QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday January 21, 2020 (Day 1)

1. (A) Show $\mathbb{Z}[\sqrt{p}]$ is not a unique factorization domain for p a prime congruent to 1 mod 4.

Solution: Consider

$$p - 1 = 2 \cdot \frac{p-1}{2} = (\sqrt{p}+1)(\sqrt{p}-1).$$

Then we claim 2 is irreducible, as it has norm 4 and there are no elements $a + b\sqrt{p}$ of norm 2, as that would imply $a^2 - pb^2 = 2$, which is impossible mod 4. But 2 clearly does not divide either factor on the right in the given ring.

Alternatively, UFDs must be normal but $(1 + \sqrt{p})/2$ is in the normalization of the above ring.

2. (AT) Determine whether $X = S^2 \vee S^3 \vee S^5$ is homotopy equivalent to (a) a manifold, (b) a compact manifold, (c) a compact, orientable manifold.

Solution: Although its Betti numbers are symmetric, its cohomology ring does not satisfy Poincaré duality, which shows it cannot be homotopy equivalent to a compact, oriented manifold.

For part (b), note that if it were homotopy equivalent to a compact, nonoriented manifold, it would have a nontrivial orientable double cover, and yet its fundamental group is trivial and so no nontrivial double covers exist.

Finally, for part (a) we can embed X in a Euclidean space, such as $\mathbb{R}^{13} = \mathbb{R}^3 \times \mathbb{R}^4 \times \mathbb{R}^6$, and take a small open neighborhood which deformation retracts back to X.

3. (AG) We say that a curve $C \subset \mathbb{P}^3$ is a *twisted cubic* if it is congruent (mod the automorphism group PGL_4 of $\mathbb{P}^3$) to the image of the map $\mathbb{P}^1 \to \mathbb{P}^3$ given by

$$\phi_0 : [X, Y] \mapsto [X^3, X^2Y, XY^2, Y^3].$$

Now let $C \subset \mathbb{P}^3$ be any irreducible, nondegenerate curve of degree 3 over an algebraically closed field. (Here, "nondegenerate" means that C is not contained in any plane.)

(a) Show that C cannot contain three collinear points.

(b) Show that C is rational, that is, birational to $\mathbb{P}^1$.

(c) Show that C is a twisted cubic.

Solution: For the first part, observe that if $p, q, r \in C$ are collinear, then for any fourth point $s \in C$ not on the line $\overline{p, q, r}$, the plane spanned by p, q, r and s will meet C in four points and hence contain C, contradicting nondegeneracy.

To see that C is rational, choose any two distinct points $p, q \in C$; let $L \subset \mathbb{P}^3$ be the line they span and let $\{H_\lambda\}_{\lambda \in \mathbb{P}^1}$ be the family of planes in $\mathbb{P}^3$ containing L. A general plane H_λ will intersect C at p, q and one other point R_λ; conversely, a general point $r_\lambda \in C$ will lie on a unique plane H_λ. This association gives a birational isomorphism of C with $\mathbb{P}^1$.

Finally, given the second part we have a rational map ϕ from $\mathbb{P}^1$ to $C \subset \mathbb{P}^3$, and since $\mathbb{P}^1$ is smooth this map is in fact regular. We can therefore write it as

$$[X, Y] \mapsto [F_0(X, Y), F_1(X, Y), F_2(X, Y), F_3(X, Y)]$$

for some 4-tuple $[F_0, F_1, F_2, F_3]$ of homogeneous cubic polynomials on $\mathbb{P}^1$. Since C is nondegenerate, the F_i are linearly independent, and hence form a basis for the 4-dimensional space of homogeneous cubic polynomials on $\mathbb{P}^1$. If we let $A \in GL_4$ be the change of basis matrix from the basis $\{X^3, X^2Y, XY^2, Y^3\}$ to the basis $\{F_0, F_1, F_2, F_3\}$, then, the action of A on $\mathbb{P}^3$ carries the image of ϕ_0 to C.

