QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday September 3, 2019 (Day 1)

1. (AT) Suppose that M is a compact connected manifold of dimension 3, and that the abelianization $(\pi_1 M)_{\text{ab}}$ is trivial. Determine the homology and cohomology groups of M (with integer coefficients).

2. (A) Prove that for every finite group G the number of groups homomorphisms $h : \mathbf{Z}^2 \to G$ is $n|G|$ where n is the number of conjugacy classes of G.

3. (AG) Let $X \subset \mathbb{P}^n$ be a projective variety over a field K, with ideal $I(X) \subset K[Z_0, \ldots, Z_n]$ and homogeneous coordinate ring $S(X) = K[Z_0, \ldots, Z_n]/I(X)$. The *Hilbert function* $h_X(m)$ is defined to be the dimension of the mth graded piece of $S(X)$ as a vector space over K.

a. Define the Hilbert polynomial $p_X(m)$ of X.

b. Prove that the degree of p_X is equal to the dimension of X.

c. For each m, give an example of a variety $X \subset \mathbb{P}^n$ such that $h_X(m) \neq p_X(m)$.

4. (CA) Use contour integration to prove that for real numbers a and b with $a > b > 0$,

$$\int_0^\pi \frac{d\theta}{a - b\cos\theta} = \frac{\pi}{\sqrt{a^2 - b^2}}.$$

5. (RA) *Dirichlet's function* D is the function on $[0, 1] \subset \mathbb{R}$ that equals 1 at every rational number and equals 0 at every irrational number. *Thomae's function* T is the function on $[0, 1]$ whose value at irrational numbers is 0 and whose value at any given rational number r is $1/q$, where $r = p/q$ with p and q relatively prime integers, $q > 0$.

1. Prove that D is nowhere continuous.
2. Show that T is continuous at the irrational numbers and discontinuous at the rational numbers.
3. Show that T is nowhere differentiable.

6. (DG)

Consider the Riemannian manifold $(\mathbb{D}, g)$ with $\mathbb{D}$ the unit disk in $\mathbb{R}^2$ and

$$g = \frac{1}{1 - x^2 - y^2}(dx^2 + dy^2)$$

Find the Riemann curvature tensor of $(\mathbb{D}, g)$. Use this to read off the Gaussian curvature of $(\mathbb{D}, g)$.

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Wednesday September 4, 2019 (Day 2)

1. (CA) Fix $a \in \mathbb{C}$ and an integer $n \geq 2$. Show that the equation $az^n + z + 1 = 0$ has a solution with $|z| \leq 2$.

2. (AG) Let $\mathbb{P}^N$ be the space of nonzero homogeneous polynomials of degree d in $n+1$ variables over a field K, modulo multiplication by nonzero scalars, and let $U \subset \mathbb{P}^N$ be the subset of irreducible polynomials F such that the zero locus $V(F) \subset \mathbb{P}^n$ is smooth.

(a) Show that U is a Zariski open subset of $\mathbb{P}^N$.

(b) What is the dimension of the complement $D = \mathbb{P}^N \setminus U$?

(c) Show that D is irreducible.

3. (RA) Let B denote the Banach space of continuous, real valued functions on $[0,1] \subset \mathbb{R}$ with the sup norm.

1. State the Arzela-Ascoli theorem in the context of B.
2. Define what is meant by a *compact operator* between two Banach spaces.
3. Prove that the operator $T : \mathcal{B} \to \mathcal{B}$ defined by

$$(Tf)(x) = \int_0^x f(y)\,dy$$

is compact.

4. (A) Let $\mathbb{F}_q$ be the finite field with q elements. Show that the number of 3×3 nilpotent matrices over $\mathbb{F}_q$ is q^6.

5. (AT) Let $\mathrm{Sym}^n X$ denote the nth symmetric power of a CW complex X, i.e. X^n/S_n, where the symmetric group S_n acts by permuting coordinates. Show that for all $n \geq 2$, the fundamental group of $\mathrm{Sym}^n X$ is abelian.

6. (DG) Let $S^2 \subset \mathbb{R}^3$ be the unit 2-sphere, with its usual orientation. Let X be the vector field generating the flow given by

$$\begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and let ω be the volume form induced by the embedding in $\mathbb{R}^3$ (so the total "volume" is 4π). Find a function $f : S^2 \to \mathbb{R}$ satisfying

$$df = \iota_X \omega$$

where $\iota_X \omega$ is the contraction of ω by X.

QUALIFYING EXAMINATION

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Thursday September 5, 2019 (Day 3)

1. (RA) Let $f : [0,1] \to \mathbb{R}$ be in the Sobolev space $H^1([0,1])$; that is, functions f such that both f and its derivative are L^2-integrable. Prove that

$$\lim_{n\to\infty} \left(n \int_0^1 f(x) e^{-2\pi i n x} dx \right) = 0.$$

2. (CA) Given that the sum

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$$

converges uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{Z}$ to a meromorphic function on the entire complex plane, prove the identity

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}.$$

3. (AG) Let $C \subset \mathbb{P}^3$ be a smooth curve of degree 5 and genus 2.

(a) By considering the restriction map $\rho : H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_C(2))$, show that C must lie on a quadric surface Q.

(b) Show that the quadric surface Q is unique.

(c) Similarly, show that C must lie on at least one cubic surface S not containing Q.

(d) Finally, deduce that there exists a line $L \subset \mathbb{P}^3$ such that the union $C \cup L$ is a complete intersection of a quadric and a cubic.

4. (A) Show that if p, q are distinct primes then the polynomial $(x^p - 1)/(x-1)$ is irreducible $\bmod q$ if an only if q is a primitive residue of p (i.e. if every integer that is not a multiple of p is congruent to q^e mod p for some integer e).
ii) Prove that $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ factors mod 23 as the product of two irreducible cubics.

5. (DG) Suppose that G is a Lie group.

(a) Consider the map $\iota : G \to G$ defined by $\iota(g) = g^{-1}$. Show that the derivative of ι at the identity element is multiplication by -1.

(b) For $g \in G$ define maps $L_g, R_g : G \to G$ by

$$L_g(x) = gx$$
$$R_g(x) = xg.$$

Show that if ω is a k-form which is *bi-invariant* in the sense that $L_g^*\omega = R_g^*\omega$ then $\iota^*\omega = (-1)^k\omega$.

(c) Show that bi-invariant forms on G are closed.

6. (AT) Suppose that m is odd. Show that if n is odd there is a fixed point free action of $\mathbb{Z}/m$ on S^n. What happens if n is even?