QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday January 19, 2021 (Day 1)

1. (AG) Let $Y \subset \mathbb{P}^2$ be an irreducible curve of degree $d > 1$ having a point of multiplicity $d-1$. Show that Y is a rational curve.

Solution: Recall that if $P \in Y \subset \mathbb{P}^2$ is a point of multiplicity m, then a generic line through P meets Y at p with multiplicity m. Then there is a dense open subset U of $\mathbb{P}^1$ such that if $L \in U$ then the intersection multiplicity of Y at L along P is m.

Without loss of generality, let $P = (0,0,1) \in Y$ be the point with multiplicty $d-1$. Then consider two maps:

$$\phi : Y \setminus P \to \mathbb{P}^1$$

the projective map, and by Bezout's theorem, there is a map

$$\psi : \mathbb{P}^1 \supset U \to Y \setminus P$$

which maps a point L of U to the unique point that L meets Y other than P.

Thus

$$\begin{gathered} Y \setminus P \xrightarrow{\phi} \mathbb{P}^1 \dashrightarrow Y \setminus P \\ (a,b,c) \mapsto (a,b,0) \mapsto (a,b,c) \end{gathered}$$

are rational. Thus this gives a birational equivalence of $Y \setminus P$ and $\mathbb{P}^1$.

2. (CA) Use the method of contour integrals to find the integral

$$\int_0^\infty \frac{\log x}{x^2+4} \mathrm{d}x.$$

Solution: Consider the contour from $-R$ to $-r$, a semicircle, r to R and a large semicircle. Then the integration from $-R$ to $-r$ and r to R becomes

$$2\int_r^R \frac{\log x}{x^2+4}\mathrm{d}x + i\int_{-R}^{-r} \frac{\pi}{x^2+4}\mathrm{d}x$$

The last term is imaginary. Hence

$$\int_0^\infty \frac{\log x}{x^2+4}\mathrm{d}x = \mathrm{d}\Big[\pi i \lim_{z\to 2i} \frac{\log z}{z+2i}\Big] = \frac{\pi \log 2}{4}$$

3. (RA) Suppose μ and ν are two positive measures on $\mathbb{R}^n$ with $n \geq 1$. For a positive function f, consider two quantities

$$A := \int \nu(dy) \left[\int f(x,y)^p \mu(dx) \right]^{1/p}$$

$$B := \left[\int \mu(dx) \left(\int f(x,y)\nu(dy) \right)^p \right]^{1/p}$$

For $1 \leq p < \infty$. Assume all quantities are integrable and finite. Do we know that $A \geq B$ or $A \leq B$ for all functions f ? Prove your assertion or give a counterexample.

Solution: By duality,

$$\left[\int \mu(dx) \left(\int f(x,y)\nu(dy) \right)^p \right]^{1/p} = \sup_{g:\|g\|_{L_q(\mu)}\leq 1} \int \mu(dx) g(x) \int f(x,y)\nu(dy)$$

$$= \sup_{g:\|g\|_{L_q(\mu)}\leq 1} \int \nu(dy) \int \mu(dx) g(x) f(x,y) \leq \int \nu(dy) \left[\int f(x,y)^p \mu(dx) \right]^{1/p}$$

Since ϵ was arbitrary we are done.

4. (A) Let $\mathfrak{p}$ be a prime ideal in a commutative ring A. Show that $\mathfrak{p}[x]$ is a prime idea in $A[x]$. If m is a maximal idea in A, is m$[x]$ a maximal ideal in $A[x]$?

Solution: Consider the projection

$$A[x] \to (A/\mathfrak{a})[x].$$

The kernel of the projection is $\mathfrak{a}[x]$ and hence $A[x]/\mathfrak{a}[x] \cong (A/\mathfrak{a})[x]$. Now consider $\mathfrak{p}$ a prime idea of A. Then $A/\mathfrak{p}$ is an integral domain, and so do $(A/\mathfrak{p})[x]$ by Hilbert Basis Theorem. That is $A[x]/\mathfrak{p}[x]$ is an integral domain as well. This implies $\mathfrak{p}[x]$ is prime in $A[x]$.

