QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday August 31, 2021 (Day 1)

1. (A) Let G be a finite group, let V be a representation of G on a finite-dimensional vector space over $\mathbb{C}$, and let $W \subset V$ be a subrepresentation. Show that there is a subrepresentation $W' \subset V$ such that

$$V = W \oplus W'.$$

Solution: Choose any complementary subspace $U \subset V$ with $V = W \oplus U$, and let

$$\pi : V \longrightarrow W$$

be the corresponding projection onto the first component. Define a new linear map $\pi' : V \longrightarrow W$ by averaging π over the group G—that is,

$$\pi'(v) := \frac{1}{|G|} \sum_{g \in G} g\pi(g^{-1}v).$$

This is a G-equivariant map such that, for any $w \in W$,

$$\pi'(w) = w.$$

Its kernel $W' := \ker \pi'$ is G-invariant, of complementary dimension to W, and has the property that $W \cap W' = 0$. Therefore

$$V = W \oplus W'.$$

2. (AG) Consider the varieties in the affine plane $\mathbb{A}^2_{\mathbb{C}}$ with coordinates (x, y) defined by the following polynomials:

1. $X_1 = V(x^2 - 1)$
2. $X_2 = V(x^2 - y)$
3. $X_3 = V(x^2 - y^2)$
4. $X_4 = V(x^2 - y^3)$
5. $X_5 = V(x^2 - y^4)$.

Prove that no two of the varieties X_i are isomorphic. (Note: we are **not** adopting the convention that varieties are assumed irreducible.)

Solution: First off, the varieties X_2 and X_4 are irreducible, whereas the other three are reducible. Since X_2 is nonsingular and X_4 is singular, they are not isomorphic to each other.

Among the varieties X_1, X_3 and X_5, the first is nonsingular whereas the other two are singular. And finally, in the case of X_3, the intersection of the two irreducible components is transverse, while in X_5 the two irreducible components are tangent at their point of intersection.

3. (AT) Let D^n be a closed disc in $\mathbb{R}^n$ and $S^{n-1} = \partial D^n$ its boundary. For any topological space X and map $\alpha : S^{n-1} \to X$, we define *the space Y obtained from X by attaching an n-cell via the map α* to be the quotient of the disjoint union $D^n \sqcup X$ by the equivalence relation generated by $p \sim \alpha(p)$ for all $p \in \partial D^n$. Assuming that the Betti numbers of X are finite, show that one of the two following statements holds:

1. the nth Betti number of Y is 1 greater than the nth Betti number of X, and all other Betti numbers are equal; or
2. the $(n-1)$st Betti number of Y is 1 less than the $(n-1)$st Betti number of X, and all other Betti numbers are equal.

Solution: We consider the covering of Y by the two open sets U and V, where $U = Y \setminus \{0\}$ is the complement in Y of the image of the origin $0 \in D^n$, and V is the image in Y of the open disc $D^n \setminus S^{n-1}$. Here V is contractible, so its reduced homology is 0, and V may be retracted back to X, so its reduced homology is the same as that of X. Finally, the intersection $U \cap V$ has the homotopy type of S^{n-1}, so its reduced homology is $\mathbb{Z}$ in degree $n-1$ and 0 otherwise. The relevant part of the Mayer-Vietoris sequence is thus

$$0 \to H_n(X) \to H_n(Y) \to H_{n-1}(S^{n-1}) \cong \mathbb{Z} \to^{\alpha_*} H_{n-1}(X) \to H_{n-1}(Y) \to 0.$$

If the rank of the map α_* is zero—that is, if the image in $H_{n-1}(X)$ of the fundamental class of S^{n-1} is torsion—then the first statement holds; if the rank of α_* is 1, the second holds.

4. (CA) Evaluate the series

$$\sum_{n=-\infty}^{\infty} \frac{n^2+n+1}{n^4+1}$$

by integrating $R(z)\cot\pi z$ for some appropriate rational function $R(z)$ over the boundary of the square $C_n \subset \mathbb{C}$ whose four vertices are $(n+\frac{1}{2})(\pm 1 \pm i)$ and then letting $n \to \infty$.

