto x \cdot n + a \mid (n, a) \in S^{n-1} \times \mathbb{R}\right\}$$

and we use the product (Borel) measure. The level curves are precisely hyperplanes. There is a technical subtlety that Crofton's formula for curves uses the notion of N_γ in definition 4.2.1, but in this setting it can be shown that for any curve the two definitions differ on hyperplanes that make up zero measure. □

The following is a generalization of the above example, and show that our integral geometric framework is no less general than what Riemannian geometry has to offer.

Proposition 4.2.6. For every Riemannian manifold (M, g), there is a measure space (Ω, Σ, μ) of smooth functions on M such that length of curves are equal under L_g and L_μ.

Proof. By the Nash embedding theorem, we may isometrically embed M into some $\mathbb{R}^n$. A curve in M is then a curve in $\mathbb{R}^n$. We take the same (Ω, Σ, μ) as in example 4.2.5, except that the functions $x \mapsto x \cdot n + a$ are restricted to M. (Figure 4.2 shows how the level curves get induced on the embedded copy of M by the hyperplanes of $\mathbb{R}^n$.) Then L_g and L_μ are equal up to a scaling factor. Simply rescale μ so that $L_g = L_\mu$. □

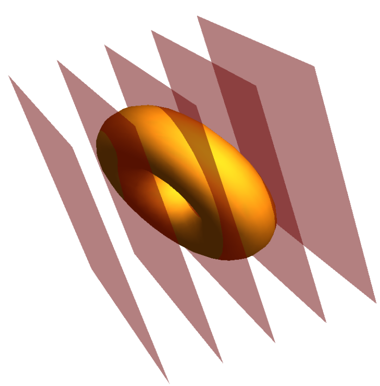


FIGURE 4.2.

Example 4.2.7. Here is an example which gives a geometry of a different flavor from the above ones. We consider having only finitely many functions in Ω. For concreteness, take $M = S^{n-1}$ and $\Omega = \left\{x \mapsto x \cdot n_i \mid 1 \leq i \leq k, n_1, \ldots, n_k \in S^{n-1}\right\}$ with the counting measure μ imposed. The zero level curves are grand circles and these grand circles partition the surface of the sphere into pieces (whether an arc of the grand circle lies in the piece on one side or the other side is determined by the direction of n_i; we want $x \cdot n_i \leq 0$ to be one piece and $x \cdot n_i > 0$ to be another piece). We say that two pieces are adjacent if their boundaries share an arc (of non-zero length) of one of these grand circles. Under the metric identification of the pseudometric d_μ, we obtain a graph where vertices represent pieces and vertices are adjacent if and only if the corresponding pieces are adjacent. The path length between two vertices is equal to k times the distance between any two points in the corresponding sphere piece under d_μ. One might ask: can we characterize the graphs that are produced by this approach? $\square$

As an idea still in its stage of infancy, it remains to be seen what kinds of interesting geometries can be obtained on a smooth manifold M by a choice of (Ω, Σ, μ). We might also examine various types of convergence of our measure μ and ask if L_μ will respectively converge in some sense. This will give us the possibility of studying more complicated geometries by approximating with geometries that we understand. The potential of this framework remains to be seen.

5. Total Curvature Measures

In mathematics you don't understand things. You just get used to them.
– John von Neumann

In this section, we return to the Euclidean set-up of chapter 3 but leave behind geometrical notions of size like length and surface area, to give a glimpse of how integral geometry is a framework that encompasses many other geometrical ideas, in particular curvature. With regards to curvature, Thomas Banchoff presented an application of integral geometry that can be found in [Ban67] and [Ban70]. However, this chapter will focus on a different approach that is a direct extension of the ideas from chapter 3.

The key will again be Steiner's formula, or rather, a variant of it called the local Steiner's formula. We previously saw how the intrinsic volume functions showed up as the coefficients of the Steiner polynomial, and we were able to interpret directly some of these coefficients for "sufficiently nice" elements of $\mathcal{K}^n$ (e.g. polygons, polyhedrons). In turn, this gives us an interpretation of the intrinsic volume functions, linking their integral geometric definitions with classical terms. We will similarly be able to interpret the coefficients of the local Steiner polynomial and watch integral geometric definitions show up as these coefficients.