4. (CA) Let $\Omega \subset \mathbb{C}$ be a connected open subset of the complex plane and $f_1, f_2, \ldots$ a sequence of holomorphic functions on Ω converging uniformly on compact sets to a function f. Suppose that $f(z_0) = 0$ for some $z_0 \in \Omega$. Show that either $f \equiv 0$, or there exists a sequence $z_1, z_2, \cdots \in \Omega$ converging to z_0, with $f_n(z_n) = 0$.

Solution: Suppose not. Then we can find a disc $\Delta \subset \Omega$ around z_0 such that $f_n(z) \neq 0$ for $z \in \overline{\Delta}$, and such that z_0 is the sole zero of f in $\overline{\Delta}$. Now, since the functions f_n and their derivatives converge uniformly on compact sets to f, we have

$$\lim_{n \to \infty} \frac{1}{2\pi i} \int_{\partial\Delta} \frac{f_n'(z)}{f_n(z)} dz = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{f'(z)}{f(z)} dz.$$

But by the residue theorem and the hypothesis that $f_n(z) \neq 0$ for $z \in \Delta$, the terms on the left are all zero, while the right hand side is equal to 1, a contradiction.

5. (RA)

(i) Specify the range of $1 \leq p < \infty$ for which

$$\varphi(f) = \int_0^1 \frac{f(t)}{\sqrt{t}}\, dt.$$

defines a linear functional $\varphi : L^p([0,1]) \to \mathbb{R}$.

(ii) For those values of p, calculate the norm of the linear functional $\varphi : L^p([0,1]) \to \mathbb{R}$. The norm of a linear functional is defined as

$$\|\varphi\| = \sup_{\substack{f \in L^p([0,1]) \\ f \neq 0}} \frac{|\varphi(f)|}{\|f\|_{L^p}}$$

Solution.
(i) We use the fact that for $1 \leq p < \infty$, we can identify the dual space $(L^p)^*$ with L^q where q is the dual index to p, i.e., $p^{-1} + q^{-1} = 1$.

By this identification, the claim can be rephrased as asking for which q-values the function $\frac{1}{\sqrt{t}} \in L^q([0,1])$. The answer is for all $q \in [1,2)$. By the relation $p^{-1} + q^{-1} = 1$ and the restriction to $p < \infty$, the answer to part (i) is the range $p \in (2, \infty)$.

(ii). Let $p \in (2,\infty)$ or equivalently $q \in (1,2)$. We use that the identification of $(L^p)^*$ with L^q is in fact isometric and calculate

$$\|\varphi\| = \left(\int_0^1 \left(\frac{1}{\sqrt{t}} \right)^q dt \right)^{1/q} = \left(\frac{1}{1 - q/2} \right)^{1/q}.$$

6. (DG)

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x,y,z) = x^2 + y^2 - 1$.

(i) Prove that $M = f^{-1}(0)$ is a two-dimensional embedded submanifold of $\mathbb{R}^3$.

(ii) For $a, b, c \in \mathbb{R}$, consider the vector field

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$$

For which values of a, b, c is X tangent to M at the point $(1,0,1)$?

Solution.
(i) We note that $f : \mathbb{R}^3 \to \mathbb{R}$ is smooth with derivative $f_* = \nabla f = (2x, 2y, 0)$. This derivative has rank 1 everywhere on M. (Indeed, the points where the rank vanishes satisfy $x = y = 0$ and are not in $M = f^{-1}(0)$.) Therefore

the inverse function theorem implies that M is an embedded submanifold of dimension $3 - 1 = 2$.

