1. (AG) Let $Y \subset \mathbb{P}^2$ be an irreducible curve of degree $d > 1$ having a point of multiplicity $d - 1$. Show that $Y$ is a rational curve.

Solution: Recall that if $P \in Y \subset \mathbb{P}^2$ is a point of multiplicity $m$, then a generic line through $P$ meets $Y$ at $p$ with multiplicity $m$. Then there is a dense open subset $U$ of $\mathbb{P}^1$ such that if $L \in U$ then the intersection multiplicity of $Y$ at $L$ along $P$ is $m$.

Without loss of generality, let $P = (0, 0, 1) \in Y$ be the point with multiplicity $d - 1$. Then consider two maps:

$$\phi : Y \setminus P \to \mathbb{P}^1$$

the projective map, and by Bezout’s theorem, there is a map

$$\psi : \mathbb{P}^1 \supset U \to Y \setminus P$$

which maps a point $L$ of $U$ to the unique point that $L$ meets $Y$ other than $P$.

Thus

$$Y \setminus P \xrightarrow{\phi} \mathbb{P}^1 \xrightarrow{\psi} Y \setminus P$$

$(a, b, c) \mapsto (a, b, 0) \mapsto (a, b, c)$

are rational. Thus this gives a birational equivalence of $Y \setminus P$ and $\mathbb{P}^1$.

2. (CA) Use the method of contour integrals to find the integral

$$\int_0^\infty \frac{\log x}{x^2 + 4} \, dx.$$

Solution: Consider the contour from $-R$ to $-r$, a semicircle, $r$ to $R$ and a large semicircle. Then the integration from $-R$ to $-r$ and $r$ to $R$ becomes

$$2 \int_r^R \frac{\log x}{x^2 + 4} \, dx + i \int_{-R}^{-r} \frac{\pi}{x^2 + 4} \, dx$$

The last term is imaginary. Hence

$$\int_0^\infty \frac{\log x}{x^2 + 4} \, dx = d \left[ \pi i \lim_{z \to 2i} \frac{\log z}{z + 2i} \right] = \frac{\pi \log 2}{4}$$
3. (RA) Suppose $\mu$ and $\nu$ are two positive measures on $\mathbb{R}^n$ with $n \geq 1$. For a positive function $f$, consider two quantities

$$A := \int \nu(dy) \left[ \int f(x,y)^p \mu(dx) \right]^{1/p}$$

$$B := \left[ \int \mu(dx) \left( \int f(x,y)\nu(dy) \right)^p \right]^{1/p}$$

For $1 \leq p < \infty$. Assume all quantities are integrable and finite. Do we know that $A \geq B$ or $A \leq B$ for all functions $f$? Prove your assertion or give a counterexample.

**Solution:** By duality,

$$\left[ \int \mu(dx) \left( \int f(x,y)\nu(dy) \right)^p \right]^{1/p} = \sup_{g: \|g\|_{L^q(\mu)} \leq 1} \int \mu(dx)g(x) \int f(x,y)\nu(dy)$$

$$= \sup_{g: \|g\|_{L^q(\mu)} \leq 1} \int \nu(dy) \int \mu(dx)g(x)f(x,y) \leq \int \nu(dy) \left[ \int f(x,y)^p \mu(dx) \right]^{1/p}$$

Since $\epsilon$ was arbitrary we are done.

4. (A) Let $p$ be a prime ideal in a commutative ring $A$. Show that $p[x]$ is a prime idea in $A[x]$. If $m$ is a maximal idea in $A$, is $m[x]$ a maximal ideal in $A[x]$?

**Solution:** Consider the projection

$$A[x] \to (A/a)[x].$$

The kernel of the projection is $a[x]$ and hence $A[x]/a[x] \cong (A/a)[x]$. Now consider $p$ a prime idea of $A$. Then $A/p$ is an integral domain, and so do $(A/p)[x]$ by Hilbert Basis Theorem. That is $A[x]/p[x]$ is an integral domain as well. This implies $p[x]$ is prime in $A[x]$.

Note that if $A$ is a field, $A[x]$ may not be a field. Hence if $m$ is maximal in $A$ does not implies $m[x]$ a maximal ideal in $A[x]$.