5.1. **Weyl's Tube Formula.** The best way to motivate the approach of this chapter is to realize that the Weyl's tube formula from differential geometry has a strikingly similar form to Steiner's formula. In loose terms, this formula considers the thickening of hypersurfaces and how the resulting volume is a function of the amount of thickening. We will present this formula in a slightly different form from how it is usually stated – a form that is more suggestive of how we should later extend Steiner's formula.

For this entire chapter, let $n \geq 2$. Recall that for a non-empty compact convex subset $K \in \mathcal{K}^n$, every point $x \in \mathbb{R}^n$ is associated with a unique closest point in K, and we write $\mathrm{proj} : \mathbb{R}^n \to \mathbb{R}^n$ for this map (the image will be in K). For $A \in \mathcal{B}(\mathbb{R}^n)$ and $\epsilon \geq 0$, we define the **A-restricted ϵ-fattening of** K as

$$K_{+\epsilon,A} := \{x \in K_{+\epsilon} \mid proj(x) \in A\}$$

For example, $K_{+0,A} = K \cap A$ and $K_{+\epsilon,\partial K} = K_{+\epsilon}$. The first order of business is to ensure that we can talk about the volume of these objects.

Proposition 5.1.1. For a fixed $K \in \mathcal{K}^n$ and $\epsilon \geq 0$, $K_{+\epsilon,A}$ is n-Lebesgue measurable for every $A \in \mathcal{B}(\mathbb{R}^n)$. In fact, the function $\lambda_n(K_{+\epsilon,\cdot}) : \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}$ is a finite Borel measure (note the $\cdot$ in the subscript of the function $\lambda_n(K_{+\epsilon,\cdot})$, which is where the argument is).

Proof. Observe that the map proj : $\mathbb{R}^n \to \mathbb{R}^n$ is continuous and thus (Lebesgue-Borel) measurable. Therefore,

$$K_{+\epsilon,A} = K_{+\epsilon} \cap \mathrm{proj}^{-1}(A)$$

is Lebesgue measurable. For the second part of the proposition, observe that the induced subset σ-algebra of $K_{+\epsilon} \subseteq \mathbb{R}^n$ is precisely the set of Lebesgue measurable subset of $\mathbb{R}^n$ contained in $K_{+\epsilon}$ (because $K_{+\epsilon}$ is itself Lebesgue measurable). Therefore, it makes sense to talk about the Lebesgue measure induced on $K_{+\epsilon}$. Now, $\mathrm{proj}|_{K_{+\epsilon}}$ is (Lebesgue-Borel) measurable and observe that $\lambda_n(K_{+\epsilon,\cdot})$ is the pushforward measure under this map. □

We say that a $K \in \mathcal{K}^n$ is of **class** $\mathcal{C}^2_{\neq 0}$ if K has non-empty interior and its boundary ∂K is a $\mathcal{C}^2$-manifold with all of its principal curvatures[12] at each point being non-zero.

Theorem 5.1.2. (Weyl's tube formula) There are some constants $c_{n,k}$ (which we will not care about), such that for any $K \in \mathcal{K}^n$ of class $\mathcal{C}^2_{\neq 0}$ and A an open subset[13] of $\mathbb{R}^n$,

$$\lambda_n(K_{+\epsilon,A} \backslash K) = \sum_{k=0}^{n-1} \epsilon^{n-k} \cdot c_{n,k} \int_{A \cap \partial K} H_{n,n-1-k} \; dS$$

where $H_{n,k}$ is the k^{th} elementrary symmetric function of the $n-1$ principal curvatures.

Proof. See, for example, [Sch14, (2.63)]. □

Let us make a quick comment about why one would care about the functions $\Phi_{n,k}(K,\cdot) : \{\text{open subsets of } \mathbb{R}^n\} \to \mathbb{R}$ defined by

$$\Phi_{n,k}(K,A) := \int_{A \cap \partial K} H_{n,k} \; dS$$

Note that if $n = 3$ and $k = 2$, then $H_{3,2}$ is the usual Gaussian curvature and $\Phi_{3,2}(K,\cdot)$ is the usual total curvature function. Thus, we may think of $\Phi_{n,k}(K,\cdot)$ as a generalized sort of total curvature function. Most of these functions do not have a name, just like how $V_{3,1}$ on polyhedrons do not have a name. However, these functions collectively give a more complete description of the curvature behavior of ∂K.