Solution Since

$$\cot\pi z = i\,\frac{e^{i\pi z}+e^{-i\pi z}}{e^{i\pi z}-e^{-i\pi z}} = i\,\frac{e^{-\pi y}e^{i\pi x}+e^{\pi y}e^{-i\pi x}}{e^{-\pi y}e^{i\pi x}-e^{\pi y}e^{-i\pi x}},$$

by looking at $y \to \infty$ and $y \to -\infty$ separately, we conclude from

$$\cot\pi(z+2) = \cot\pi z$$

that $\cot\pi z$ is uniformly bounded on C_n (independent of n). Let

$$f(z) = \frac{z^2+z+1}{z^4+1}\,\pi\cot\pi z.$$

From

$$\lim_{n\to\infty}\sup_{z\in C_n}\frac{z^2+z+1}{z^4+1} = O\left(\frac{1}{n^2}\right)$$

and the length of C_n of order $O(n)$, it follows that

$$\lim_{n\to\infty}\int_{C_n} f(z)\,dz = 0$$

and the sum of the residues of $f(z)$ on $\mathbb{C}$ vanishes. The poles of f are all simple poles and are at $z \in \mathbb{Z}$ and the four roots $e^{\frac{ik\pi}{4}}$ $(k = 1,3,5,7)$ of $z^4+1=0$. The residue at $z=n$ is $\frac{n^2+n+1}{n^4+1}$ and the residue at $e^{\frac{ik\pi}{4}}$ is

$$\left(\frac{z^2+z+1}{4z^3}\pi\cot\pi z\right)_{z=e^{\frac{ik\pi}{4}}}.$$

The sum of the four residues at $e^{\frac{ik\pi}{4}}$ $(k=1,3,5,7)$ is

$$-\frac{i\pi}{\sqrt{2}}\left(\cot\left(\pi e^{\frac{i\pi}{4}}\right)+\cot\left(\pi e^{\frac{3i\pi}{4}}\right)\right).$$

Thus,

$$\sum_{n=-\infty}^{\infty}\frac{n^2+n+1}{n^4+1} = \frac{i\pi}{\sqrt{2}}\left(\cot\left(\pi e^{\frac{i\pi}{4}}\right)+\cot\left(\pi e^{\frac{3i\pi}{4}}\right)\right).$$

5. (DG) Let $c>0$. Consider the catenary C defined by

$$x = c\cosh\left(\frac{z}{c}\right)$$

in the xz-plane. Let S be the catenoid in the xyz-space obtained by rotating the catenary C with respect to the z-axis. Use θ, z as coordinates for S, where θ is from the polar coordinates (r, θ) of the xy-plane. In terms of (θ, z), write down the first and second fundamental forms of S and the mean curvature and Gaussian curvature of S.

Solution: The parametric equations for S are

$$\begin{aligned} x &= c\cosh\left(\frac{z}{c}\right)\cos\theta, \\ y &= c\cosh\left(\frac{z}{c}\right)\sin\theta, \\ z &= z. \end{aligned}$$

The first fundamental form $I = Ed\theta^2 + 2Fd\theta dz + Gdz^2$ is

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= \left(-c\cosh\left(\frac{z}{c}\right)\sin\theta d\theta + \sinh\left(\frac{z}{c}\right)\cos\theta dz\right)^2 \\ &\quad + \left(c\cosh\left(\frac{z}{c}\right)\cos\theta d\theta + \sinh\left(\frac{z}{c}\right)\sin\theta dz\right)^2 + dz^2 \\ &= c^2\cosh^2\left(\frac{z}{c}\right)d\theta^2 + \cosh^2\left(\frac{z}{c}\right)dz^2 \end{aligned}$$

with

$$\begin{aligned} E &= c^2\cosh^2\left(\frac{z}{c}\right), \\ F &= 0, \\ G &= \cosh^2\left(\frac{z}{c}\right). \end{aligned}$$