Note that if A is a field, $A[x]$ may not be a field. Hence if $\mathfrak{m}$ is maximal in A does not implies m$[x]$ a maximal ideal in $A[x]$.

5. (AT) What are the homology groups of the 5-manifold $\mathbb{RP}^2 \times \mathbb{RP}^3$,

(a) with coefficients in $\mathbb{Z}$?
(b) with coefficients in $\mathbb{Z}/2$?
(c) with coefficients in $\mathbb{Z}/3$?

Solution: $\mathbb{RP}^2$ and $\mathbb{RP}^3$ have cell complexes with sequences

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0 \quad \text{and} \quad 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0$$

where the maps are alternately 0 and multiplication by 2; from this the homology groups of $\mathbb{RP}^2$ and $\mathbb{RP}^3$ can be calculated as $\mathbb{Z}, \mathbb{Z}/2, 0$ and $\mathbb{Z}, \mathbb{Z}/2, 0, \mathbb{Z}$ respectively. The rest is just Künneth; the answers are

(a): $\mathbb{Z}$, $(\mathbb{Z}/2)^2$, $(\mathbb{Z}/2)^2$, $\mathbb{Z}$, $\mathbb{Z}/2$, 0;

(b): $\mathbb{Z}/2$, $(\mathbb{Z}/2)^2$, $(\mathbb{Z}/2)^3$, $(\mathbb{Z}/2)^3$, $(\mathbb{Z}/2)^2$, $\mathbb{Z}/2$,

(c): $\mathbb{Z}/3$, 0, 0, $\mathbb{Z}/3$, 0, 0

6. (DG) Let $a > b > 0$ be positive numbers. Let C be the circle of radius b centered at $(a, 0)$ in the (x, z)-plane. Let T be the torus obtained by revolving the circle C about the z-axis in the (x, y, z)-space. The torus T can be identified as the product of two circles whose points are described by the two angle-variables φ, θ (or arc-length-variables) of the two circles. Compute, in terms of a, b, φ, θ, the Gaussian curvature of T and determine the subsets T^+, T^-, T^0 of T where the Gaussian curvature of T is respectively positive, negative, and zero.

Solution: Parametrize T by two circles with angle-variable φ, θ as follows. The circle C with angle-variable φ can be described as

$$(x, z) = b(\cos\varphi, \sin\varphi) + (a, 0) = (a + b\cos\varphi, b\sin\varphi).$$

The result of rotating a point P on C by an angle θ about the z-axis is the same as replacing the x-coordinate x_P of P by $(x, y) = (x_P\cos\theta,\ x_P\sin\theta)$. It follows that the parametrization of T is given by

$$\vec{r}(\varphi, \theta) = ((a + b\cos\varphi)\cos\theta,\ (a + b\cos\varphi)\sin\theta,\ b\sin\varphi)\,.$$

The first and second partial derivatives of $\vec{r}(\theta, \varphi)$ and the unit normal vector $\vec{N}$ of T are given by

$$\begin{aligned}
\vec{r}_\varphi &= (-b\sin\varphi\cos\theta,\ -b\sin\varphi\sin\theta,\ b\cos\varphi)\,,\\
\vec{r}_\theta &= (-(a + b\cos\varphi)\sin\theta,\ (a + b\cos\varphi)\cos\theta,\ 0)\,,\\
\vec{r}_\varphi \times \vec{r}_\theta &= (-(a + b\cos\varphi)\cos\theta\, b\cos\varphi, -(a + b\cos\varphi)\sin\theta\, b\cos\varphi,\ -(a + b\cos\varphi) b\sin\varphi)\,,\\
\vec{n} &= \frac{\vec{r}_\varphi \times \vec{r}_\theta}{\|\vec{r}_\varphi \times \vec{r}_\theta\|} = (-\cos\theta\cos\varphi,\ -\sin\theta\cos\varphi,\ -\sin\varphi)\,,\\
\vec{r}_{\varphi\varphi} &= (-b\cos\varphi\cos\theta,\ -b\cos\varphi\sin\theta,\ -b\sin\varphi)\,,\\
\vec{r}_{\varphi\theta} &= (b\sin\varphi\sin\theta,\ -b\sin\varphi\cos\theta,\ 0)\,,\\
\vec{r}_{\theta\theta} &= (-(a + b\cos\varphi)\cos\theta,\ -(a + b\cos\varphi)\sin\theta,\ 0)\,.
\end{aligned}$$