5. (AT) What are the homology groups of the 5-manifold $\mathbb{RP}^2 \times \mathbb{RP}^3$,

(a) with coefficients in $\mathbb{Z}$?
(b) with coefficients in $\mathbb{Z}/2$?
(c) with coefficients in $\mathbb{Z}/3$?
Solution: $\mathbb{R}P^2$ and $\mathbb{R}P^3$ have cell complexes with sequences

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0 \quad \text{and} \quad 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0$$

where the maps are alternately 0 and multiplication by 2; from this the homology groups of $\mathbb{R}P^2$ and $\mathbb{R}P^3$ can be calculated as $\mathbb{Z}, \mathbb{Z}/2, 0$ and $\mathbb{Z}, \mathbb{Z}/2, 0, \mathbb{Z}$ respectively. The rest is just Künneth; the answers are

(a): $\mathbb{Z}$, $(\mathbb{Z}/2)^2$, $\mathbb{Z}/2$, $\mathbb{Z}$, $\mathbb{Z}/2$, $\mathbb{Z}$

(b): $\mathbb{Z}/2$, $(\mathbb{Z}/2)^2$, $(\mathbb{Z}/2)^3$, $(\mathbb{Z}/2)^3$, $(\mathbb{Z}/2)^2$, $\mathbb{Z}/2$

(c): $\mathbb{Z}/3$, 0, 0, $\mathbb{Z}/3$, 0, 0

6. (DG) Let $a > b > 0$ be positive numbers. Let $C$ be the circle of radius $b$ centered at $(a,0)$ in the $(x,z)$-plane. Let $T$ be the torus obtained by revolving the circle $C$ about the $z$-axis in the $(x,y,z)$-space. The torus $T$ can be identified as the product of two circles whose points are described by the two angle-variables $\varphi, \theta$ (or arc-length-variables) of the two circles. Compute, in terms of $a, b, \varphi, \theta$, the Gaussian curvature of $T$ and determine the subsets $T^+, T^-$, $T^0$ of $T$ where the Gaussian curvature of $T$ is respectively positive, negative, and zero.

Solution: Parametrize $T$ by two circles with angle-variable $\varphi, \theta$ as follows.

The circle $C$ with angle-variable $\varphi$ can be described as

$$(x, z) = b(\cos \varphi, \sin \varphi) + (a, 0) = (a + b \cos \varphi, b \sin \varphi).$$

The result of rotating a point $P$ on $C$ by an angle $\theta$ about the $z$-axis is the same as replacing the $x$-coordinate $x_P$ of $P$ by $(x, y) = (x_P \cos \theta, x_P \sin \theta)$. It follows that the parametrization of $T$ is given by

$$\vec{r}(\varphi, \theta) = ((a + b \cos \varphi) \cos \theta, (a + b \cos \varphi) \sin \theta, b \sin \varphi).$$

The first and second partial derivatives of $\vec{r}(\varphi, \theta)$ and the unit normal vector $\vec{N}$ of $T$ are given by

$$\vec{r}_{\varphi} = (-b \sin \varphi \cos \theta, -b \sin \varphi \sin \theta, b \cos \varphi),$$

$$\vec{r}_{\theta} = (-a + b \cos \varphi) \sin \theta, (a + b \cos \varphi) \cos \theta, 0),$$

$$\vec{r}_{\varphi} \times \vec{r}_{\theta} = ((-a + b \cos \varphi) \cos \theta b \cos \varphi, -(a + b \cos \varphi) \sin \theta b \cos \varphi, -(a + b \cos \varphi) b \sin \varphi),$$

$$\vec{n} = \frac{\vec{r}_{\varphi} \times \vec{r}_{\theta}}{\|\vec{r}_{\varphi} \times \vec{r}_{\theta}\|} = (-\cos \theta \cos \varphi, -\sin \theta \cos \varphi, -\sin \varphi),$$

$$\vec{r}_{\varphi\varphi} = (-b \cos \varphi \cos \theta, -b \cos \varphi \sin \theta, -b \sin \varphi),$$