(ii) First, we note that $(1, 0, 1) \in M$. The vector field X is tangent to M at the point $(1, 0, 1)$ if and only if $X(f) = 0$ at $(1, 0, 1)$. We compute

$$X(f)_{(1,0,1)} = (2ax + 2by)_{(1,0,1)} = 2a$$

which vanishes if and only if $a = 0$. The values of $b, c \in \mathbb{R}$ are arbitrary.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday January 22, 2020 (Day 2)

1. (AT) Let $\Delta = \{z \in \mathbb{C} : |z| \leq 1$ be the closed unit disc in the complex plane, and let X be the space obtained by identifying z with $e^{2\pi i/3}z$ for all z with $|z| = 1$.

1. Find the homology groups $H_k(X, \mathbb{Z})$ of X with coefficients in $\mathbb{Z}$.
2. Find the homology groups $H_k(X, \mathbb{Z}/3)$ of X with coefficients in $\mathbb{Z}/3$.

Solution: X can be realized as a CW complex with one 0-cell, one 1-cell and one 2-cell; the 1-skeleton is just a circle S^1 and the attaching map for the 2-cell is the map $z \mapsto z^3$. The associated cell complex is thus

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ \alpha\ } \mathbb{Z} \xrightarrow{\ \beta\ } \mathbb{Z} \longrightarrow 0,$$

with $\beta = 0$ and α given by multiplication by 3. The homology of X is then the homology of this complex, which is to say

$$H_2(X, \mathbb{Z}) = 0; \quad H_1(X, \mathbb{Z}) = \mathbb{Z}/3; \quad \text{and} \quad H_0(X, \mathbb{Z}) = \mathbb{Z}.$$

Similarly, to find the homology with coefficients in $\mathbb{Z}/3$ we tensor this complex with $\mathbb{Z}/3$; now all the maps are 0 and we have

$$H_2(X, \mathbb{Z}/3) = H_1(X, \mathbb{Z}/3) = H_0(X, \mathbb{Z}/3) = \mathbb{Z}/3.$$

2. (AG) Let C be a smooth, geometrically irreducible curve of genus 1 defined over $\mathbb{Q}$, and suppose L and M are line bundles on C of degrees 3 and 5, also defined over $\mathbb{Q}$. Show that C has a rational point, that is, $C(\mathbb{Q}) \neq \emptyset$.

Solution. Consider the line bundle $N = L^2 \otimes M^{-1}$, which has degree 1. By Riemann-Roch, $h^0(N) = 1$, so N has a global section σ; the zero locus of σ is then a single point $p \in C$, which is necessarily defined over $\mathbb{Q}$.

3. (A) Let g be an element of the finite group G. Prove that the following are equivalent:

1. g is in the center of G.
2. For every irreducible representation (V, ρ) of G, the image $\rho(g)$ is a multiple of the identity.
3. For every irreducible representation (V, ρ) of G, the character of g has absolute value $\dim(G)$.

Solution. (1) implies (2): $\rho(g)$ is a G-endomorphism of V, so is a multiple of the identity by Schur's Lemma.

(2) implies (1): for any $h \in G$, $\rho([g,h]) = \rho(id)$ for all irreducible ρ, and thus for all ρ, including the regular representation. But then $[g,h] = id$.

(2) implies (3): Say $\rho(g) = c \cdot I_V$. Then $\rho(g)$ has trace $\dim(V) \cdot |c|$. Since some power of g (and thus of $\rho(g)$) is the identity, c is a root of unity. Hence $|c| = 1$ and the trace has absolute value $\dim(V)$.

(3) implies (2): Since some power of $\rho(g)$ is the identity, $\rho(g)$ is diagonalizable and all eigenvalues are roots of unity. Hence the trace has absolute value at most $\dim(V)$ (triangle inequality), and equals $\dim(V)$ only when the eigenvalues are all equal to each other. But then $\rho(g)$ is a multiple of the identity, and we're done.