The first fundamental form $Ed\varphi^2 + 2Fd\varphi d\theta + Gd\theta^2$ and the second fundamental form $Ld\varphi^2 + 2Md\varphi d\theta + Nd\theta^2$ of the torus T are given by

$$\begin{aligned}
E &= \vec{r}_\varphi \cdot \vec{r}_\varphi = b^2, \\
F &= \vec{r}_\varphi \cdot \vec{r}_\theta = 0, \\
G &= \vec{r}_\theta \cdot \vec{r}_\theta = (a + b\cos\varphi)^2, \\
L &= \vec{r}_{\varphi\varphi} \cdot \vec{n} = b, \\
M &= \vec{r}_{\varphi\theta} \cdot \vec{n} = 0, \\
N &= \vec{r}_{\theta\theta} \cdot \vec{n} = (a + b\cos\varphi)\cos\varphi.
\end{aligned}$$

The Gaussian curvature is given by

$$\frac{LN - M^2}{EF - G^2} = \frac{b(a + b\cos\varphi)\cos\varphi}{b^2(a + b\cos\varphi)^2} = \frac{\cos\varphi}{b(a + b\cos\varphi)},$$

which

(i) is positive for $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$,

(ii) is zero for $\varphi = \frac{\pi}{2}$ or $\frac{-\pi}{2}$, and

(iii) is negative for $\frac{\pi}{2} < \varphi < \pi$ or $-\pi < \varphi < -\frac{\pi}{2}$.

In other words,

(i) the Gaussian curvature of the torus T is positive on the part T^+ obtained by rotating the right-half $C \cap \{x > a\}$ of the circle C about the z-axis.

(ii) The Gaussian curvature of the torus T is negative on the part T^- obtained by rotating the left-half $C \cap \{x < a\}$ of the circle C about the z-axis.

(iii) The Gaussian curvature of the torus T is zero on the part T^0 obtained by rotating the the highest point $C \cap \{z = b\}$ and the lowest point $C \cap \{z = -b\}$ of the circle C about the z-axis.

QUALIFYING EXAMINATION

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Department of Mathematics

Wednesday January 20, 2021 (Day 2)

1. (CA) Let q be any positive integer. Let Ω be a connected open subset of $\mathbb{C}$. Suppose $f_n(z)$ is a sequence of holomorphic functions on Ω such that for any positive number n and for any $c \in \mathbb{C}$, the set $f_n^{-1}(c)$ has no more than q distinct elements. Suppose the sequence $f_n(z)$ converges to a function $f(z)$ uniformly on compact subsets of Ω. Prove that either $f(z)$ is constant or $f(z)$ satisfies the property that for any $c \in \mathbb{C}$ the set $f^{-1}(c)$ has no more than q distinct elements.

Solution. Assume that f is nonconstant and achieves the value c at $q+1$ distinct points $z_1, \cdots, z_{q+1}$ of Ω and we are going to derive a contradiction. Choose some $\varepsilon > 0$ such that

(i) the $q+1$ closed disks $|z - z_j| \leq \varepsilon$ for $1 \leq j \leq q+1$ are inside Ω and are disjoint,

(ii) for $1 \leq j \leq q+1$ the function $f(z) - f(z_j)$ has exactly one zero on the closed disk $|z - z_j| \leq \varepsilon$ which is not on the boundary $|z - z_j| = \varepsilon$.