$$\vec{r}_{\varphi\theta} = (b \sin \varphi \sin \theta, -b \sin \varphi \cos \theta, 0),$$

$$\vec{r}_{\theta\theta} = ((-a + b \cos \varphi) \cos \theta, -(a + b \cos \varphi) \sin \theta, 0).$$
The first fundamental form \( E\,d\phi^2 + 2F\,d\phi\,d\theta + G\,d\theta^2 \) and the second fundamental form \( L\,d\phi^2 + 2M\,d\phi\,d\theta + N\,d\theta^2 \) of the torus \( T \) are given by

\[
E = \vec{r}_\phi \cdot \vec{r}_\phi = b^2,
F = \vec{r}_\phi \cdot \vec{r}_\theta = 0,
G = \vec{r}_\theta \cdot \vec{r}_\theta = (a + b \cos \varphi)^2,
L = \vec{r}_{\varphi\varphi} \cdot \vec{n} = b,
M = \vec{r}_{\varphi\theta} \cdot \vec{n} = 0,
N = \vec{r}_{\theta\theta} \cdot \vec{n} = (a + b \cos \varphi) \cos \varphi.
\]

The Gaussian curvature is given by

\[
\frac{LN - M^2}{EG - F^2} = \frac{b(a + b \cos \varphi) \cos \varphi}{b^2(a + b \cos \varphi)^2} = \frac{\cos \varphi}{b(a + b \cos \varphi)},
\]

which

(i) is positive for \(-\frac{\pi}{2} < \varphi < \frac{\pi}{2}\),
(ii) is zero for \(\varphi = \frac{\pi}{2}\) or \(-\frac{\pi}{2}\), and
(iii) is negative for \(\frac{\pi}{2} < \varphi < \pi\) or \(-\pi < \varphi < -\frac{\pi}{2}\).

In other words,

(i) the Gaussian curvature of the torus \( T \) is positive on the part \( T^+ \) obtained by rotating the right-half \( C \cap \{x > a\} \) of the circle \( C \) about the \( z \)-axis.
(ii) The Gaussian curvature of the torus \( T \) is negative on the part \( T^- \) obtained by rotating the left-half \( C \cap \{x < a\} \) of the circle \( C \) about the \( z \)-axis.
(iii) The Gaussian curvature of the torus \( T \) is zero on the part \( T^0 \) obtained by rotating the the highest point \( C \cap \{z = b\} \) and the lowest point \( C \cap \{z = -b\} \) of the circle \( C \) about the \( z \)-axis.
1. (CA) Let $q$ be any positive integer. Let $\Omega$ be a connected open subset of $\mathbb{C}$. Suppose $f_n(z)$ is a sequence of holomorphic functions on $\Omega$ such that for any positive number $n$ and for any $c \in \mathbb{C}$, the set $f_n^{-1}(c)$ has no more than $q$ distinct elements. Suppose the sequence $f_n(z)$ converges to a function $f(z)$ uniformly on compact subsets of $\Omega$. Prove that either $f(z)$ is constant or $f(z)$ satisfies the property that for any $c \in \mathbb{C}$ the set $f^{-1}(c)$ has no more than $q$ distinct elements.

Solution. Assume that $f$ is nonconstant and achieves the value $c$ at $q + 1$ distinct points $z_1, \cdots, z_{q+1}$ of $\Omega$ and we are going to derive a contradiction. Choose some $\epsilon > 0$ such that

(i) the $q + 1$ closed disks $|z - z_j| \leq \epsilon$ for $1 \leq j \leq q + 1$ are inside $\Omega$ and are disjoint,

(ii) for $1 \leq j \leq q + 1$ the function $f(z) - f(z_j)$ has exactly one zero on the closed disk $|z - z_j| \leq \epsilon$ which is not on the boundary $|z - z_j| = \epsilon$.

