QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday January 21, 2020 (Day 1)

1. (A) Show $\mathbb{Z}[\sqrt{p}]$ is not a unique factorization domain for p a prime congruent to 1 mod 4.

2. (AT) Determine whether $X = S^2 \vee S^3 \vee S^5$ is homotopy equivalent to (a) a manifold, (b) a compact manifold, (c) a compact, orientable manifold.

3. (AG) We say that a curve $C \subset \mathbb{P}^3$ is a *twisted cubic* if it is congruent (mod the automorphism group PGL_4 of $\mathbb{P}^3$) to the image of the map $\mathbb{P}^1 \to \mathbb{P}^3$ given by

$$\phi_0 : [X, Y] \mapsto [X^3, X^2Y, XY^2, Y^3].$$

Now let $C \subset \mathbb{P}^3$ be any irreducible, nondegenerate curve of degree 3 over an algebraically closed field. (Here, "nondegenerate" means that C is not contained in any plane.)

(a) Show that C cannot contain three collinear points.

(b) Show that C is rational, that is, birational to $\mathbb{P}^1$.

(c) Show that C is a twisted cubic.

4. (CA) Let $\Omega \subset \mathbb{C}$ be a connected open subset of the complex plane and $f_1, f_2, \ldots$ a sequence of holomorphic functions on Ω converging uniformly on compact sets to a function f. Suppose that $f(z_0) = 0$ for some $z_0 \in \Omega$. Show that either $f \equiv 0$, or there exists a sequence $z_1, z_2, \cdots \in \Omega$ converging to z_0, with $f_n(z_n) = 0$.

5. (RA)

(i) Specify the range of $1 \leq p < \infty$ for which

$$\varphi(f) = \int_0^1 \frac{f(t)}{\sqrt{t}}\, dt.$$

defines a linear functional $\varphi : L^p([0,1]) \to \mathbb{R}$.

(ii) For those values of p, calculate the norm of the linear functional $\varphi : L^p([0,1]) \to \mathbb{R}$. The norm of a linear functional is defined as

$$\|\varphi\| = \sup_{\substack{f \in L^p([0,1]) \\ f \neq 0}} \frac{|\varphi(f)|}{\|f\|_{L^p}}$$

6. (DG)

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x, y, z) = x^2 + y^2 - 1$.

(i) Prove that $M = f^{-1}(0)$ is a two-dimensional embedded submanifold of $\mathbb{R}^3$.

(ii) For $a, b, c \in \mathbb{R}$, consider the vector field

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$$

For which values of a, b, c is X tangent to M at the point $(1, 0, 1)$?

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Wednesday January 22, 2020 (Day 2)

1. (AT) Let $\Delta = \{z \in \mathbb{C} : |z| \leq 1$ be the closed unit disc in the complex plane, and let X be the space obtained by identifying z with $e^{2\pi i/3}z$ for all z with $|z| = 1$.

1. Find the homology groups $H_k(X, \mathbb{Z})$ of X with coefficients in $\mathbb{Z}$.
2. Find the homology groups $H_k(X, \mathbb{Z}/3)$ of X with coefficients in $\mathbb{Z}/3$.

2. (AG) Let C be a smooth, geometrically irreducible curve of genus 1 defined over $\mathbb{Q}$, and suppose L and M are line bundles on C of degrees 3 and 5, also defined over $\mathbb{Q}$. Show that C has a rational point, that is, $C(\mathbb{Q}) \neq \emptyset$.

3. (A) Let g be an element of the finite group G. Prove that the following are equivalent:

1. g is in the center of G.
2. For every irreducible representation (V, ρ) of G, the image $\rho(g)$ is a multiple of the identity.
3. For every irreducible representation (V, ρ) of G, the character of g has absolute value $\dim(G)$.