4. (RA)

Let $g \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ and write $\hat{g}$ for its Fourier transform defined by

$$\hat{g}(k) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ik\cdot x} g(x)\, dx$$

For $m > 0$, define the function $f : \mathbb{R}^3 \to \mathbb{C}$ by

$$f(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ik\cdot x} \frac{\hat{g}(k)}{k^2 + m^2}\, dx$$

Show that f solves the partial differential equation $-\Delta f + m^2 f = g$ in the distributional sense, i.e., show that for every test function $\varphi \in C_0^\infty(\mathbb{R}^3)$,

$$\langle -\Delta\varphi + m^2\varphi, f\rangle = \langle \varphi, g\rangle.$$

Here $\langle \cdot, \cdot \rangle$ denotes the $L^2(\mathbb{R}^3)$-inner product.

Solution.
We observe that $-\Delta\varphi + m^2\varphi$ and f lie in $L^2(\mathbb{R}^3)$. The former holds since $-\Delta\varphi + m^2\varphi \in C_0^\infty(\mathbb{R}^3)$ and the latter holds because f is the inverse Fourier

transform of an L^2-function. Hence, we can apply unitarity of the Fourier transform on $L^2(\mathbb{R}^3)$ (Parseval's theorem) and linearity to find

$$\langle -\Delta\varphi + m^2\varphi, f\rangle = \langle \widehat{-\Delta\varphi + m^2\varphi}, \hat{f}\rangle = \langle \widehat{-\Delta\varphi} + m^2\hat{\varphi}, \hat{f}\rangle$$

On the right-hand side, we use the pointwise identity $\widehat{-\Delta\varphi(k)} = k^2\hat{\varphi}(k)$ and the Fourier inversion theorem on $\hat{f}$ to find

$$\langle \widehat{-\Delta\varphi + m^2\varphi}, \hat{f}\rangle = \langle (k^2+m^2)\hat{\varphi}, \frac{1}{k^2+m^2}\hat{g}\rangle = \langle \hat{\varphi}, \hat{g}\rangle$$

On the last expression, we use Parseval's theorem again, which is allowed because $\varphi, g \in L^2(\mathbb{R}^3)$, and we obtain

$$\langle \hat{\varphi}, \hat{g}\rangle = \langle \varphi, g\rangle$$

as desired.

5. (DG)

Consider $\mathbb{R}^2$ as a Riemannian manifold equipped with the metric

$$g = (1+x^2)\mathrm{d}x^2 + \mathrm{d}y^2.$$

(i) Compute the Christoffel symbols of the Levi-Civita connection for g.

(ii) Compute the parallel transport of an arbitrary vector $(a, b) \in \mathbb{R}^2$ along the curve $\gamma(t) = (t, t)$ starting at $t = 0$.

(iii) Is γ a geodesic?

(iv) Are there two parallel vector fields $X(t), Y(t)$ to the curve γ, such that $g(X(t), Y(t)) = 2t$?

Solution.

(i). We have

$$g^{-1} = \begin{pmatrix} \frac{1}{1+x^2} & 0 \\ 0 & 1 \end{pmatrix}.$$

Denoting $x^1 = x$, $x^2 = y$, the only non-vanishing Christoffel symbol is

$$\Gamma^1_{11} = \frac{1}{2}(g^{-1})_{11}\partial_1 g_{11} = \frac{x}{1+x^2}.$$

(ii). The equation for parallel transport $\nabla_{\gamma'}(a^1, a^2) = 0$, with $\gamma(t) = (t, t)$, becomes

$$\frac{\mathrm{d}a^1}{\mathrm{d}t} + \frac{t}{1+t^2}a^1 = 0, \qquad \frac{\mathrm{d}a^2}{\mathrm{d}t} = 0.$$

The second equation is trivial and the first one can be solved by separation of variables. Implementing the initial conditions $(a^1(0), a^2(0)) = (a, b)$ gives the solutions $a^1(t) = \frac{a}{\sqrt{1+t^2}}$ and $a^2(t) = b$. The parallel transport is therefore

$$(a^1(t), a^2(t)) = \left(\frac{a}{\sqrt{1+t^2}}, b\right).$$

(iii). By part (i), the two ODE describing the geodesic $(x(t), y(t))$ are given by

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{x}{1+x^2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = 0, \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = 0.$$

While $\gamma(t) = (t, t)$ solves the second equation, it satisfies

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{x}{1+x^2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = \frac{t}{1+t^2} \neq 0$$

and is therefore not a geodesic.