5.2. **Local Steiner's Formula.** The polynomial form of Weyl's tube formula could not have been more suggestive that one should study, for a general $K \in \mathcal{K}^n$, how $\lambda_n(K_{+\epsilon,A})$ behaves as a function of ϵ. For a fixed K and a fixed Borel A, the answer will again be a polynomial of degree $\leq n$, but this time the coefficients are functions of K and A.

Before proceeding, there is an important property about the measures $\lambda_n(K_{+\epsilon,\cdot})$ that needs to be recorded as a lemma for future use. The result is intuitive but the proof will take us too far afield that we shall be content with providing a reference. If M is a metric space and Σ its Borel σ-algebra, recall that a sequence of its finite

[12] Principal curvatures refer to the eigenvalues of the Weingarten map (which is the differential of the Gauss map).

[13] We chose openness rather than Borel because an open subset intersecting ∂K is an open submanifold of ∂K for which the surface integral will make sense.

signed measures $\mu_1, \mu_2, \dots$ converges weakly to the finite signed measure μ, denoted $\mu_i \stackrel{w}{\to} \mu$, if

$$\int f \ d\mu_i \to \int f \ d\mu$$

for all bounded, continuous (real-valued) functions f. Recall also the fact from functional analysis is that the weak limit of a sequence of finite signed Borel measures will be unique. Here is the lemma that we will need.

Lemma 5.2.1. Suppose $K, K_1, K_2, \dots \in \mathcal{K}^n$ are such that $K_i \to K$ under the Hausdorff metric. Then we have the following weak convergence of measures:

$$\lambda_n((K_i)_{+\epsilon,\cdot}) \stackrel{w}{\to} \lambda_n(K_{+\epsilon,\cdot})$$

Proof. See [Sch97, 4.1.1]. □

Let us now proceed to find an expression of $\lambda_n(K_{+\epsilon,A})$ in terms of ϵ. For the case of K being a polytope P, a discussion almost identical to that of the original Steiner's formula would give us the desired result. Namely, write

$$P_{+\epsilon,A} = \coprod_{F \in \mathcal{F}} (P_{+\epsilon} \cap \operatorname{proj}^{-1}(\operatorname{relint}(F) \cap A))$$

and conclude that

$$\lambda_n(P_{+\epsilon,A}) = \sum_{k=0}^{n} \left(\epsilon^{n-k} \cdot \sum_{F \in \mathcal{F}_k} \lambda_{n-k}(N(F) \cap B_n) \cdot \lambda_k(F \cap A) \right)$$

Setting (the notation is chosen to be analogous to that of the original Steiner's formula)

$$\tilde{W}_{n,k}(P, A) := \sum_{F \in \mathcal{F}_k} \lambda_{n-k}(N(F) \cap B_n) \cdot \lambda_k(F \cap A)$$

for $0 \le k \le n-1$ and $\tilde{W}_{n,n}(P, A) := \lambda_n(P \cap A)$, we have the following theorem.

Theorem 5.2.2. (Local Steiner's formula for polytopes) For a polytope $P \in \mathcal{P}^n$ and any $A \in \mathcal{B}(\mathbb{R}^n)$, $\lambda_n(P_{+\epsilon,A})$ as a function of ϵ is a $\le n$-degree polynomial. In fact, with $\tilde{W}_{n,0}, \dots, \tilde{W}_{n,n} : \mathcal{P}^n \times \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}$ defined as above,

$$\lambda_n(P_{+\epsilon,A}) = \sum_{k=0}^{n} \epsilon^{n-k} \cdot \tilde{W}_{n,k}(P, A)$$

Proof. The proof is given by the discussion above. □

The extension from $\mathcal{P}^n$ to $\mathcal{K}^n$ is similar in spirit to how it was done for the original Steiner's formula, but with a small twist.