To compute the unit normal vector $\vec{n}$, we compute the partial derivatives of the radius vector $\vec{r}$ with respect θ and z,

$$\begin{aligned} \vec{r}_\theta &= \left(-c\cosh\left(\frac{z}{c}\right)\sin\theta, c\cosh\left(\frac{z}{c}\right)\cos\theta, 0\right) \\ \vec{r}_z &= \left(\sinh\left(\frac{z}{c}\right)\cos\theta, \sinh\left(\frac{z}{c}\right)\sin\theta, 1\right), \end{aligned}$$

to form

$$\vec{r}_\theta \times \vec{r}_z = \left(c\cosh\left(\frac{z}{c}\right)\cos\theta, c\cosh\left(\frac{z}{c}\right)\sin\theta, -c\sinh\left(\frac{z}{c}\right)\cosh\left(\frac{z}{c}\right)\right).$$

The length of $\vec{r}_\theta \times \vec{r}_z$ is equal to $\sqrt{EG - F^2} = c\cosh^2\left(\frac{z}{c}\right)$ so that

$$\vec{n} = \left(\cosh\left(\frac{z}{c}\right)^{-1}\cos\theta, \cosh\left(\frac{z}{c}\right)^{-1}\sin\theta, -\sinh\left(\frac{z}{c}\right)\cosh\left(\frac{z}{c}\right)^{-1}\right).$$

To obtain the coefficients L, M, N of the second fundamental form $II = Ldz^2 + 2Mdzd\theta + Nd\theta^2$, we compute the partial derivatives of the radius vector $\vec{r}$,

$$\vec{r}_{\theta\theta} = \left(-c\cosh\left(\frac{z}{c}\right)\cos\theta,\ -c\cosh\left(\frac{z}{c}\right)\sin\theta,\ 0\right),$$
$$\vec{r}_{\theta z} = \left(-\sinh\left(\frac{z}{c}\right)\sin\theta,\ \sinh\left(\frac{z}{c}\right)\cos\theta,\ 0\right),$$
$$\vec{r}_{zz} = \left(\frac{1}{c}\cosh\left(\frac{z}{c}\right)\cos\theta,\ \frac{1}{c}\cosh\left(\frac{z}{c}\right)\sin\theta,\ 0\right).$$

The coefficients L, M, N of the second fundamental form are given by

$$L = \vec{n}\cdot\vec{r}_{\theta\theta} = -c,$$
$$M = \vec{n}\cdot r_{\theta z} = 0,$$
$$N = \vec{n}\cdot\vec{r}_{zz} = \frac{1}{c}.$$

The mean curvature of S is

$$\frac{1}{2}\frac{LG - 2MF + NE}{EG - F^2} = \frac{1}{2}\frac{(-c)\cosh^2\left(\frac{z}{c}\right) + \frac{1}{c}c^2\cosh^2\left(\frac{z}{c}\right)}{c^2\cosh^2\left(\frac{z}{c}\right)\cosh^2\left(\frac{z}{c}\right)} = 0.$$

The Gaussian curvature of S is

$$\frac{LN - M^2}{EG - F^2} = \frac{(-c)\frac{1}{c}}{c^2\cosh^2\left(\frac{z}{c}\right)\cosh^2\left(\frac{z}{c}\right)} = \frac{-1}{c^2\cosh^4\left(\frac{z}{c}\right)}.$$

6. (RA) Suppose $f : [-1,1] \to \mathbf{R}$ is a continuous function such that

$$\int_{-1}^{1} x^{2n} f(x)\,dx = 0$$

for each $n = 0, 1, 2, 3, \ldots$. Prove that f is an odd function (i.e., that $f(-x) = -f(x)$ for all $x \in [-1,1]$).

Solution: Let $g : [-1,1] \to \mathbf{R}$ be the continuous function defined by $g(x) = f(x) + f(-x)$. We prove that g is the zero function, which is equivalent to the desired $f(-x) = -f(x)$.

