1. (AG) Let $Y \subset \mathbb{P}^2$ be an irreducible curve of degree $d > 1$ having a point of multiplicity $d - 1$. Show that $Y$ is a rational curve.

2. (CA) Use the method of contour integrals to find the integral
$$\int_{0}^{\infty} \frac{\log x}{x^2 + 4} \, dx.$$

3. (RA) Suppose $\mu$ and $\nu$ are two positive measures on $\mathbb{R}^n$ with $n \geq 1$. For a positive function $f$, consider two quantities
$$A := \int \nu(dy) \left[ \int f(x, y)^p \mu(dx) \right]^{1/p}$$
$$B := \left[ \int \mu(dx) \left( \int f(x, y)^p \nu(dy) \right)^p \right]^{1/p}$$

For $1 \leq p < \infty$. Assume all quantities are integrable and finite. Do we know that $A \geq B$ or $A \leq B$ for all functions $f$? Prove your assertion or give a counterexample.

4. (A) Let $p$ be a prime ideal in a commutative ring $A$. Show that $p[x]$ is a prime ideal in $A[x]$. If $m$ is a maximal ideal in $A$, is $m[x]$ a maximal ideal in $A[x]$?

5. (AT) What are the homology groups of the 5-manifold $\mathbb{RP}^2 \times \mathbb{RP}^3$,
   (a) with coefficients in $\mathbb{Z}$?
   (b) with coefficients in $\mathbb{Z}/2$?
   (c) with coefficients in $\mathbb{Z}/3$?
6. (DG) Let $a > b > 0$ be positive numbers. Let $C$ be the circle of radius $b$ centered at $(a,0)$ in the $(x,z)$-plane. Let $T$ be the torus obtained by revolving the circle $C$ about the $z$-axis in the $(x,y,z)$-space. The torus $T$ can be identified as the product of two circles whose points are described by the two angle-variables $\phi, \theta$ (or arc-length-variables) of the two circles. Compute, in terms of $a, b, \phi, \theta$, the Gaussian curvature of $T$ and determine the subsets $T^+, T^-, T^0$ of $T$ where the Gaussian curvature of $T$ is respectively positive, negative, and zero.
1. (CA) Let $q$ be any positive integer. Let $\Omega$ be a connected open subset of $\mathbb{C}$. Suppose $f_n(z)$ is a sequence of holomorphic functions on $\Omega$ such that for any positive number $n$ and for any $c \in \mathbb{C}$, the set $f_n^{-1}(c)$ has no more than $q$ distinct elements. Suppose the sequence $f_n(z)$ converges to a function $f(z)$ uniformly on compact subsets of $\Omega$. Prove that either $f(z)$ is constant or $f(z)$ satisfies the property that for any $c \in \mathbb{C}$ the set $f^{-1}(c)$ has no more than $q$ distinct elements.

2. (AG) Let $X$ be a degree 3 hypersurface in $\mathbb{P}^3$. Show that $X$ contains a line. (You may use the fact that the Fermat cubic surface $V(x^3 + y^3 + z^3 + w^3)$ contains a positive finite number of lines.)

3. (RA) Suppose $X_j$ are independent identically distributed Poisson distributions with intensity $\lambda$, i.e.,

$$P(X_j = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \in \mathbb{N} \cup \{0\}$$

Show that for any $y \geq \lambda$,

$$P\left(\frac{X_1 + \cdots + X_n}{n} \geq y\right) \leq e^{-n[y \log(y/\lambda) - y + \lambda]}$$

and for any $y \leq \lambda$,

$$P\left(\frac{X_1 + \cdots + X_n}{n} \leq y\right) \leq e^{-n[y \log(y/\lambda) - y + \lambda]}$$

Hint: Consider the moment generating function.

4. (A) Determine the Galois group of the polynomial $f(x) = x^3 - 2$. Let $K$ be the splitting field of $f$ over $\mathbb{Q}$. Describe the set of all intermediate fields $L$, $\mathbb{Q} < L < K$ and the Galois correspondence.

5. (AT) Let $X \subset \mathbb{R}^3$ be the union of the unit sphere $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ and the line segment $I = \{(x, 0, 0) \mid -1 \leq x \leq 1\}$. 
(a) What are the homology groups of $X$?
(b) What are the homotopy groups $\pi_1(X)$ and $\pi_2(X)$?