Theorem 5.2.3. (Local Steiner's formula) There are functions $\tilde{W}_{n,0}, \dots, \tilde{W}_{n,n} : \mathcal{K}^n \times B(\mathbb{R}^n) \to \mathbb{R}$ such that for any $K \in \mathcal{K}^n$, we have

$$\lambda_n(K_{+\epsilon,A}) = \sum_{k=0}^{n} \epsilon^{n-k} \cdot \tilde{W}_{n,k}(K, A)$$

Proof. As in the proof for theorem 3.4.4, we begin by writing

$$\tilde{W}_{n,k}(P, A) = \sum_{j=0}^{n} c_{kj} \cdot \lambda_n(P_{+j,A})$$

where c_{kj} are the entries of the matrix

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ 2^n & 2^{n-1} & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ n^n & n^{n-1} & \dots & 1 \end{bmatrix}^{-1}$$

Now, for a fixed $K \in \mathcal{K}^n$, we already know that $\lambda_n(K_{+j,\cdot})$ is a finite Borel measure. Thus, we can define the following (possibly signed) finite Borel measures

$$\tilde{W}_{n,k}(K,\cdot) := \sum_{j=0}^{n} c_{kj} \cdot \lambda_n(K_{+j,\cdot})$$

For a fixed K, choose polytopes $P_1, P_2, \dots$ such that $P_i \to K$ (possible by proposition 3.4.2). By lemma 5.2.1, we have $\lambda_n((P_i)_{+\epsilon,\cdot}) \xrightarrow{w} \lambda_n(K_{+\epsilon,\cdot})$ and thus $\tilde{W}_{n,k}(P_i,\cdot) \xrightarrow{w} \tilde{W}_{n,k}(K,\cdot)$. At the same time, we already have the local Steiner result for polytopes (theorem 5.2.2), so

$$\lambda_n((P_i)_{+\epsilon,\cdot}) = \sum_{k=0}^{n} \epsilon^{n-k} \cdot \tilde{W}_{n,k}(P_i,\cdot) \xrightarrow{w} \sum_{k=0}^{n} \epsilon^{n-k} \cdot \tilde{W}_{n,k}(K,\cdot)$$

By uniqueness of weak limit, we conclude that

$$\lambda_n(K_{+\epsilon,\cdot}) = \sum_{k=0}^{n} \epsilon^{n-k} \cdot \tilde{W}_{n,k}(K,\cdot)$$

That is,

$$\lambda_n(K_{+\epsilon,A}) = \sum_{k=0}^{n} \epsilon^{n-k} \cdot \tilde{W}_{n,k}(K,A)$$

for all $A \in \mathcal{B}(\mathbb{R}^n)$. $\square$

By a direct comparison of Weyl's tube formula and local Steiner's formula, we see that if in particular K is of class $\mathcal{C}^2_{\neq 0}$ and A is open, then $\tilde{W}_{n,k}(K,A)$ is simply a rescaled version of $\Phi_{n,n-1-k}(K,A) = \int_{A\cap\partial K} H_{n,n-1-k}\, dS$, where the constant of proportionality is independent of K and A. Thus, we can directly interpret the coefficients of the local Steiner polynomial as (generalized) total curvatures.

5.3. ***V*-measures and Integral Geometry.** We will now use integral geometry to define a class of Borel measures called V-measures, so named because they are a simple modification of the intrinsic volume functions of chapter 3. We will introduce a second argument, writing $V_{n,k}(K,A)$ where $A \in \mathcal{B}(\mathbb{R}^n)$. (This is slight overuse of notation but we will soon see that $V_{n,k}(K) = V_{n,k}(K,\mathbb{R}^n)$.) For a fixed K, it will turn out that $V_{n,k}(K,\cdot)$ is a Borel measure.

Compared to the order of presentation in chapter 3, the appearance of integral geometry has been delayed till the end (i.e. now) because it would not have been easy to motivate why anyone would be interested in looking at these V-measures. The definitions will appear less arbitrary if one keeps the local Steiner's formula in mind – in the end, these V-measures appear as the coefficients in the local Steiner polynomial.

From the onset, let us be clear that to work out some technical details of this section requires a deep journey into the study of compact convex bodies, and can be enough material for another thesis. Our main goal is to give a glimpse of how integral geometry can be connected to curvature. To focus on the key integral geometric ideas, we would simply mention the details that need to be checked and refer interested readers to [Sch14], an extensive monograph on compact convex bodies.