(iv). No. The scalar product of two vectors is preserved by parallel transport, since ∇ is the Levi-Civita connection.

6. (CA) Evaluate the contour integral of the following functions around the circle $|z| = 2020$ oriented counterclockwise:

(i) $\frac{1}{\sin z}$;

(ii) $\frac{1}{e^{2z} - e^z}$.

Note that $\frac{2020}{\pi} \sim 642.98597$.

Solution:

(i) $f(z) = \frac{1}{\sin z}$ is analytic in $\{z \neq n\pi : n \in \mathbb{Z}\}$. It has a pole of order one at $n\pi$ (reason: $(\sin z)'|_{z=n\pi} = \cos(n\pi) = (-1)^n \neq 0$). So

$$\mathrm{Res}_{z=n\pi}\frac{1}{\sin z} = \frac{1}{\cos(n\pi)} = (-1)^n.$$

Therefore,

$$\int_{|z|=2020} \frac{dz}{\sin z} = 2\pi i \sum_{|n\pi| \leq 2020} \mathrm{Res}_{z=n\pi}\frac{1}{\sin z} = 2\pi i \sum_{|n| \leq 642} (-1)^n = 2\pi i.$$

(ii) $f(z) = \frac{1}{e^{2z}-e^z}$ is analytic in $\{e^{2z} - e^z \neq 0\} = \{e^z \neq 1\} = \{z \neq 2n\pi i : n \in \mathbb{Z}\}$. Since $(e^{2z} - e^z)'|_{z=2n\pi i} = 1 \neq 0$, $f(z)$ has a pole of order one at $2n\pi i$. So

$$\mathrm{Res}_{z=2n\pi} \frac{1}{e^{2z} - e^z} = \left.\frac{1}{2e^{2z} - e^z}\right|_{z=2n\pi} = 1.$$

Therefore,

$$\begin{aligned}\int_{|z|=2020} \frac{1}{e^{2z} - e^z} &= 2\pi i \sum_{|2n\pi i|\leq 2020} \mathrm{Res}_{z=2n\pi i} \frac{1}{e^{2z} - e^z} \\ &= 2\pi i \sum_{|n\pi|\leq 321} 1 \\ &= 1286\pi i.\end{aligned}$$

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday January 23, 2020 (Day 3)

1. (A) Let V be an n-dimensional vector space over an arbitrary field K, and let $T_1, \ldots, T_n : V \to V$ be pairwise commuting nilpotent operators on V.

1. Show that the composition $T_1 T_2 \cdots T_n = 0$.
2. Does this conclusion still hold if we drop the hypothesis that the T_i are pairwise commuting?

Solution. The key step for Part a is the

Lemma 1 *If S and T are commuting nilpotent operators on V then either* $\mathrm{im}(ST) \subsetneq \mathrm{im}(T)$ *or* $T = 0$.

Proof Since S and T commute, $\mathrm{im}(ST) = \mathrm{im}(TS) \subset \mathrm{im}(T)$; that is, S carries the image of T to itself. If we had the equality $\mathrm{im}(ST) = \mathrm{im}(T)$, with $\mathrm{im}(T) \neq 0$, then the restriction $S|_{\mathrm{im}(T)}$ would be invertible, and no power of S could be 0.

Part (1) follows immediately. For part (2), the simplest counterexample is to take $V = K^2$, $S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

2. (RA)

(a) Let H be a Hilbert space, $K \subset H$ a closed subspace, and x a point in H. Show that there exists a unique y in K that minimizes the distance $\|x - y\|$ to x.

(b) Give an example to show that the conclusion can fail if H is an inner product space which is not complete.