6. (DG) Let $X$ be a Riemannian manifold and $\sigma$ be an isometry of $X$. Let $Y$ be the set of fixed points of $\sigma$ in the sense that $Y$ is the set of all points $y$ of $X$ such that $\sigma(y) = y$. Prove that $Y$ is regular and is totally geodesic (in the sense that any geodesic in $Y$ with respect to the metric induced from $X$ is also a geodesic in $X$).
1. (AG) Let $X \subset \mathbb{P}^3$ be a curve that is not contained in any proper linear subspace of $\mathbb{P}^3$. Show that if $\deg X$ is a prime number, then the homogeneous ideal $I(X)$ cannot be generated by two elements.

2. (RA) Let $E$ be the space of even $C^\infty$ functions $\mathbb{R}/\mathbb{Z} \to \mathbb{R}$. Prove that for every $f \in E$ there exists a unique $g \in E$ such that

$$f(x) = \int_0^1 \int_0^1 g(y) g(z) g(x - y - z) \, dy \, dz$$

for all $x \in \mathbb{R}/\mathbb{Z}$. [Hint: write the integral formula for $f$ as a convolution.]

3. (CA) Suppose $f(z)$ is analytic and bounded for $|z| < 1$. Let $\zeta = x + iy$. If $|z| < 1$, prove that

$$f(z) = \frac{1}{\pi} \int_{|\zeta| < 1} \frac{f(\zeta)}{(1 - z \bar{\zeta})^2} \, dxdy$$

4. (AT) Suppose $f$ is an orientation-preserving self-homeomorphism of $\mathbb{C}P^n$ such that the graph $\Gamma_f \subset \mathbb{C}P^n \times \mathbb{C}P^n$ intersects the diagonal transversely. Compute all possibilities for the number of its fixed points.

5. (DG) Let $G$ be an open subset of $\mathbb{R}^n$. For $1 \leq p \leq n - 1$ denote by $\wedge^p T_G$ the exterior product of $p$ copies of the tangent bundle $T_G$ of $G$. For $1 \leq j \leq m$ let $\eta_j$ be a $C^\infty$ section of $\wedge^p T_G$ over $G$. For a $C^\infty$ vector field $\xi$ on an open subset of $G$, denote by $\mathcal{L}_\xi \eta_j$ the Lie derivative of $\eta_j$ with respect to $\xi$, which means that if $\varphi_{\xi,t}$ is the local diffeomorphism defined by $\xi$ so that the tangent vector $\frac{d}{dt} \varphi_{\xi,t}$ equals the value of $\xi$ at $\varphi_{\xi,t}$, then

$$\mathcal{L}_\xi \eta_j = \lim_{t \to 0} \frac{1}{t} ((\varphi_{\xi,t})_* \eta_j - \eta_j),$$

where $(\varphi_{\xi,t})_* \eta_j$ is the pushforward of $\eta_j$ under $\varphi_{\xi,t}$. Let $\Phi_{\eta_j} : T_G \to \wedge^{p+1} T_G$ be defined by exterior product with $\eta_j$. Assume that the intersection $\cap_{j=1}^{m} \text{Ker} \Phi_{\eta_j}$
of the kernel \( \text{Ker} \Phi_{\eta_j} \) of \( \Phi_{\eta_j} \) for \( 1 \leq j \leq m \) is a subbundle of \( T_G \) of rank \( q \) over \( G \). Suppose for any \( C^\infty \) tangent vector field \( \zeta \) in any open subset \( W \) there exist \( C^\infty \) functions \( g_{j,k,\zeta} \) on \( W \) for \( 1 \leq j, k \leq m \) such that

\[
\mathcal{L}_\zeta \eta_j = \sum_{k=1}^{m} g_{j,k,\zeta} \eta_k
\]
on \( W \). Prove that for every point \( x \) of \( G \) there exist some open neighborhood \( U_x \) of \( x \) in \( G \) and \( C^\infty \) functions \( f_1, \ldots, f_{n-q} \) on \( U_x \) such that the fiber of \( \cap_{j=1}^{m} \text{Ker} \Phi_{\eta_j} \) at \( y \) is equal to \( \cap_{k=1}^{n-q} \text{Ker} df_k \) at \( y \) for \( y \in U_x \).

6. (A) Let \( \pi \) be a finite dimensional representation of a finite group \( G \) with the character \( \chi_\pi \). Prove that \( \pi \) is irreducible if and only if

\[
\frac{1}{|G|} \sum_{g \in G} |\chi_\pi(g)|^2 = 1.
\]