Recall that in chapter 3, the function $V_{n,0}$ is essentially an indicator of whether the set is empty or not. Crofton's formula (proposition 3.2.4) then define the other functions $V_{n,1}, \dots, V_{n,n}$ in terms of some expectation involving $V_{n,0}$. Motivated by the study of the local Steiner's formula, we now define the V-measure $V_{n,0}$ to care only about points of $\partial K \cap A$ as follows.

Definition 5.3.1. Let $K \in \mathcal{K}^n$, $A \in \mathcal{B}(\mathbb{R}^n)$. Let $N(K, A) \subseteq \mathbb{R}^n$ be the set consisting of the zero vector and all (not necessarily unit) outer normal vectors of K at points of $\partial K \cap A$. Define

$$V_{n,0}(K, A) := \frac{\lambda_n(N(K, A) \cap B^n)}{\lambda_n(B^n)}$$

where $\lambda_n(B^n)$ is just a normalization factor. For this definition to make sense, one should verify that $N(K, A) \cap B^n$ is always n-Lebesgue measurable. One can also check that $V_{n,0}(K, \cdot)$ satisfies the axioms of a measure.

Note that for non-empty $K \in \mathcal{K}^n$, the set of all its outer unit normal vectors, when varied across its entire boundary, is S^{n-1}. Thus, $V_{n,0}(K, \mathbb{R}^n) = 1 = V_{n,0}(K)$ for non-empty K. Intuitively, $V_{n,0}(K, A)$ is now the "fraction" of ∂K that also lies in A, judged by looking at the set of outer unit normal vectors of $\partial K \cap A$ as a fraction of the set of outer unit normal vectors of ∂K. For some motivation on why such a quantity might be related to curvature, think of how a patch with high total curvature will have unit normal vectors that sweep through a larger proportion of S^{n-1}.

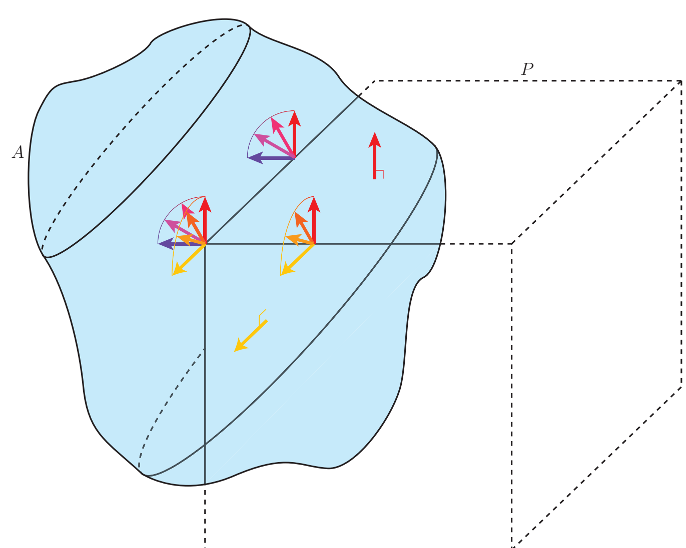


FIGURE 5.1.

Example 5.3.2. An interesting example to study would be where K is a polytope P. Recall that we also used the notation $N(F)$ for the normal cone of F. In this case, note that the vertices of P are the only ones that "matter" (see figure 5.1). More specifically,

$$V_{n,0}(P, A) = \sum_{v \in \mathcal{F}_0, v \in A} \frac{\lambda_n(N(v) \cap B^n)}{\lambda_n(B^n)}$$

This is true even if P is a "lower dimensional" polytope. One can think of the contributions to $V_{n,0}(P, \cdot)$ as being concentrated at the vertices of P, with the contribution being effected if and only if the vertex is in A. Later when we show the link between $V_{n,0}$ and total curvature, one can think of all the "Gaussian curvature" of P being solely concentrated at the vertices. □

Here is the integral geometric definition of the other V-measures.

Definition 5.3.3. (Local Crofton formula) For $1 \leq k \leq n$, define

$$V_{n,k}(K, A) := \int_{\mathrm{Aff}(n,n-k)} V_{n,0}(K \cap P, A)\ \alpha_{n,n-k}(dP)$$

As usual, one has to check that $V_{n,0}(K \cap \cdot, A)$ is a measurable function. One can also check that $V_{n,k}(K, \cdot)$ satisfies the axioms of a measure.