First note that $\int_{-1}^{1} x^m g(x)\,dx = 0$ for each $m = 0, 1, 2, 3, \ldots$; this is automatic for m odd, and follows from the hypothesis for m even. By linearity it follows that $\int_{-1}^{1} P(x)g(x)\,dx = 0$ for all polynomials P. By the Weierstrass approximation theorem there exists a sequence $\{P_k\}_{k=1}^{\infty}$ of polynomials such that $P_k(x) \to g(x)$ uniformly for all $x \in [-1,1]$. Since g is bounded (continuous function on a compact set), it follows that

$$\int_{-1}^{1} g(x)^2\,dx = \lim_{k\to\infty}\int_{-1}^{1} P(x)g(x)\,dx.$$

Hence $\int_{-1}^{1} g(x)^2 \, dx = 0$. Since g is continuous and real valued, it thus vanishes identically, and we are done.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Wednesday September 1, 2021 (Day 2)

1. (AT)

(a) Let X and Y be compact, connected, oriented n-manifolds, and $f : X \to Y$ a continuous map. Define the *degree* of the map f.

(b) Let S^n be the unit sphere in $\mathbb{R}^{n+1}$, and let $r_i : S^n \to S^n$ be the reflection in the ith axis; that is, the map

$$(x_0, \dots, x_n) \mapsto (x_0, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_n)$$

What is the degree of r_i?

(c) Let S^n be the unit sphere in $\mathbb{R}^{n+1}$, and let $a : S^n \to S^n$ be the antipodal map sending x to $-x$. What is the degree of a?

Solution: For the first part, by hypothesis we have $H_n(X) = H_n(Y) \cong \mathbb{Z}$, where we choose the identification so that the generator $1 \in \mathbb{Z}$ corresponds to the fundamental class given by the orientation. The map $f_* : H_n(X) \to H_n(Y)$ is then simply multiplication by an integer d; the degree of the map is defined to be this integer d.

As for the second part, the reflection r_i is an orientation-reversing automorphism of S^n, so its degree is -1. And for the third part, we note that a is the composition of the $n+1$ reflections $r_0, \dots, r_n$, so its degree is $(-1)^{n+1}$.

2. (CA) Suppose that $f : \{z : 0 < |z| < 1\} \to \mathbb{C}$ is holomorphic and $|f(z)| \leq A|z|^{-3/2}$ for some constant A. Prove that there is a complex constant α such that $g(z) := f(z) - \alpha z^{-1}$ can be extended to a holomorphic function on $\{z : |z| < 1\}$.

Solution: For $0 < a < |z| < b < 1$, we can write

$$2\pi i f(z) = \int_{|w|=b} \frac{f(w)}{z-w} \mathrm{d}w - \int_{|w|=a} \frac{f(w)}{z-w} \mathrm{d}w$$

Notice that

$$\int_{|w|=a} \frac{f(w)}{z-w} \mathrm{d}w = \frac{1}{z} \int_{|w|=a} f(w) \mathrm{d}w - \frac{1}{z} \int_{|w|=a} O(w/z) f(w) \mathrm{d}w$$

By assumption, the last term can be estimated by

$$\frac{1}{|z|}\int_{|w|=a} O(w/z)|f(w)||\mathrm{d}w| \leq \frac{A}{|z|^2}\sqrt{a}$$

As $a \to 0$, the last term vanishes. Thus we have

$$2\pi i f(z) + \frac{c}{z} = \int_{|w|=b} \frac{f(w)}{z-w}\mathrm{d}w, \quad c = \int_{|w|=a} f(w)\mathrm{d}w$$

Notice that c is independent of the choice of a. The right hand side defines a holomorphic function near $z = 0$.

3. (DG) Which of the following smooth manifolds:

1. S^2,
2. $\mathbb{RP}^2$ and
3. $S^1 \times S^1$

admit a closed, non-exact differential 1-form? In each case, either argue why such form does not exist or give an example.

Solution: By deRham's theorem, if M is a manifold, then the real-valued singular cohomology groups $H^*(M,\mathbb{R})$are isomorphic to the cohomology of the complex of differential forms. Thus, it follows that M admits a closed, non-exact differential 1-form if and only if $H^1(M,\mathbb{R}) \neq 0$.