4. (RA) Let $g \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ and write $\hat{g}$ for its Fourier transform defined by

$$\hat{g}(k) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ik\cdot x} g(x)\, dx$$

For $m > 0$, define the function $f : \mathbb{R}^3 \to \mathbb{C}$ by

$$f(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ik\cdot x} \frac{\hat{g}(k)}{k^2 + m^2}\, dx$$

Show that f solves the partial differential equation $-\Delta f + m^2 f = g$ in the distributional sense, i.e., show that for every test function $\varphi \in C_0^\infty(\mathbb{R}^3)$,

$$\langle -\Delta\varphi + m^2\varphi, f\rangle = \langle \varphi, g\rangle.$$

Here $\langle \cdot, \cdot \rangle$ denotes the $L^2(\mathbb{R}^3)$-inner product.

5. (DG)

Consider $\mathbb{R}^2$ as a Riemannian manifold equipped with the metric

$$g = (1 + x^2)\mathrm{d}x^2 + \mathrm{d}y^2.$$

(i) Compute the Christoffel symbols of the Levi-Civita connection for g.

(ii) Compute the parallel transport of an arbitrary vector $(a, b) \in \mathbb{R}^2$ along the curve $\gamma(t) = (t, t)$ starting at $t = 0$.

(iii) Is γ a geodesic?

(iv) Are there two parallel vector fields $X(t), Y(t)$ to the curve γ, such that $g(X(t), Y(t)) = 2t$?

6. (CA) Evaluate the contour integral of the following functions around the circle $|z| = 2020$ oriented counterclockwise:

(i) $\frac{1}{\sin z}$;

(ii) $\frac{1}{e^{2z} - e^z}$.

Note that $\frac{2020}{\pi} \sim 642.98597$.

QUALIFYING EXAMINATION

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Thursday January 23, 2020 (Day 3)

1. (A) Let V be an n-dimensional vector space over an arbitrary field K, and let $T_1, \ldots, T_n : V \to V$ be pairwise commuting nilpotent operators on V.

1. Show that the composition $T_1 T_2 \cdots T_n = 0$.
2. Does this conclusion still hold if we drop the hypothesis that the T_i are pairwise commuting?

2. (RA)

(a) Let H be a Hilbert space, $K \subset H$ a closed subspace, and x a point in H. Show that there exists a unique y in K that minimizes the distance $\|x - y\|$ to x.

(b) Give an example to show that the conclusion can fail if H is an inner product space which is not complete.

3. (AG)

1. Let the homogeneous coordinates of $\mathbb{P}^m$ be $x_0, \ldots, x_m$, and the homogeneous coordinates of $\mathbb{P}^n$ be $y_0, \ldots, y_n$, $N = (m+1)(n+1) - 1$, and the homogeneous coordinates of $\mathbb{P}^N$ be $z_{i,j}$ for $i = 0, \ldots, m$, $j = 0, \ldots, n$. Consider the Segre embedding
$$f : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N,$$
given by $z_{i,j} = x_i y_j$. Show that the degree of the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^n$ is $\binom{n+m}{n}$.
2. Let Y be a variety of dimension k in $\mathbb{P}^n$, with Hilbert polynomial h_Y. Define the arithmetic genus of Y to be $g = (-1)^k (p_Y(0) - 1)$. Show that the arithmetic genus of the hypersurface H of degree d in $\mathbb{P}^n$ is $\binom{d-1}{n}$.

4. (CA) Find the Laurent series expansion of the meromorphic function
$$f(z) = \frac{1}{(z-1)(z-2)}$$
around the origin, valid in the annulus $\{z : 1 < |z| < 2\}$.

5. (DG)

Define the set
$$H = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

(i) Equip H with a C^∞ differentiable structure so that it is diffeomorphic to $\mathbb{R}^2$.

(ii) Show that H is a Lie group under matrix multiplication.

(iii) Show that

$$\left\{ \frac{\partial}{\partial y}, \frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \right\}$$

forms a basis of left-invariant vector fields of the associated Lie algebra.

6. (AT) Suppose that X is a space written as a union of two simply connected open subsets U_1 and U_2.

(a) Show that $H_1 X$ is a free abelian group.

(b) Find an example in which $\pi_1 X$ is a non-trivial group. Why does this not contradict the the Seifert-van Kampen theorem?

(c) Find an example in which $\pi_1 X$ is non-abelian.