Note that

$$V_{n,k}(K, \mathbb{R}^n) = \int_{\mathrm{Aff}(n,n-k)} V_{n,0}(K \cap P)\ \alpha_{n,n-k}(dP) = V_{n,k}(K)$$

Intuitively, $V_{n,k}(K)$ previously counts each intersection with a plane "fully", but $V_{n,k}(K, A)$ counts only a "fraction" of every intersection, by looking at the "fraction" of $\partial(K \cap P)$ that also lies in A, via the use of $V_{n,0}$. Note that $V_{n,n}(P, A) = \lambda_n(K \cap A)$.

Example 5.3.4. For the case of K being a polytope P, it turns out that

$$V_{n,k}(P, A) = c_{n,k} \sum_{F \in \mathcal{F}_k(P)} \frac{\lambda_{n-k}(N(F) \cap B^n)}{\lambda_{n-k}(B^{n-k})} \cdot \lambda_k(F \cap A)$$

where $c_{n,k}$ are constants independent of P and A. Compared to example 5.3.2, this example is more messy to show rigorously, but figure 5.2 illustrates (with $n = 3,\ k = 1$) why the above form is intuitively believable. Intuitively, each intersection of P with a $(n-k)$-plane will generally be $(n-k)$-dimensional polytope, and its $V_{n,0}$ output is determined by its vertices (example 5.3.2), which are actually points on the k-face of the original polytope P.

Later when we show the link between $V_{3,1}$ and integral of mean curvature, one can think of all the "mean curvature" of P as being solely concentrated at the edges of the cube. □

Finally, this proposition will provide the link between V-measures and (generalized) total curvature.

Proposition 5.3.5. The V-measures $V_{n,k}$ are respectively equal to the coefficients of the local Steiner polynomial $\tilde{W}_{n,k}$, up to a scaling factor.

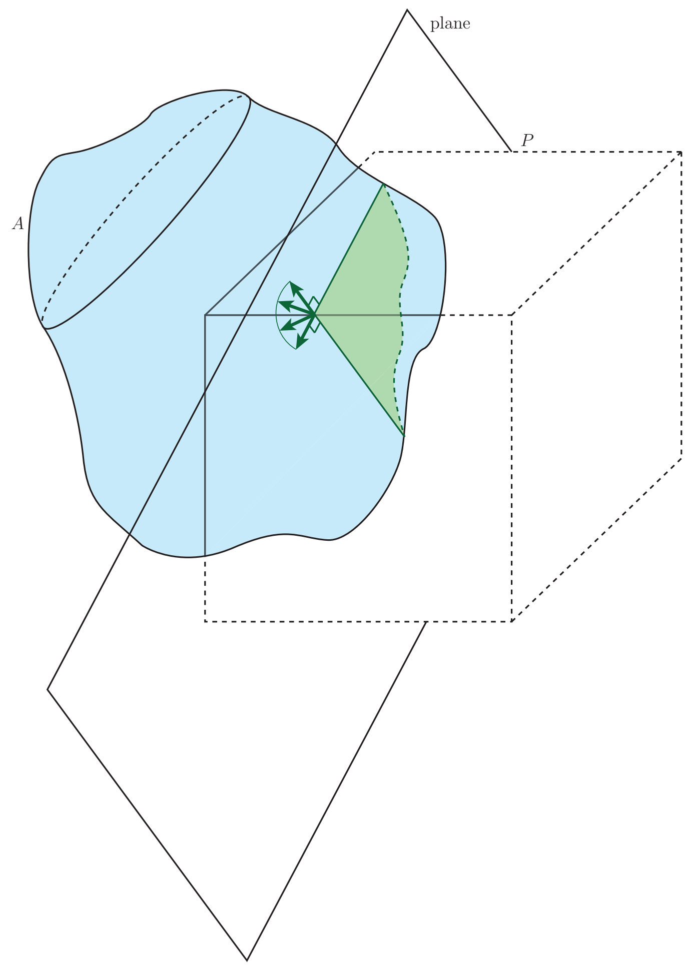


FIGURE 5.2.