This is possible, because $f$ is nonconstant and is holomorphic on the connected open subset $\Omega$ of $\mathbb{C}$ (as the uniform limit on compact subsets of holomorphic functions on $\Omega$). Let $\eta > 0$ be the minimum of $|f(z) - c|$ on $|z - z_j| = \epsilon$ for $1 \leq j \leq q + 1$. By uniform convergence of $f_n \to f$ on compact sets of $\Omega$ there is some $n$ (as a matter of fact, any sufficiently large $n$) for which $|f_n(z) - f(z)| < \eta$ on $|z - z_j| = \epsilon$ for $1 \leq j \leq q + 1$. Since $|f_n(z) - f(z)| < |f(z) - c|$ on $|z - z_j| = \epsilon$ for $1 \leq j \leq q + 1$, by applying Rouche’s theorem to $f_n(z) - c = (f(z) - c) + (f_n(z) - f(z))$, we conclude that $f_n(z) - c$ has the same number of zeroes as the function $f(z) - c$ on each of the $q + 1$ disjoint disks $|z - z_j| < \epsilon$. This contradicts the assumption that the set $f_n^{-1}(c)$ has no more than $q$ distinct elements.

2. (AG) Let $X$ be a degree 3 hypersurface in $\mathbb{P}^3$. Show that $X$ contains a line.

(You may use the fact that the Fermat cubic surface $V(x^3 + y^3 + z^3 + w^3)$ contains a positive finite number of lines.)

Solution: Let $P = |\mathcal{O}_{\mathbb{P}^3}(3)| \cong \mathbb{P}^{19}$ be the projective space of cubics in $\mathbb{P}^3$ (Note $(3+3) - 1 = 19$,) and let $G = \text{Gr}(\mathbb{P}^1, \mathbb{P}^3)$ which is of dimension $(1 + 1) \times ((3 + 1) - (1 + 1)) = 4.$
Consider the incidence variety \( Z = \{ (\ell, S) \in G \times P \mid \ell \subseteq S \} \). Then we have

\[
\begin{array}{c}
G \xleftarrow{p_1} Z \xrightarrow{p_2} P
\end{array}
\]

Let the coordinate on \( \mathbb{P}^3 \) be \((x, y, z, w)\). A cubic surface \( S \subset \mathbb{P}^3 \) contains the line \( \ell = \{z = w = 0\} \) if and only if the defining equation of \( S \) having the terms \( x^3, x^2y, xy^2, y^3 \) vanish. This shows that \( p_1^{-1}(\ell) \) is irreducible of dimension \( \dim P - 4 \). Hence

\[ \dim Z = \dim P. \]

If there is a degree 3 hypersurface \( X \) not containing a line, then \( p_2(Z) \subset P \) is of codimensional \( \geq 1 \) in \( P \). Hence for any \( S \in P \), the fibre \( p_2^{-1}(S) \) is either empty or of positive dimension. However this contradicts to the del Pezzo surfaces having only finitely many line.

3. (RA) Suppose \( X_j \) are independent identically distributed Poisson distributions with intensity \( \lambda \), i.e.,

\[ P(X_j = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \in \mathbb{N} \cup \{0\} \]

Show that for any \( y \geq \lambda \),

\[ P\left( \frac{X_1 + \cdots + X_n}{n} \geq y \right) \leq e^{-n[y \log(y/\lambda) - y + \lambda]} \]

and for any \( y \leq \lambda \),

\[ P\left( \frac{X_1 + \cdots + X_n}{n} \leq y \right) \leq e^{-n[y \log(y/\lambda) - y + \lambda]} \]

Hint: Consider the moment generating function.

**Solution:** By the Markov inequality, for any \( \lambda \leq y \),

\[ P\left( \frac{X_1 + \cdots + X_n}{n} \geq y \right) \leq \inf_{t \geq 0} e^{-tny} e^{t(X_1 + \cdots + X_n)} = \inf_{t \geq 0} e^{-n[y \log(y/\lambda) - y + \lambda]} = e^{-n[y \log(y/\lambda) - y + \lambda]} \]

Similarly, for any \( y \leq \lambda \),

\[ P\left( \frac{X_1 + \cdots + X_n}{n} \leq y \right) \leq \inf_{t \geq 0} e^{tny} e^{-t(X_1 + \cdots + X_n)} = \inf_{t \geq 0} e^{-n[y \log(y/\lambda) - y + \lambda]} = e^{-n[y \log(y/\lambda) - y + \lambda]} \]
4. (A) Determine the Galois group of the polynomial \( f(x) = x^3 - 2 \). Let \( K \) be the splitting field of \( f \) over \( \mathbb{Q} \). Describe the set of all intermediate fields \( L, \mathbb{Q} < L < K \) and the Galois correspondence.