Since $H^1(S^2,\mathbb{R}) = 0$ and $H^1(\mathbb{RP}^2,\mathbb{R}) = 0$, these are no such forms in these cases.

In the last case, we have $H^1(S^1 \times S^1,\mathbb{R}) \simeq \mathbb{R} \oplus \mathbb{R}$, so such forms exist. As an explicit example, let us choose a diffeomorphism

$$(\theta, \rho) : S^1 \times S^1 \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}.$$

Then, θ can be considered as a real-valued function, well-defined up to adding an integral constant, so that $d\theta$ is a well-defined differential 1-form on $S^1 \times S^1$. By definition, $d\theta$ is locally a differential of a real-valued function, so it is closed. On the other hand, it is not exact, as its integral around the loop corresponding to $\mathbb{R}/\mathbb{Z} \times \{e\}$ is not zero.

4. (RA) Let $\mathbf{T}$ be the torus $(\mathbf{R}/\mathbf{Z})^2$, and let $a : \mathbf{T} \to \mathbf{R}$ be any continuous function. Prove that the $\mathbf{R}$-vector space of solutions of the partial differential equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = af$$

in functions $f : \mathbf{T} \to \mathbf{R}$ is finite dimensional.

Solution: Call that vector space V, and write the differential equation as $(1-a)f = (1-\Delta)f$ where Δ is the Laplacian $\partial^2/\partial x^2 + \partial^2/\partial y^2$. Let A be the operator $(1-\Delta)^{-1}$ on $L^2(\mathbf{T})$, which is compact because it is diagonalized by the Fourier basis with eigenvalues $\left(1 + 4\pi^2(m^2+n^2)\right)^{-1}$, only finitely many of which are outside $(0,\epsilon)$ for any $\epsilon > 0$. Then V is the fixed subspace of $A(1-a)$, which is also compact (composition of the compact operator A with the bounded operator $1-a$). Hence V is finite dimensional (for example, because the closure of its unit ball is compact), Q.E.D.

5. (A) Consider the polynomial $f(x) = x^4 + 1$.

(a) Prove that the Galois group G of f over $\mathbb{Q}$ has order 4.

(b) Show that G is in fact isomorphic to the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(c) Is there any prime $p > 2$ such that f is irreducible over the finite field of order p?

Solution: Let $\alpha \in \mathbb{C}$ be a root of f. Then the full set of roots of f is given by

$$\{\pm\alpha, \pm i\alpha\}.$$

Since $\alpha^2 = \pm i$, it follows that $\mathbb{Q}[\alpha]$ is the splitting field of f over $\mathbb{Q}$, and we have

$$|G| = [\mathbb{Q}[\alpha] : \mathbb{Q}] = 4.$$

For the second part, note that the Galois group G acts transitively on the roots of f, so it contains elements σ and τ such that

$$\sigma(\alpha) = -\alpha \qquad \text{and} \qquad \tau(\alpha) = \alpha^3.$$

Then

$$\begin{aligned} &\sigma^2(\alpha) = \alpha, \text{ and} \\ &\tau^2(\alpha) = \alpha^9 = (-1)^2 \cdot \alpha = \alpha. \end{aligned}$$

Since G is a group of order 4 which contains two elements of order 2, it must be isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Finally, the arguments in parts (a) and (b) show that, if $\mathbb{F}$ is a field of characteristic not equal to 2 or 3 over which f is irreducible, the Galois group of f over $\mathbb{F}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. However, the Galois group of any finite extension of $\mathbb{F}_p$ is cyclic. Therefore, f cannot be irreducible over $\mathbb{F}_p$.

Alternatively, we could argue that the cyclic group $\mathbb{F}_{p^2}^\times$ has order p^2-1, which is always congruent to $0 \pmod 8$. This implies that there is an element $\alpha \in \mathbb{F}_{p^2}^\times$ of order 8. Then $\alpha^4 = -1$, so α is a root of the polynomial $f(x) \in \mathbb{F}_p[x]$.