Proof. (Sketch) Suppose $K, K_1, K_2, \dots \in \mathcal{K}^n$ are such that $K_i \to K$ under the Hausdorff metric. In the course of proving theorem 5.2.3, we saw that lemma 5.2.1 implied $\tilde{W}_{n,k}(K_i, \cdot) \xrightarrow{w} \tilde{W}_{n,k}(K, \cdot)$. It can be shown that a similar weak convergence property holds for the V-measures, namely $V_{n,k}(K_i, \cdot) \xrightarrow{w} V_{n,k}(K, \cdot)$. If we further believe the formula for polytopes discussed in example 5.3.4, we see that $\tilde{W}_{n,k}(P, \cdot)$ and $V_{n,k}(P, \cdot)$ agree whenever P is a polytope. For a fixed K, choose polytopes $P_1, P_2, \dots$ such that $P_i \to K$ and by uniqueness of weak limits, we conclude that $\tilde{W}_{n,k}(K, \cdot) = V_{n,k}(K, \cdot)$. □

Just as how the intrinsic volume functions can be thought of as notions of size applicable to compact convex sets, the V-measures can be thought of as notions of total curvature applicable to compact convex sets and whose measure domain applies to all Borel subsets of $\mathbb{R}^n$, not just open ones. Therefore, we can talk about total curvature of patches of ∂K that look nothing like a manifold.

6. Recent Developments

The worst thing you can do is to completely solve a problem.
– Daniel Kleitman

To conclude, we list a number of recent developments in integral geometry. Many of these extensions have been motivated by applications in stochastic geometry [SW08]. This list is by no means exhaustive – we only cover extensions that are more directly related to the content discussed in this thesis. A more thorough survey that also covers other aspects of integral geometry can be found in [HS02].

More general sets. While convex sets (or even polyconvex sets) constitute a wide collection of objects, there have been further attempts to establish some form of Crofton's formula or curvature measures on collections of sets with more pathological behaviors. For example, Fu [Fu94] examined "subanalytic sets" while Brocker and Kuppe [BK00] further advanced this subject by considering "compact tame Whitney-stratified sets" (this collection includes semi-algebraic sets, subanalytic sets, and sets belonging to an σ-minimal system). These sets show up in model theory and computer science.

Non-Euclidean Spaces. Starting from a given non-Euclidean space X, one can focus on a collection of sets $\mathcal{K} \subseteq 2^X$ and ask if we can similarly[14] characterize valuations on $\mathcal{K}$ in integral geometric terms. Little is known in the general context but Klain and Rota [KR97, chapters 3 and 11] provides some answers for the case of X being a sphere and X being a finite set.

Crofton formula for functions. A function $f : \Omega \to \mathbb{R}$, where $\Omega \subseteq \mathbb{R}^n$ is open and convex, is a convex function if and only if its epigraph is convex. This suggests the possibility of studying such a convex function by the techniques we explored. Indeed, for $\epsilon \geq 0$ and A is Borel set contained in Ω, one may define the **ϵ-fattening of A by means of f** as

$$\{x + \epsilon v \mid x \in A,\ v \in \partial f(x)\}$$

where $\partial f\,(x)$ denotes the "subdifferential" of f at x. It turns out that the fattening is a Lebesgue measurable subset of $\mathbb{R}^n$, and its Lebesgue measure as a function of ϵ is a polynomial of degree $\leq n$. By considering the special case where f happens to be of class $\mathcal{C}^2$, these coefficients can be interpreted in terms of some expression involving the eigenvalues of $\mathrm{Hess}(f)$ at various points (the coefficients are thus called Hessian measures). One can make various integral geometric definitions of measures that turn out to be Hessian measures. A concrete discussion can be found in [CH05]. For comparison, a study of Hessian measures without the integral geometric approach can be found in [TW97], [TW99] and [TW02].

Translative integral geometry. In some applications of stochastic geometry, an assumption that space is isotopic is superfluous or invalid. Thus, rather than

[14] We characterized all continuous invariant valuations when $X = \mathbb{R}^n$ and $\mathcal{K} = \mathcal{K}^n$ as linear combinations of intrinsic volume functions.

considering invariance under $E(n)$, one only considers invariance under the translation group $T(n)$. The study of expectations under this setting leads to translative integral geometry. This subject is treated in [AF14, 1.1].

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