QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday August 31, 2021 (Day 1)

1. (A) Let G be a finite group, let V be a representation of G on a finite-dimensional vector space over $\mathbb{C}$, and let $W \subset V$ be a subrepresentation. Show that there is a subrepresentation $W' \subset V$ such that

$$V = W \oplus W'.$$

2. (AG) Consider the varieties in the affine plane $\mathbb{A}^2_{\mathbb{C}}$ with coordinates (x, y) defined by the following polynomials:

1. $X_1 = V(x^2 - 1)$
2. $X_2 = V(x^2 - y)$
3. $X_3 = V(x^2 - y^2)$
4. $X_4 = V(x^2 - y^3)$
5. $X_5 = V(x^2 - y^4)$.

Prove that no two of the varieties X_i are isomorphic. (Note: we are **not** adopting the convention that varieties are assumed irreducible.)

3. (AT) Let D^n be a closed disc in $\mathbb{R}^n$ and $S^{n-1} = \partial D^n$ its boundary. For any topological space X and map $\alpha : S^{n-1} \to X$, we define *the space Y obtained from X by attaching an n-cell via the map α* to be the quotient of the disjoint union $D^n \sqcup X$ by the equivalence relation generated by $p \sim \alpha(p)$ for all $p \in \partial D^n$. Assuming that the Betti numbers of X are finite, show that one of the two following statements holds:

1. the nth Betti number of Y is 1 greater than the nth Betti number of X, and all other Betti numbers are equal; or
2. the $(n-1)$st Betti number of Y is 1 less than the $(n-1)$st Betti number of X, and all other Betti numbers are equal.

4. (CA) Evaluate the series

$$\sum_{n=-\infty}^{\infty} \frac{n^2 + n + 1}{n^4 + 1}$$

by integrating $R(z)\cot \pi z$ for some appropriate rational function $R(z)$ over the boundary of the square $C_n \subset \mathbb{C}$ whose four vertices are $(n+\frac{1}{2})(\pm 1 \pm i)$ and then letting $n \to \infty$.

5. (DG) Let $c > 0$. Consider the catenary C defined by

$$x = c\cosh\left(\frac{z}{c}\right)$$

in the xz-plane. Let S be the catenoid in the xyz-space obtained by rotating the catenary C with respect to the z-axis. Use θ, z as coordinates for S, where θ is from the polar coordinates (r, θ) of the xy-plane. In terms of (θ, z), write down the first and second fundamental forms of S and the mean curvature and Gaussian curvature of S.

6. (RA) Suppose $f : [-1, 1] \to \mathbf{R}$ is a continuous function such that

$$\int_{-1}^{1} x^{2n} f(x)\, dx = 0$$

for each $n = 0, 1, 2, 3, \ldots$. Prove that f is an odd function (i.e., that $f(-x) = -f(x)$ for all $x \in [-1, 1]$).

QUALIFYING EXAMINATION

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Wednesday September 1, 2021 (Day 2)

1. (AT)

(a) Let X and Y be compact, connected, oriented n-manifolds, and $f : X \to Y$ a continuous map. Define the *degree* of the map f.

(b) Let S^n be the unit sphere in $\mathbb{R}^{n+1}$, and let $r_i : S^n \to S^n$ be the reflection in the ith axis; that is, the map

$$(x_0, \dots, x_n) \mapsto (x_0, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_n)$$

What is the degree of r_i?

(c) Let S^n be the unit sphere in $\mathbb{R}^{n+1}$, and let $a : S^n \to S^n$ be the antipodal map sending x to $-x$. What is the degree of a?

2. (CA) Suppose that $f : \{z : 0 < |z| < 1\} \to \mathbb{C}$ is holomorphic and $|f(z)| \leq A|z|^{-3/2}$ for some constant A. Prove that there is a complex constant α such that $g(z) := f(z) - \alpha z^{-1}$ can be extended to a holomorphic function on $\{z : |z| < 1\}$.

3. (DG) Which of the following smooth manifolds:

1. S^2,
2. $\mathbb{RP}^2$ and
3. $S^1 \times S^1$

admit a closed, non-exact differential 1-form? In each case, either argue why such form does not exist or give an example.

4. (RA) Let $\mathbf{T}$ be the torus $(\mathbf{R}/\mathbf{Z})^2$, and let $a : \mathbf{T} \to \mathbf{R}$ be any continuous function. Prove that the $\mathbf{R}$-vector space of solutions of the partial differential equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = af$$

in functions $f : \mathbf{T} \to \mathbf{R}$ is finite dimensional.