Solution: Let \( K \) be the splitting field of \( f \) over \( \mathbb{Q} \). Then \( K = \mathbb{Q}(\sqrt[3]{2}, \zeta) \) is generated by \( \sqrt[3]{2}, \sqrt[3]{2}\zeta, \) and \( \sqrt[3]{2}\zeta^2 \), where \( \zeta \) is the primitive cubic root of unity.

The discriminant of \( f(x) = x^3 - 2 \) is \( D = -27(2)^2 \) and square root of \( D \) is not rational. Hence the Galois group \( \text{Gal}(K/\mathbb{Q}) = S_3 \) which is the permutation group of order 6. As a permutation group, \( S_3 \) can be expressed as \{id, (12), (13), (23), (123), (132)\}. The four proper subgroup of \( S_3 \) are \{id, (12)\}, \{id, (13)\}, \{id, (23)\}, and \{id, (123), (132)\}.

The intermediate fields \( L \) are \( \mathbb{Q}, \mathbb{Q}(\sqrt[3]{2}), \mathbb{Q}(\sqrt[3]{2}\zeta), \mathbb{Q}(\sqrt[3]{2}\zeta^2), \) and \( K \). They correspond to the subgroups \{id\}, \{id, (12)\}, \{id, (13)\}, \{id, (23)\}, \{id, (123), (132)\}, and \( S_3 \).

5. (AT) Let \( X \subseteq \mathbb{R}^3 \) be the union of the unit sphere \( S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\} \) and the line segment \( I = \{(x, 0, 0) \mid -1 \leq x \leq 1\} \).

   (a) What are the homology groups of \( X \)?

   (b) What are the homotopy groups \( \pi_1(X) \) and \( \pi_2(X) \)?

Solution: Under the attaching map \( I \hookrightarrow X \), the boundary \( \varphi(I) \) is homologous to 0, so attaching \( I \) simply adds one new, non-torsion generator to \( H^1 \); thus

\[
H_0(X) = H^1(X) = H^2(X) = \mathbb{Z},
\]

and all other homology groups are 0. Similarly, \( \pi_1(X) = \mathbb{Z} \). For \( \pi_2(X) \), note that the universal cover of \( X \) is a string of spheres attached in a sequence by line segments; \( \pi_2(X) \) is thus the free abelian group on countably many generators.

Solution: The space \( X \) can be realized as a CW complex with one 0-cell, one 1-cell and one 2-cell, with the 1-skeleton the unit circle \( S^1 \) in \( \mathbb{C} \) and the 2-cell attached via the map \( S^1 \to S^1 \) given by \( z \mapsto z^5 \). The cellular complex is thus

\[
\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}
\]

with the first map multiplication by 5 and the second map 0; the homology groups with coefficients in \( \mathbb{Z} \) are thus

\[
H^0(X, \mathbb{Z}) = \mathbb{Z}, \quad H^1(X, \mathbb{Z}) = \mathbb{Z}/5, \quad \text{and} \quad H^2(X, \mathbb{Z}) = 0.
\]
If we use coefficients in $\mathbb{Z}/5$, then both maps are 0 and we have
\[ H^0(X, \mathbb{Z}) = \mathbb{Z}/5; \quad H^1(X, \mathbb{Z}) = \mathbb{Z}/5; \quad \text{and} \quad H^2(X, \mathbb{Z}) = \mathbb{Z}/5. \]

6. (DG) Let $X$ be a Riemannian manifold and $\sigma$ be an isometry of $X$. Let $Y$ be the set of fixed points of $\sigma$ in the sense that $Y$ is the set of all points $y$ of $X$ such that $\sigma(y) = y$. Prove that $Y$ is regular and is totally geodesic (in the sense that any geodesic in $Y$ with respect to the metric induced from $X$ is also a geodesic in $X$).