Suppose that $f(x)$ is irreducible over $\mathbb{F}_p$. Then $\mathbb{F}_p(\alpha)$ is the splitting field of f over $\mathbb{F}_p$ and it has degree 4. We get

$$2 = [\mathbb{F}_{p^2} : \mathbb{F}_p] = [\mathbb{F}_{p^2} : \mathbb{F}_p(\alpha)][\mathbb{F}_p(\alpha) : \mathbb{F}_p] = [\mathbb{F}_{p^2} : \mathbb{F}_p(\alpha)] \cdot 4$$

—a contradiction!

6. (AG) Let $C \subset \mathbb{P}^3$ be a smooth, irreducible, nondegenerate curve of degree 4.

(a) If the genus of C is 0, show that C is contained in a quadric surface.

(b) If the genus of C is 1, show that C is equal to the intersection of two quadric surfaces.

(c) Show that the genus of C cannot be greater than 1.

Solution: For both (a) and (b), the key is to look at the restriction map $\rho : H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_C(2))$.

In either case, we know that $h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = 10$. As for $\mathcal{O}_C(2)$, this is a line bundle/invertible sheaf of degree $2 \times 4 = 8$ on C; if the genus of C is 0, then by Riemann-Roch we have $h^0(\mathcal{O}_C(2)) = 9$, and so the map ρ must have a kernel; thus C lies on a quadric. Similarly, if the genus of C is 1, we have $h^0(\mathcal{O}_C(2)) = 8$, so C must lie on at least two linearly independent quadrics Q and Q'. Since C is nondegenerate, these quadrics must be irreducible, and so by Bezout we must have $Q \cap Q' = C$.

There are many ways to do part (c); probably the simplest would be to argue that for a general point $p \in C$, the projection map $\pi_p : C \to \mathbb{P}^2$ maps C birationally onto a plane cubic curve, which will either have genus 1 (if it's smooth) or 0 (if it's singular).

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Thursday September 2, 2021 (Day 3)

1. (DG) Let a_{ij} for $1 \le i \le n-1$ and $1 \le j \le n$ be real constants. For $1 \le i \le n-1$ consider the vector field

$$X_i = \left(\underbrace{0, \cdots, 0, 1, 0 \cdots, 0}_{1 \text{ in } i^{\text{th}} \text{ position}}, \sum_{j=1}^{n} a_{ij} x_j \right)$$

on $\mathbb{R}^n$ (with coordinates $x_1, \cdots, x_n$). Let Π be the distribution of the tangent subspace of dimension $n-1$ in $\mathbb{R}^n$ spanned by $X_1, \cdots, X_{n-1}$. Determine the necessary and sufficient condition for Π to be integrable. Express the condition in terms of symmetry properties of the $(n-1) \times (n-1)$ matrix $(a_{ij})_{1 \le i,j \le n-1}$ and the relation among the ratios $\frac{a_{ik}}{a_{jk}}$ for $1 \le i < j \le n-1$ and $1 \le k \le n$.

Solution: Write

$$X_i = \frac{\partial}{\partial x_i} + \left(\sum_{j=1}^{n} a_{ij} x_j \right) \frac{\partial}{\partial x_n}$$

for $1 \le i \le n-1$. By Frobenius theorem, integrability of Π is equivalent to $[X_i, X_j]$ being spanned by $X_1, \cdots, X_{n-1}$ for $1 \le i < j \le n-1$. Since

$$[X_i, X_j] = \left(a_{ji} + \left(\sum_{k=1}^{n} a_{ik} x_k \right) a_{jn} - a_{ij} - \left(\sum_{k=1}^{n} a_{jk} x_k \right) a_{in} \right) \frac{\partial}{\partial x_n}$$

has zero coefficients for $\frac{\partial}{\partial x_k}$ for $1 \le k \le n-1$, the integrability condition can be rewritten as the vanishing of $[X_i, X_j]$ for $1 \le i < j \le n-1$, which means

$$a_{ji} + \left(\sum_{k=1}^{n} a_{ik} x_k \right) a_{jn} = a_{ij} + \left(\sum_{k=1}^{n} a_{jk} x_k \right) a_{in}.$$

