QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Tuesday January 19, 2021 (Day 1)

1. (AG) Let $Y \subset \mathbb{P}^2$ be an irreducible curve of degree $d > 1$ having a point of multiplicity $d-1$. Show that Y is a rational curve.

2. (CA) Use the method of contour integrals to find the integral

$$\int_0^\infty \frac{\log x}{x^2+4} \mathrm{d}x.$$

3. (RA) Suppose μ and ν are two positive measures on $\mathbb{R}^n$ with $n \geq 1$. For a positive function f, consider two quantities

$$A := \int \nu(dy) \left[\int f(x,y)^p \mu(dx) \right]^{1/p}$$

$$B := \left[\int \mu(dx) \left(\int f(x,y) \nu(dy) \right)^p \right]^{1/p}$$

For $1 \leq p < \infty$. Assume all quantities are integrable and finite. Do we know that $A \geq B$ or $A \leq B$ for all functions f ? Prove your assertion or give a counterexample.

4. (A) Let $\mathfrak{p}$ be a prime ideal in a commutative ring A. Show that $\mathfrak{p}[x]$ is a prime idea in $A[x]$. If m is a maximal idea in A, is m$[x]$ a maximal ideal in $A[x]$?

5. (AT) What are the homology groups of the 5-manifold $\mathbb{RP}^2 \times \mathbb{RP}^3$,

(a) with coefficients in $\mathbb{Z}$?

(b) with coefficients in $\mathbb{Z}/2$?

(c) with coefficients in $\mathbb{Z}/3$?

6. (DG) Let $a > b > 0$ be positive numbers. Let C be the circle of radius b centered at $(a, 0)$ in the (x, z)-plane. Let T be the torus obtained by revolving the circle C about the z-axis in the (x, y, z)-space. The torus T can be identified as the product of two circles whose points are described by the two angle-variables φ, θ (or arc-length-variables) of the two circles. Compute, in terms of a, b, φ, θ, the Gaussian curvature of T and determine the subsets T^+, T^-, T^0 of T where the Gaussian curvature of T is respectively positive, negative, and zero.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday January 20, 2021 (Day 2)

1. (CA) Let q be any positive integer. Let Ω be a connected open subset of $\mathbb{C}$. Suppose $f_n(z)$ is a sequence of holomorphic functions on Ω such that for any positive number n and for any $c \in \mathbb{C}$, the set $f_n^{-1}(c)$ has no more than q distinct elements. Suppose the sequence $f_n(z)$ converges to a function $f(z)$ uniformly on compact subsets of Ω. Prove that either $f(z)$ is constant or $f(z)$ satisfies the property that for any $c \in \mathbb{C}$ the set $f^{-1}(c)$ has no more than q distinct elements.

2. (AG) Let X be a degree 3 hypersurface in $\mathbb{P}^3$. Show that X contains a line. (You may use the fact that the Fermat cubic surface $V(x^3 + y^3 + z^3 + w^3)$ contains a positive finite number of lines.)

3. (RA) Suppose X_j are independent identically distributed Poisson distributions with intensity λ, i.e.,

$$P(X_j = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k \in \mathbb{N} \cup \{0\}$$

Show that for any $y \geq \lambda$,

$$P(\frac{X_1 + \cdots + X_n}{n} \geq y) \leq e^{-n[y\log(y/\lambda) - y + \lambda]}$$

and for any $y \leq \lambda$,

$$P(\frac{X_1 + \cdots + X_n}{n} \leq y) \leq e^{-n[y\log(y/\lambda) - y + \lambda]}$$

Hint: Consider the moment generating function.

4. (A) Determine the Galois group of the polynomial $f(x) = x^3 - 2$. Let K be the splitting field of f over $\mathbb{Q}$. Describe the set of all intermediate fields L, $\mathbb{Q} < L < K$ and the Galois correspondence.

5. (AT) Let $X \subset \mathbb{R}^3$ be the union of the unit sphere $S^2 = \{(x, y, z) \mid x^2+y^2+z^2 = 1\}$ and the line segment $I = \{(x, 0, 0) \mid -1 \leq x \leq 1\}$.

(a) What are the homology groups of X?

(b) What are the homotopy groups $\pi_1(X)$ and $\pi_2(X)$?

6. (DG) Let X be a Riemannian manifold and σ be an isometry of X. Let Y be the set of fixed points of σ in the sense that Y is the set of all points y of X such that $\sigma(y) = y$. Prove that Y is regular and is totally geodesic (in the sense that any geodesic in Y with respect to the metric induced from X is also a geodesic in X).