This is possible, because f is nonconstant and is holomorphic on the connected open subset Ω of $\mathbb{C}$ (as the uniform limit on compact subsets of holomorphic functions on Ω). Let $\eta > 0$ be the minimum of $|f(z) - c|$ on $|z - z_j| = \varepsilon$ for $1 \leq j \leq q+1$. By uniform convergence of $f_n \to f$ on compact sets of Ω there is some n (as a matter of fact, any sufficiently large n) for which $|f_n(z) - f(z)| < \eta$ on $|z - z_j| = \varepsilon$ for $1 \leq j \leq q+1$. Since $|f_n(z) - f(z)| < |f(z) - c|$ on $|z - z_j| = \varepsilon$ for $1 \leq j \leq q+1$, by applying Rouché's theorem to $f_n(z) - c = (f(z) - c) + (f_n(z) - f(z))$, we conclude that $f_n(z) - c$ has the same number of zeroes as the function $f(z) - c$ on each of the $q+1$ disjoint disks $|z - z_j| < \varepsilon$. This contradicts the assumption that the set $f_n^{-1}(c)$ has no more than q distinct elements.

2. (AG) Let X be a degree 3 hypersurface in $\mathbb{P}^3$. Show that X contains a line. (You may use the fact that the Fermat cubic surface $V(x^3 + y^3 + z^3 + w^3)$ contains a positive finite number of lines.)

Solution: Let $P = |\mathcal{O}_{\mathbb{P}^3}(3)| \cong \mathbb{P}^{19}$ be the projective space of cubics in $\mathbb{P}^3$ (Note $\binom{3+3}{3} - 1 = 19$.) and let $G = \mathrm{Gr}(\mathbb{P}^1, \mathbb{P}^3)$ which is of dimension $(1+1) \times ((3+1) - (1+1)) = 4$.

Consider the incidence variety $Z = \{(\ell, S) \in G \times P \mid \ell \subseteq S\}$. Then we have

$$\begin{array}{ccc} & Z & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ G & & P \end{array}$$

Let the coordinate on $\mathbb{P}^3$ be (x, y, z, w). A cubic surface $S \subset \mathbb{P}^3$ contains the line $\ell = \{z = w = 0\}$ if and only if the defining equation of S having the terms x^3, x^2y, xy^2, y^3 vanish. This shows that $p_1^{-1}(\ell)$ is irreducible of dimension $\dim P - 4$. Hence

$$\dim Z = \dim P.$$

If there is a degree 3 hypersurface X not containing a line, then $p_2(Z) \subseteq P$ is of codimensional ≥ 1 in P. Hence for any $S \in P$, the fibre $p_2^{-1}(S)$ is either empty or of positive dimension. However this contradicts to the del Pezzo surfaces having only finitely many line.

3. (RA) Suppose X_j are independent identically distributed Poisson distributions with intensity λ, i.e.,

$$P(X_j = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k \in \mathbb{N} \cup \{0\}$$

Show that for any $y \geq \lambda$,

$$P(\frac{X_1 + \cdots + X_n}{n} \geq y) \leq e^{-n[y\log(y/\lambda) - y + \lambda]}$$

and for any $y \leq \lambda$,

$$P(\frac{X_1 + \cdots + X_n}{n} \leq y) \leq e^{-n[y\log(y/\lambda) - y + \lambda]}$$

Hint: Consider the moment generating function.

Solution: By the Markov inequality, for any $\lambda \leq y$,

$$P(\frac{X_1 + \cdots + X_n}{n} \geq y) \leq \inf_{t\geq 0} e^{-tny}\mathbb{E}e^{t(X_1+\cdots+X_n)} = \inf_{t\geq 0} e^{-n[ty - \lambda(e^t - 1)]}$$
$$= e^{-n[y\log(y/\lambda) - y + \lambda]}$$

Similarly, for any $y \leq \lambda$,

$$P(\frac{X_1 + \cdots + X_n}{n} \leq y) \leq \inf_{t\geq 0} e^{tny}\mathbb{E}e^{-t(X_1+\cdots+X_n)} = \inf_{t\geq 0} e^{n[ty + \lambda(e^{-t} - 1)]}$$
$$= e^{-n[y\log(y/\lambda) - y + \lambda]}$$

4. (A) Determine the Galois group of the polynomial $f(x) = x^3 - 2$. Let K be the splitting field of f over $\mathbb{Q}$. Describe the set of all intermediate fields L, $\mathbb{Q} < L < K$ and the Galois correspondence.