5. (A) Consider the polynomial $f(x) = x^4 + 1$.

(a) Prove that the Galois group G of f over $\mathbb{Q}$ has order 4.

(b) Show that G is in fact isomorphic to the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(c) Is there any prime $p > 2$ such that f is irreducible over the finite field of order p?

6. (AG) Let $C \subset \mathbb{P}^3$ be a smooth, irreducible, nondegenerate curve of degree 4.

(a) If the genus of C is 0, show that C is contained in a quadric surface.

(b) If the genus of C is 1, show that C is equal to the intersection of two quadric surfaces.

(c) Show that the genus of C cannot be greater than 1.

QUALIFYING EXAMINATION

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Thursday September 2, 2021 (Day 3)

1. (DG) Let a_{ij} for $1 \leq i \leq n-1$ and $1 \leq j \leq n$ be real constants. For $1 \leq i \leq n-1$ consider the vector field

$$X_i = \left(\underbrace{0, \cdots, 0, 1, 0 \cdots, 0}_{1 \text{ in } i^{\text{th}} \text{ position}}, \sum_{j=1}^{n} a_{ij} x_j \right)$$

on $\mathbb{R}^n$ (with coordinates $x_1, \cdots, x_n$). Let Π be the distribution of the tangent subspace of dimension $n-1$ in $\mathbb{R}^n$ spanned by $X_1, \cdots, X_{n-1}$. Determine the necessary and sufficient condition for Π to be integrable. Express the condition in terms of symmetry properties of the $(n-1) \times (n-1)$ matrix $(a_{ij})_{1 \leq i,j \leq n-1}$ and the relation among the ratios $\frac{a_{ik}}{a_{jk}}$ for $1 \leq i < j \leq n-1$ and $1 \leq k \leq n$.

2. (RA) Suppose U and V are two random variables. We say that U and V are *uncorrelated* if $\mathrm{Cov}(U, V) = \mathbb{E}[UV] - \mathbb{E}[U]\mathbb{E}[V] = 0$.

(a) Is it true that if U and V are uncorrelated, then U and V are independent? Prove it or give a counter example.

(b) Suppose X and Y are distributed by the following bivariate normal distribution with density

$$f(x, y) = \frac{1}{2\pi} \frac{1}{\sqrt{1-\rho^2}} e^{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}},$$

where $0 < \rho < 1$ is a parameter. Let $U = X + aY$ and $V = X + bY$ with $a, b \neq 0$. Find the condition that $\mathrm{Cov}(U, V) = 0$. In this case, prove that U and V are independent (you cannot just cite a theorem).

3. (A) Suppose R is a commutative ring with unit, I an ideal in R, and M a finitely-generated R-module. If $IM = M$, prove that there exists $r \in R$ such that $r - 1 \in I$ and $rM = 0$.

4. (AG) Let $\mathbb{P}^{n^2-1}$ be the variety of nonzero $n \times n$ complex matrices modulo scalars. Consider the set

$$X := \left\{ [A] \in \mathbb{P}^{n^2-1} \mid A \text{ is nilpotent} \right\}.$$

(a) Show that X is a closed subvariety of $\mathbb{P}^{n^2-1}$.

(b) Show that X is irreducible, and find its dimension.

5. (AT) Let M be a connected closed 4-manifold such that $\pi_1(M)$ is perfect; that is, does not have any non-trivial abelian quotients. Determine the possible cohomology groups $H^*(M, \mathbb{Z})$.

6. (CA) Let $a < b$ and $f(z)$ be a continuous function on the closed strip $\{a \leq x \leq b\}$ which is holomorphic on its interior $\{a < x < b\}$, where $z = x + iy$, such that $|f(z)| = O\left(e^{\varepsilon|y|}\right)$ on $\{a \leq x \leq b\}$ for every $\varepsilon > 0$ as $|y| \to \infty$. If $|f(z)| \leq M$ on the boundary $\{x = a \;\text{ or }\; x = b\}$ of the strip $\{a \leq x \leq b\}$ and on the interval $[a, b]$ for some positive number M, prove that $|f(z)| \leq M$ on the entire closed strip $\{a \leq x \leq b\}$.

Hint: Consider

$$g_\varepsilon(z) = e^{\varepsilon iz} f(z) \quad \text{and } h_\varepsilon(z) = e^{-\varepsilon iz} f(z).$$