Solution. For any geodesic $C$ in $X$, if some point $P$ of $C$ and the tangent vector of $C$ at $P$ is fixed by $\sigma$, then the entire geodesic $C$ is pointwise fixed by $\sigma$ by the uniqueness theorem of ordinary differential equation of second order. Moreover, in a sufficiently small neighborhood of a given point $P$ of $X$, any two points are joined by a unique geodesic and as a consequence the unique geodesic is pointwise fixed by $\sigma$ if the two points are fixed by $\sigma$. The exponential map at any point $P$ of $X$ maps an open neighborhood of the tangent space $T_{X,P}$ of $X$ at $P$ to an open neighborhood of $P$ in $X$. For $P$ in $Y$, by using the exponential map at $P$ we can conclude that some open neighborhood of $P$ in $Y$ is the diffeomorphic image of some vector subspace of $T_{Y,P}$ under the exponential map.
1. (AG) Let $X \subset \mathbb{P}^3$ be a curve that is not contained in any proper linear subspace of $\mathbb{P}^3$. Show that if $\deg X$ is a prime number, then the homogeneous ideal $I(X)$ cannot be generated by two elements.

Solution: Assume for a contradiction that we have $I(X) = (f, g)$ for two homogeneous polynomials $f, g \in \mathbb{k}[x, y, z, w]$. Clearly, $g$ does not vanish identically on any irreducible component of $V(f)$, since otherwise the zero locus of $(f, g)$ would have codimension 1.

By Bezout’s theorem, $\deg X = \deg f \cdot \deg g$. This implies either $\deg f$ or $\deg g = 1$. However this means one of $f$ or $g$ is linear which contradicts to the assumption.

2. (RA) Let $E$ be the space of even $C^\infty$ functions $\mathbb{R}/\mathbb{Z} \to \mathbb{R}$. Prove that for every $f \in E$ there exists a unique $g \in E$ such that

$$f(x) = \int_0^1 \int_0^1 g(y) g(z) g(x - y - z) \, dy \, dz$$

for all $x \in \mathbb{R}/\mathbb{Z}$. [Hint: write the integral formula for $f$ as a convolution.]

Solution. The right-hand side of the displayed equation is the value at $x$ of the convolution $f * f * f$. We shall use the following standard facts:

- The $C^\infty$ functions $\mathbb{R}/\mathbb{Z} \to \mathbb{C}$ are exactly the Fourier series

$$f(x) = \sum_{n \in \mathbb{Z}} a_n(f) e^{2\pi i n x}$$

whose coefficient sequence $\{a_n\}_{n=-\infty}^{\infty}$ is “Schwartz”, i.e., such that for every $M > 0$ the sequence $\{|n|^M a_n\}$ is bounded.

- If $\sum_{n \in \mathbb{Z}} |a_n(f)| < \infty \sum_{n \in \mathbb{Z}} |a_n(g)| < \infty$ then

$$f * g(x) = \sum_{n \in \mathbb{Z}} a_n(f) a_n(g) e^{2\pi i n x}$$

for all $x$.

---

1This may be a neologism. A “Schwartz function” on a real vector space is one that is both $O(||x||^{-M})$ (all $M$) and $C^\infty$. For a function on $\mathbb{Z}$ only the decay condition makes sense.
- A real-valued $C^\infty$ function $f$ on $\mathbb{R}/\mathbb{Z}$ is even if and only if $a_n \in \mathbb{R}$ and $a_n = a_{-n}$ for all $n$.

Thus we are to show that for every even real Schwartz sequence $\{a_n\} = \{a_n(f)\}$ there exists a unique even real Schwartz sequence $\{b_n\} = \{a_n(g)\}$ such that $a_n = b_n^3$ for all $n$. This is clear because every real number has a unique real cube root and $\{a_n^{1/3}\}$ is even (resp. Schwartz) if and only if $\{a_n\}$ is.