Solution: Let K be the splitting field of f over $\mathbb{Q}$. Then $K = \mathbb{Q}(\sqrt[3]{2}, \zeta)$ is generated by $\sqrt[3]{2}$, $\sqrt[3]{2}\zeta$, and $\sqrt[3]{2}\zeta^2$, where ζ is the primitive cubic root of unity.

The discriminant of $f(x) = x^3 - 2$ is $D = -27(2)^2$ and square root of D is not rational. Hence the Galois group $\mathrm{Gal}(K/\mathbb{Q}) = S_3$ which is the permutation group of order 6. As a permutation group, S_3 can be expressed as $\{\mathrm{id}, (12), (13), (23), (123), (132)\}$. The four proper subgroup of S_3 are $\{\mathrm{id}, (12)\}$, $\{\mathrm{id}, (13)\}$, $\{\mathrm{id}, (23)\}$, and $\{\mathrm{id}, (123), (132)\}$.

The intermediate fields L are $\mathbb{Q}$, $\mathbb{Q}(\sqrt[3]{2})$, $\mathbb{Q}(\sqrt[3]{2}\zeta)$, $\mathbb{Q}(\sqrt[3]{2}\zeta^2)$ $\mathbb{Q}(\zeta)$, and K. They corresponds to the subgroups $\{\mathrm{id}\}$, $\{\mathrm{id}, (12)\}$, $\{\mathrm{id}, (13)\}$, $\{\mathrm{id}, (23)\}$, $\{\mathrm{id}, (123), (132)\}$, and S_3.

5. (AT) Let $X \subset \mathbb{R}^3$ be the union of the unit sphere $S^2 = \{(x, y, z) \mid x^2+y^2+z^2 = 1\}$ and the line segment $I = \{(x, 0, 0) \mid -1 \leq x \leq 1\}$.

(a) What are the homology groups of X?

(b) What are the homotopy groups $\pi_1(X)$ and $\pi_2(X)$?

Solution: Under the attaching map $I \hookrightarrow X$, the boundary $\varphi(I)$ is homologous to 0, so attaching I simply adds one new, non-torsion generator to H^1; thus

$$H_0(X) = H^1(X) = H^2(X) = \mathbb{Z},$$

and all other homology groups are 0. Similarly, $\pi_1(X) = \mathbb{Z}$. For $\pi_2(X)$, note that the universal cover of X is a string of spheres attached in a sequence by line segments; $\pi_2(X)$ is thus the free abelian group on countably many generators.

Solution: The space X can be realized as a CW complex with one 0-cell, one 1-cell and one 2-cell, with the 1-skeleton the unit circle S^1 in $\mathbb{C}$ and the 2-cell attached via the map $S^1 \to S^1$ given by $z \mapsto z^5$. The cellular complex is thus

$$\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}$$

with the first map multiplication by 5 and the second map 0; the homology groups with coefficients in $\mathbb{Z}$ are thus

$$H^0(X, \mathbb{Z}) = \mathbb{Z}; \quad H^1(X, \mathbb{Z}) = \mathbb{Z}/5, \quad \text{and} \quad H^2(X, \mathbb{Z}) = 0.$$

If we use coefficients in $\mathbb{Z}/5$, then both maps are 0 and we have

$$H^0(X, \mathbb{Z}) = \mathbb{Z}/5; \quad H^1(X, \mathbb{Z}) = \mathbb{Z}/5, \quad \text{and} \quad H^2(X, \mathbb{Z}) = \mathbb{Z}/5.$$

6. (DG) Let X be a Riemannian manifold and σ be an isometry of X. Let Y be the set of fixed points of σ in the sense that Y is the set of all points y of X such that $\sigma(y) = y$. Prove that Y is regular and is totally geodesic (in the sense that any geodesic in Y with respect to the metric induced from X is also a geodesic in X).