Solution: (a): If $y, y' \in K$ both minimize distance to x, then by the parallelogram law:

$$\|x - \frac{y + y'}{2}\|^2 + \|\frac{y - y'}{2}\|^2 = \frac{1}{2}(\|x - y\|^2 + \|x - y'\|^2) = \|x - y\|^2$$

But $\frac{y+y'}{2}$ cannot be closer to x than y, by assumption, so $y = y'$.

Let $C = \inf_{y \in K} \|x - y\|$, then $0 \leq C < \infty$ because K is non-empty. We can find a sequence $y_n \in K$ such that $\|x - y_n\| \to C$, which we want to show is

Cauchy. The midpoints $\frac{y_n+y_m}{2}$ are in K by convexity, so $\|x - \frac{y_n+y_m}{2}\| \geq C$ and using the parallelogram law as above one sees that $\|y_n - y_m\| \to 0$ as $n, m \to \infty$. By completeness of H the sequence y_n converges to a limit y, which is in K, since K is closed. Finally, continuity of the norm implies that $\|x - y\| = C$.

(b): For example choose $H = C([0,1]) \subset L^2([0,1])$, K the subspace of functions with support contained in $[0, \frac{1}{2}]$, and and $x = 1$ the constant function.

If f_n is a sequence in K converging to $f \in H$ in L^2-norm, then

$$\int_{1/2}^{1} |f|^2 = 0$$

thus f vanishes on $[1/2, 1]$, showing that K is closed. The distance $\|x - y\|$ can be made arbitrarily close to $1/\sqrt{2}$ for $y \in K$ by approximating $\chi_{[0,1/2]}$ by continuous functions, but the infimum is not attained.

3. (AG)

1. Let the homogeneous coordinates of $\mathbb{P}^m$ be $x_0, \ldots, x_m$, and the homogeneous coordinates of $\mathbb{P}^n$ be $y_0, \ldots, y_n$, $N = (m+1)(n+1) - 1$, and the homogeneous coordinates of $\mathbb{P}^N$ be $z_{i,j}$ for $i = 0, \ldots, m$, $j = 0, \ldots, n$. Consider the Segre embedding

$$f : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N,$$

   given by $z_{i,j} = x_i y_j$. Show that the degree of the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^n$ is $\binom{n+m}{n}$.

2. Let Y be a variety of dimension k in $\mathbb{P}^n$, with Hilbert polynomial h_Y. Define the arithmetic genus of Y to be $g = (-1)^k (p_Y(0) - 1)$. Show that the arithmetic genus of the hypersurface H of degree d in $\mathbb{P}^n$ is $\binom{d-1}{n}$.

Solution:

1. Note that the Hilbert polynomial $p_{f(\mathbb{P}^m \times \mathbb{P}^n)}$ of the Segre embedding of $\mathbb{P}^m \times \mathbb{P}^n$ is the product of the Hilbert polynomials $p_{\mathbb{P}^m}$, $p_{\mathbb{P}^n}$, of $\mathbb{P}^m$ and $\mathbb{P}^n$. Then

$$p_{f(\mathbb{P}^m \times \mathbb{P}^n)}(d) = p_{\mathbb{P}^m}(d) \cdot p_{\mathbb{P}^n}(d) = \binom{m+d}{d}\binom{n+d}{d}.$$

   (Or one can note that a homogeneous of degree d in $\mathbb{P}^N$ pulls back to a bihomogeneous polynomial of bidgree (d,d).)
   Thus

$$\deg(f(\mathbb{P}^m \times \mathbb{P}^n)) = (m+n)! \cdot \left(\frac{1}{n!}\frac{1}{m!}\right) = \binom{m+n}{n}.$$

2. Note that the Hilbert polynomial is $p_H(m) = \binom{m+n}{n} - \binom{m-d+n}{n}$. Then

$$p_H(0) = 1 - \binom{n-d}{n} = 1 - (-1)^n \binom{d-1}{n}.$$

4. (CA) Find the Laurent series expansion of the meromorphic function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

around the origin, valid in the annulus $\{z : 1 < |z| < 2\}$.