3. (CA) Suppose $f(z)$ is analytic and bounded for $|z| < 1$. Let $\zeta = x + iy$. If $|z| < 1$, prove that

$$f(z) = \frac{1}{\pi} \int \int_{|\zeta| < 1} \frac{f(\zeta)}{(1 - z\zeta)^2} \, dx \, dy$$

Solution: By Green’s theorem and analyticity of $f$, for $|z| < 1$, we have up to a constant,

$$\int \int_{|\zeta| < 1} \frac{f(\zeta)}{(1 - z\zeta)^2} \, dx \, dy = \int_{|\zeta| < 1} \left( \frac{1}{z} \partial_{\zeta} - \frac{1}{1 - z\zeta} \right) f(\zeta) d\zeta \, d\bar{\zeta}$$

$$= \frac{1}{z} \int_{|\zeta| < 1} d\left[ \frac{f(\zeta)}{(1 - z\zeta)} \right] = \frac{1}{z} \int_{|\zeta| = 1} \frac{\zeta f(\zeta)}{(\zeta - z)} \, d\zeta = f(z)$$

4. (AT) Suppose $f$ is an orientation-preserving self-homeomorphism of $\mathbb{C}P^n$ such that the graph $\Gamma_f \subset \mathbb{C}P^n \times \mathbb{C}P^n$ intersects the diagonal transversely. Compute all possibilities for the number of its fixed points.

Solution: We apply the Lefschetz fixed-point theorem, recalling that $H^*(\mathbb{C}P^n; \mathbb{C}) \simeq \mathbb{C}[u]/(u^{n+1})$ for $u$ a generator in degree 2. If $\lambda$ is the eigenvalue by which $f$ acts on $H^2$, then $f$ acts on $H^{2k}$ with eigenvalue $\lambda^{2k}$. But $\lambda \in \mathbb{Z}$, as the action of $f$ is defined on $H^2(\mathbb{C}P^n; \mathbb{Z})$, and $\lambda^n = 1$, as $f$ acts trivially on the volume form by virtue of preserving orientation. Hence if $n$ is odd, $\lambda = 1$ while if $n$ is even, $\lambda \in \{\pm 1\}$. In either case, the Lefschetz fixed-point theorem tells us the number of fixed points – or, more generally, the Euler characteristic of the fixed point locus $F$ – is $\chi(F) = 1 + \lambda + \cdots + \lambda^n$, so if $\lambda = 1$, we obtain $\chi(F) = n + 1$ while if $\lambda = -1$, we obtain $\chi(F) = 1$. To show both possibilities are realized, we may simply take $f$ a ‘general’ rotation for the case of $\lambda = 1$. With more details – we have the action of $(\mathbb{C}^*)^n$ on $\mathbb{C}P^n$, and if we take a rotation in $(S^1)^n$ such that all the angles are rationally independent of one another and of $2\pi$, then the fixed point locus will be $n + 1$ points. On the
other hand, when \( n \) is even, then complex-conjugation composed with a rotation such as above is an orientation-preserving self-homeomorphism which may be checked to have a unique fixed point.

5. (DG) Let \( G \) be an open subset of \( \mathbb{R}^n \). For \( 1 \leq p \leq n-1 \) denote by \( \wedge^p T_G \) the exterior product of \( p \) copies of the tangent bundle \( T_G \) of \( G \). For \( 1 \leq j \leq m \) let \( \eta_j \) be a \( C^\infty \) section of \( \wedge^p T_G \) over \( G \). For a \( C^\infty \) vector field \( \xi \) on an open subset of \( G \), denote by \( L_\xi \eta_j \) the Lie derivative of \( \eta_j \) with respect to \( \xi \), which means that if \( \varphi_{\xi,t} \) is the local diffeomorphism defined by \( \xi \) so that the tangent vector \( \frac{d}{dt} \varphi_{\xi,t} \) equals the value of \( \xi \) at \( \varphi_{\xi,t} \), then

\[
L_\xi \eta_j = \lim_{t \to 0} \frac{1}{t} \left( (\varphi_{\xi,t}^*) \eta_j - \eta_j \right),
\]