Solution. For any geodesic C in X, if some point P of C and the tangent vector of C at P is fixed by σ, then the entire geodesic C is pointwise fixed by σ by the uniqueness theorem of ordinary differential equation of second order. Moreover, in a sufficiently small neighborhood of a given point P of X, any two points are joined by a unique geodesic and as a consequence the unique geodesic is pointwise fixed by σ if the two points are fixed by σ. The exponential map at any point P of X maps an open neighborhood of the tangent space $T_{X,P}$ of X at P to an open neighborhood of P in X. For P in Y, by using the exponential map at P we can conclude that some open neighborhood of P in Y is the diffeomorphic image of some vector subspace of $T_{Y,P}$ under the exponential map.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Thursday January 21, 2021 (Day 3)

1. (AG) Let $X \subset \mathbb{P}^3$ be a curve that is not contained in any proper linear subspace of $\mathbb{P}^3$. Show that if $\deg X$ is a prime number, then the homogeneous ideal $I(X)$ cannot be generated by two elements.

Solution: Assume for a contradiction that we have $I(X) = (f, g)$ for two homogeneous polynomials $f, g \in \Bbbk[x, y, z, w]$. Clearly, g does not vanish identically on any irreducible component of $V(f)$, since otherwise the zero locus of (f, g) would have codimension 1.

By Bezout's theorem, $\deg X = \deg f \cdot \deg g$. This implies either $\deg f$ or $\deg g = 1$. However this means one of f or g is linear which contradicts to the assumption.

2. (RA) Let $\mathcal{E}$ be the space of even $\mathcal{C}^\infty$ functions $\mathbf{R}/\mathbf{Z} \to \mathbf{R}$. Prove that for every $f \in \mathcal{E}$ there exists a unique $g \in \mathcal{E}$ such that

$$f(x) = \int_0^1 \int_0^1 g(y)\, g(z)\, g(x - y - z)\, dy\, dz$$

for all $x \in \mathbf{R}/\mathbf{Z}$. [Hint: write the integral formula for f as a convolution.]

Solution. The right-hand side of the displayed equation is the value at x of the convolution $f * f * f$. We shall use the following standard facts:

- The $\mathcal{C}^\infty$ functions $\mathbf{R}/\mathbf{Z} \to \mathbf{C}$ are exactly the Fourier series

$$f(x) = \sum_{n \in \mathbf{Z}} a_n(f) e^{2\pi i n x}$$

  whose coefficient sequence $\{a_n\}_{-\infty}^{\infty}$ is "Schwartz",[1] i.e., such that for every $M > 0$ the sequence $\{|n|^M a_n\}$ is bounded.
- If $\sum_{n \in \mathbf{Z}} |a_n(f)| < \infty$ $\sum_{n \in \mathbf{Z}} |a_n(g)| < \infty$ then

$$(f * g)(x) = \sum_{n \in \mathbf{Z}} a_n(f)\, a_n(g)\, e^{2\pi i n x}$$

  for all x.

[1]This may be a neologism. A "Schwartz function" on a real vector space is one that is both $O(\|x\|^{-M})$ (all M) and $\mathcal{C}^\infty$. For a function on $\mathbf{Z}$ only the decay condition makes sense.

- A real-valued $\mathcal{C}^\infty$ function f on $\mathbf{R}/\mathbf{Z}$ is even if and only if $a_n \in \mathbf{R}$ and $a_n = a_{-n}$ for all n.

Thus we are to show that for every even real Schwartz sequence $\{a_n\} = \{a_n(f)\}$ there exists a unique even real Schwartz sequence $\{b_n\} = \{a_n(g)\}$ such that $a_n = b_n^3$ for all n. This is clear because every real number has a unique real cube root and $\{a_n^{1/3}\}$ is even (resp. Schwartz) if and only if $\{a_n\}$ is.