Solution: We use partial fractions to write f as a sum of functions with only a single pole, yielding

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

We now take the power series expansion of the second term valid in the disc $|z| < 2$, that is,

$$\frac{1}{z-2} = \frac{-1}{2} \cdot \frac{1}{1-z/2} = \sum_{n=0}^{\infty} \frac{-1}{2^{n+1}} z^n$$

and the Laurent series expansion of the first term valid in the annulus $|z| > 1$; that is,

$$\frac{-1}{z-1} = -z^{-1} \frac{1}{1-z^{-1}} = -\sum_{n=0}^{\infty} z^{-n-1}$$

The sum of these two is the Laurent series expansion.

5. (DG)

Define the set

$$H = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

(i) Equip H with a C^∞ differentiable structure so that it is diffeomorphic to $\mathbb{R}^2$.

(ii) Show that H is a Lie group under matrix multiplication.

(iii) Show that

$$\left\{ \frac{\partial}{\partial y}, \frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \right\}$$

forms a basis of left-invariant vector fields of the associated Lie algebra.

Solution (i). We use a single, global coordinate chart

$$\varphi : H \to \mathbb{R}^2$$
$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \mapsto (x, y)$$

The differentiable structure defined by H and φ is then diffeomorphic to $\mathbb{R}^2$.
(ii). Let $A, B \in H$ with

$$A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}.$$

Elementary computation shows that

$$AB = \begin{pmatrix} 1 & a+x & b+ax+y \\ 0 & 1 & a+x \\ 0 & 0 & 1 \end{pmatrix} \in H, \qquad A^{-1} = \begin{pmatrix} 1 & -a & a^2-b \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{pmatrix} \in H,$$

so H is a group. In coordinates, these maps can be written as

$$(a, b, x, y) \mapsto (a + x, b + ax + y), \qquad (a, b) \mapsto (-a, a^2 - b)$$

which are clearly C^∞, so H is a Lie group.
(iii). Since $\dim H = 2$ and the vector fields are obviously linearly independent, it suffices to show they are left-invariant. It is convenient to identify elements $A \in H$ with their coordinate vectors, say $A = (a, b)$. As computed above, the left translation L_A of $B = (x, y)$ can be written as $L_A(B) = (a+x, b+ax+y)$. It has the Jacobian

$$(L_A)_* = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

From this we can check directly that both vector fields satisfy $(L_A)_* X_B = X_{AB}$. Indeed,

$$(L_A)_* \left(\frac{\partial}{\partial y}\right)_B = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left(\frac{\partial}{\partial y}\right)_{L_A(B)}$$

and

$$(L_A)_* \left(\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)_B = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} 1 \\ x+a \end{pmatrix} = \left(\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)_{L_A(B)}$$

6. (AT) Suppose that X is a space written as a union of two simply connected open subsets U_1 and U_2.

(a) Show that H_1X is a free abelian group.

(b) Find an example in which $\pi_1 X$ is a non-trivial group. Why does this not contradict the the Seifert-van Kampen theorem?

(c) Find an example in which $\pi_1 X$ is non-abelian.

Solution: The Mayer-Vietoris sequence

$$H_1U \oplus H_1V \to H_1X \to H_0(U \cap V)$$

identifies H_1X with a subgroup of $H_0(U \cap V)$ which is a free abelian group. The claim follows from the fact that a subgroup of a free abelian group is free abelian. The circle S^1 is the union of two contractible open sets and has fundamental group $\mathbb{Z}$. (This doesn't contradict Seifert-van Kampen theorem because the intersection is not connected.)

Finally, if S is any discrete set, the suspension of S is the union of two cones on S, each of which is contractible. The fundamental group of the suspension of S is the free group on the points of S, so taking S to consist simply of three points (so that the suspension is a figure 8) provides an example for part (c).