where \( (\varphi_{\xi,t}^*) \eta_j \) is the pushforward of \( \eta_j \) under \( \varphi_{\xi,t} \). Let \( \Phi_{\eta_j} : T_G \to \wedge^{p+1} T_G \) be defined by exterior product with \( \eta_j \). Assume that the intersection \( \cap_{j=1}^m \text{Ker} \Phi_{\eta_j} \) of the kernel \( \text{Ker} \Phi_{\eta_j} \) for \( 1 \leq j \leq m \) is a subbundle of \( T_G \) of rank \( q \) over \( G \). Suppose for any \( C^\infty \) tangent vector field \( \zeta \) in any open subset \( W \) there exist \( C^\infty \) functions \( g_{j,k,\xi} \) on \( W \) for \( 1 \leq j,k \leq m \) such that

\[
L_\xi \eta_j = \sum_{k=1}^m g_{j,k,\xi} \eta_k
\]
on \( W \). Prove that for every point \( x \) of \( G \) there exist some open neighborhood \( U_x \) of \( x \) in \( G \) and \( C^\infty \) functions \( f_1, \ldots, f_{n-q} \) on \( U_x \) such that the fiber of \( \cap_{j=1}^m \text{Ker} \Phi_{\eta_j} \) at \( y \) is equal to \( \cap_{k=1}^{n-q} \text{Ker} df_k \) at \( y \) for \( y \in U_x \).

\textbf{Solution.} For any \( C^\infty \) tangent vector fields \( \xi, \zeta \) on an open subset \( W \) of \( G \), the product formula

\[
L_\xi \left( \eta_j \wedge \zeta \right) = (L_\xi \eta_j) \wedge \zeta + \eta_j \wedge L_\xi \zeta
\]

for Lie differentiation holds. Moreover, \( L_\xi \zeta \) is equal to the Lie bracket \([\xi,\zeta]\) of the tangent vector fields \( \xi,\zeta \).

If \( \xi,\zeta \) are \( C^\infty \) sections of \( \cap_{j=1}^m \text{Ker} \Phi_{\eta_j} \) over an open subset \( W \) of \( G \), then

\[
0 = L_\xi \left( \eta_j \wedge \zeta \right)
= (L_\xi \eta_j) \wedge \zeta + \eta_j \wedge L_\xi \zeta
= \left( \sum_{k=1}^m g_{j,k,\xi} \eta_k \right) \wedge \zeta + \eta_j \wedge L_\xi \zeta
= \eta_j \wedge L_\xi \zeta
\]
for $1 \leq j \leq m$, which implies that $[\xi, \zeta]$ is a section of $\cap_{j=1}^{m} \text{Ker} \Phi_{\eta_j}$ over $W$. The conclusion now follows from applying Frobenius integrability theorem to the subbundle $\cap_{j=1}^{m} \text{Ker} \Phi_{\eta_j}$ of $T_G$ over $G$.

6. (A) Let $\pi$ be a finite dimensional representation of a finite group $G$ with the character $\chi_\pi$. Prove that $\pi$ is irreducible if and only if

$$\frac{1}{|G|} \sum_{g \in G} |\chi_\pi(g)|^2 = 1.$$ 

**Solution:** A finite dimensional representation is semi-simple and hence we can express

$$\chi_\pi = \sum_{\sigma \in \hat{G}} n_\pi(\sigma) \chi_\sigma,$$

where $\hat{G}$ denotes the collection of the isomorphism classes of irreducible representations of $G$, and $n_\pi(\sigma) = \dim \text{Hom}_G(\pi, \sigma)$. Then

$$\frac{1}{|G|} \sum_{g \in G} |\chi_\pi(g)|^2 = \sum_{\rho, \tau \in \hat{G}} n_\pi(\sigma) n_\pi(\tau) \frac{1}{|G|} \sum_{g \in G} \chi_\sigma(g) \overline{\chi_\tau(g)} = \sum_{\sigma \in \hat{G}} n_\pi(\sigma)^2.$$

Note that $n_\pi(\sigma) = 1$ if and only if $n_\pi(\sigma) = 1$ for some $\sigma \in \hat{G}$ and $n_\pi(\tau) = 0$ for $\tau \neq \sigma$, $\tau \in \hat{G}$. This is equivalent of saying that $\pi$ is isomorphic to the irreducible representation $\sigma$. 