3. (CA) Suppose $f(z)$ is analytic and bounded for $|z| < 1$. Let $\zeta = x + iy$. If $|z| < 1$, prove that

$$f(z) = \frac{1}{\pi} \int\int_{|\zeta|<1} \frac{f(\zeta)}{(1 - z\bar{\zeta})^2} dxdy$$

Solution: By Green's theorem and analyticity of f, for $|z| < 1$, we have up to a constant,

$$\int\int_{|\zeta|<1} \frac{f(\zeta)}{(1 - z\bar{\zeta})^2} dxdy = \int_{|\zeta|<1} \Big(\frac{1}{z}\partial_{\bar{\zeta}} \frac{1}{(1 - z\bar{\zeta})}\Big) f(\zeta) d\zeta d\bar{\zeta}$$

$$= \frac{1}{z} \int_{|\zeta|<1} d\Big[\frac{f(\zeta)}{(1 - z\bar{\zeta})} d\zeta\Big] = \frac{1}{z} \int_{|\zeta|=1} \frac{\zeta f(\zeta)}{(\zeta - z)} d\zeta = f(z)$$

4. (AT) Suppose f is an orientation-preserving self-homeomorphism of $\mathbb{CP}^n$ such that the graph $\Gamma_f \subset \mathbb{CP}^n \times \mathbb{CP}^n$ intersects the diagonal transversely . Compute all possibilities for the number of its fixed points.

Solution: We apply the Lefschetz fixed-point theorem, recalling that $H^*(\mathbb{CP}^n; \mathbb{C}) \simeq \mathbb{C}[u]/(u^{n+1})$ for u a generator in degree 2. If λ is the eigenvalue by which f acts on H^2, then f acts on H^{2k} with eigenvalue λ^{2k}. But $\lambda \in \mathbb{Z}$, as the action of f is defined on $H^2(\mathbb{CP}^n; \mathbb{Z})$, and $\lambda^n = 1$, as f acts trivially on the volume form by virtue of preserving orientation. Hence if n is odd, $\lambda = 1$ while if n is even, $\lambda \in \{\pm 1\}$. In either case, the Lefschetz fixed-point theorem tells us the number of fixed points – or, more generally, the Euler characteristic of the fixed point locus F – is $\chi(F) = 1 + \lambda + \cdots + \lambda^n$, so if $\lambda = 1$, we obtain $\chi(F) = n+1$ while if $\lambda = -1$, we obtain $\chi(F) = 1$. To show both possibilities are realized, we may simply take f a 'general' rotation for the case of $\lambda = 1$. With more details – we have the action of $(\mathbb{C}^*)^n$ on $\mathbb{CP}^n$, and if we take a rotation in $(S^1)^n$ such that all the angles are rationally independent of one another and of 2π, then the fixed point locus will be $n + 1$ points. On the other hand, when n is even, then complex-conjugation composed with a rotation such as above is an orientation-preserving self-homeomorphism which may be checked to have a unique fixed point.

5. (DG) Let G be an open subset of $\mathbb{R}^n$. For $1 \leq p \leq n-1$ denote by $\wedge^p T_G$ the exterior product of p copies of the tangent bundle T_G of G. For $1 \leq j \leq m$ let $\boldsymbol{\eta}_j$ be a C^∞ section of $\wedge^p T_G$ over G. For a C^∞ vector field ξ on an open subset of G, denote by $\mathcal{L}_\xi \boldsymbol{\eta}_j$ the Lie derivative of $\boldsymbol{\eta}_j$ with respect to ξ, which means that if $\varphi_{\xi,t}$ is the local diffeomorphism defined by ξ so that the tangent vector $\frac{d}{dt}\varphi_{\xi,t}$ equals the value of ξ at $\varphi_{\xi,t}$, then

$$\mathcal{L}_\xi \boldsymbol{\eta}_j = \lim_{t \to 0} \frac{1}{t}\left((\varphi_{\xi,t})_* \boldsymbol{\eta}_j - \boldsymbol{\eta}_j\right),$$

where $(\varphi_{\xi,t})_* \boldsymbol{\eta}_j$ is the pushforward of $\boldsymbol{\eta}_j$ under $\varphi_{\xi,t}$. Let $\Phi_{\boldsymbol{\eta}_j} : T_G \to \wedge^{p+1} T_G$ be defined by exterior product with $\boldsymbol{\eta}_j$. Assume that the intersection $\cap_{j=1}^m \operatorname{Ker} \Phi_{\boldsymbol{\eta}_j}$ of the kernel $\operatorname{Ker} \Phi_{\boldsymbol{\eta}_j}$ of $\Phi_{\boldsymbol{\eta}_j}$ for $1 \leq j \leq m$ is a subbundle of T_G of rank q over G. Suppose for any C^∞ tangent vector field ζ in any open subset W there exist C^∞ functions $g_{j,k,\zeta}$ on W for $1 \leq j, k \leq m$ such that