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Thursday January 21, 2021 (Day 3)

1. (AG) Let $X \subset \mathbb{P}^3$ be a curve that is not contained in any proper linear subspace of $\mathbb{P}^3$. Show that if $\deg X$ is a prime number, then the homogeneous ideal $I(X)$ cannot be generated by two elements.

2. (RA) Let $\mathcal{E}$ be the space of even $\mathcal{C}^\infty$ functions $\mathbf{R}/\mathbf{Z} \to \mathbf{R}$. Prove that for every $f \in \mathcal{E}$ there exists a unique $g \in \mathcal{E}$ such that

$$f(x) = \int_0^1 \int_0^1 g(y)\, g(z)\, g(x-y-z)\, dy\, dz$$

for all $x \in \mathbf{R}/\mathbf{Z}$. [Hint: write the integral formula for f as a convolution.]

3. (CA) Suppose $f(z)$ is analytic and bounded for $|z| < 1$. Let $\zeta = x + iy$. If $|z| < 1$, prove that

$$f(z) = \frac{1}{\pi} \int \int_{|\zeta|<1} \frac{f(\zeta)}{(1 - z\bar{\zeta})^2} dxdy$$

4. (AT) Suppose f is an orientation-preserving self-homeomorphism of $\mathbb{CP}^n$ such that the graph $\Gamma_f \subset \mathbb{CP}^n \times \mathbb{CP}^n$ intersects the diagonal transversely . Compute all possibilities for the number of its fixed points.

5. (DG) Let G be an open subset of $\mathbb{R}^n$. For $1 \leq p \leq n-1$ denote by $\wedge^p T_G$ the exterior product of p copies of the tangent bundle T_G of G. For $1 \leq j \leq m$ let $\boldsymbol{\eta}_j$ be a C^∞ section of $\wedge^p T_G$ over G. For a C^∞ vector field ξ on an open subset of G, denote by $\mathcal{L}_\xi \boldsymbol{\eta}_j$ the Lie derivative of $\boldsymbol{\eta}_j$ with respect to ξ, which means that if $\varphi_{\xi,t}$ is the local diffeomorphism defined by ξ so that the tangent vector $\frac{d}{dt}\varphi_{\xi,t}$ equals the value of ξ at $\varphi_{\xi,t}$, then

$$\mathcal{L}_\xi \boldsymbol{\eta}_j = \lim_{t \to 0} \frac{1}{t} \left((\varphi_{\xi,t})_* \boldsymbol{\eta}_j - \boldsymbol{\eta}_j \right),$$

where $(\varphi_{\xi,t})_* \boldsymbol{\eta}_j$ is the pushforward of $\boldsymbol{\eta}_j$ under $\varphi_{\xi,t}$. Let $\Phi_{\boldsymbol{\eta}_j} : T_G \to \wedge^{p+1} T_G$ be defined by exterior product with $\boldsymbol{\eta}_j$. Assume that the intersection $\cap_{j=1}^m \mathrm{Ker}\, \Phi_{\boldsymbol{\eta}_j}$

of the kernel $\operatorname{Ker}\Phi_{\boldsymbol{\eta}_j}$ of $\Phi_{\boldsymbol{\eta}_j}$ for $1 \leq j \leq m$ is a subbundle of T_G of rank q over G. Suppose for any C^∞ tangent vector field ζ in any open subset W there exist C^∞ functions $g_{j,k,\zeta}$ on W for $1 \leq j, k \leq m$ such that

$$\mathcal{L}_\zeta \boldsymbol{\eta}_j = \sum_{k=1}^{m} g_{j,k,\zeta} \boldsymbol{\eta}_k$$

on W. Prove that for every point x of G there exist some open neighborhood U_x of x in G and C^∞ functions $f_1, \cdots, f_{n-q}$ on U_x such that the fiber of $\cap_{j=1}^{m} \operatorname{Ker}\Phi_{\boldsymbol{\eta}_j}$ at y is equal to $\cap_{k=1}^{n-q} \operatorname{Ker} df_k$ at y for $y \in U_x$.

6. (A) Let π be a finite dimensional representation of a finite group G with the character χ_π. Prove that π is irreducible if and only if

$$\frac{1}{|G|} \sum_{g \in G} |\chi_\pi(g)|^2 = 1.$$