$$\mathcal{L}_\zeta \boldsymbol{\eta}_j = \sum_{k=1}^m g_{j,k,\zeta} \boldsymbol{\eta}_k$$

on W. Prove that for every point x of G there exist some open neighborhood U_x of x in G and C^∞ functions $f_1, \cdots, f_{n-q}$ on U_x such that the fiber of $\cap_{j=1}^m \operatorname{Ker} \Phi_{\boldsymbol{\eta}_j}$ at y is equal to $\cap_{k=1}^{n-q} \operatorname{Ker} df_k$ at y for $y \in U_x$.

Solution. For any C^∞ tangent vector fields ξ, ζ on an open subset W of G, the product formula

$$\mathcal{L}_\xi \left(\boldsymbol{\eta}_j \wedge \zeta\right) = \left(\mathcal{L}_\xi \boldsymbol{\eta}_j\right) \wedge \zeta + \boldsymbol{\eta}_j \wedge \mathcal{L}_\xi \zeta$$

for Lie differentiation holds. Moreover, $\mathcal{L}_\xi \zeta$ is equal to the Lie bracket $[\xi, \zeta]$ of the tangent vector fields ξ, ζ.

If ξ, ζ are C^∞ sections of $\cap_{j=1}^m \operatorname{Ker} \Phi_{\boldsymbol{\eta}_j}$ over an open subset W of G, then

$$\begin{aligned} 0 &= \mathcal{L}_\xi \left(\boldsymbol{\eta}_j \wedge \zeta\right) \\ &= \left(\mathcal{L}_\xi \boldsymbol{\eta}_j\right) \wedge \zeta + \boldsymbol{\eta}_j \wedge \mathcal{L}_\xi \zeta \\ &= \left(\sum_{k=1}^m g_{j,k,\xi} \boldsymbol{\eta}_k\right) \wedge \zeta + \boldsymbol{\eta}_j \wedge \mathcal{L}_\xi \zeta \\ &= \boldsymbol{\eta}_j \wedge \mathcal{L}_\xi \zeta \end{aligned}$$

for $1 \leq j \leq m$, which implies that $[\xi, \zeta]$ is a section of $\cap_{j=1}^m \operatorname{Ker} \Phi_{\boldsymbol{\eta}_j}$ over W. The conclusion now follows from applying Frobenius integrability theorem to the subbundle $\cap_{j=1}^m \operatorname{Ker} \Phi_{\boldsymbol{\eta}_j}$ of T_G over G.

6. (A) Let π be a finite dimensional representation of a finite group G with the character χ_π. Prove that π is irreducible if and only if

$$\frac{1}{|G|} \sum_{g \in G} |\chi_\pi(g)|^2 = 1.$$

Solution: A finite dimensional representation is semi-simple and hence we can express

$$\chi_\pi = \sum_{\sigma \in \widehat{G}} n_\pi(\sigma) \chi_\sigma,$$

where $\widehat{G}$ denotes the collection of the isomorphism classes of irreducible representations of G, and $n_\pi(\sigma) = \dim \operatorname{Hom}_G(\pi, \sigma)$. Then

$$\frac{1}{|G|} \sum_{g \in G} |\chi_\pi(g)|^2 = \sum_{\rho, \tau \in \widehat{G}} n_\pi(\sigma) n_\pi(\tau) \frac{1}{|G|} \sum_{g \in G} \chi_\sigma(g) \overline{\chi_\tau(g)} = \sum_{\sigma \in \widehat{G}} n_\pi(\sigma)^2.$$

Note that $n_\pi(\sigma) = 1$ if and only if $n_\pi(\sigma) = 1$ for some $\sigma \in \widehat{G}$ and $n_\pi(\tau) = 0$ for $\tau \neq \sigma$, $\tau \in \widehat{G}$. This is equivalent of saying that π is isomorphic to the irreducible representation σ.