Equating the coefficients, we obtain $a_{ji} = a_{ij}$ and $a_{ik} a_{jn} = a_{jk} a_{in}$ for $1 \le i < j \le n-1$ and $1 \le k \le n$. The necessary and sufficient condition is that the $(n-1) \times (n-1)$ matrix $(a_{ij})_{1 \le i,j \le n-1}$ is symmetric and for $1 \le i < j \le n-1$ the n ratios $\frac{a_{ik}}{a_{jk}}$ for $1 \le k \le n$ are equal in the sense of equality after cross-multiplication.

2. (RA) Suppose U and V are two random variables. We say that U and V are *uncorrelated* if $\mathrm{Cov}(U,V) = \mathbb{E}[UV] - \mathbb{E}[U]\mathbb{E}[V] = 0$.

(a) Is it true that if U and V are uncorrelated, then U and V are independent? Prove it or give a counter example.

(b) Suppose X and Y are distributed by the following bivariate normal distribution with density

$$f(x,y) = \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}}e^{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}},$$

where $0 < \rho < 1$ is a parameter. Let $U = X + aY$ and $V = X + bY$ with $a, b \neq 0$. Find the condition that $\mathrm{Cov}(U,V) = 0$. In this case, prove that U and V are independent (you cannot just cite a theorem) .

Solution: Define the matrix

$$A^{-1} = \frac{1}{(1-\rho^2)}\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

and denote the column vectors $\mathbf{x} = (x,y)^t$ and $\mathbf{s} = (s,t)^t$. Then the characteristic function

$$\phi(s,t) = \mathbb{E}e^{isX+itY} = \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}}\int e^{i\mathbf{s}\cdot\mathbf{x}}e^{-\mathbf{x}^tA^{-1}\mathbf{x}/2}\mathrm{d}x\mathrm{d}y = e^{-\mathbf{s}^tA\mathbf{s}/2}$$

This gives $\mathrm{Cov}(X,X) = \mathrm{Cov}(Y,Y) = 1$ and $\mathrm{Cov}(X,Y) = \rho$. Now the characteristic function of U, V can be computed from $\phi(s,t)$, i.e.,

$$\mathbb{E}e^{i\alpha U + i\beta V} = \phi(\alpha+\beta, a\alpha + b\beta)$$

The condition $\mathrm{Cov}(U,V) = 0$ will imply that $\mathbb{E}e^{i\alpha U+i\beta V} = \mathbb{E}e^{i\alpha U}\mathbb{E}e^{i\beta V}$ and hence U, V are independent.

3. (A) Suppose R is a commutative ring with unit, I an ideal in R, and M a finitely-generated R-module. If $IM = M$, prove that there exists $r \in R$ such that $r - 1 \in I$ and $rM = 0$.

Solution: Let M be generated by $x_1, \ldots, x_n$. Then IM consists of module elements of the form $\sum_{j=1}^n a_j x_j$ with each $a_j \in I$. Thus $M = IM$ means that each x_i can be written as $\sum_{j=1}^n a_{ij}x_j$ for some $a_{ij} \in I$. Let A be the $n \times n$ matrix (a_{ij}), and $\vec{x}$ the column vector (x_i); then we have $(\mathbf{1} - A)\vec{x} = 0$. Multiplying from the left by $\mathrm{adj}\, A$, we deduce that $\det(\mathbf{1} - A)\cdot\vec{x} = 0$, and thus that $\det(\mathbf{1} - A) \cdot M = 0$. But the ring element $\det(\mathbf{1} - A)$ is in $1 + I$ because $\mathbf{1} - A \equiv \mathbf{1} \bmod I$. $\diamondsuit$

Remark. This is not just a random trick: the result is (depending on preferred terminology) either Nakayama's lemma or a key step in the proof of Nakayama's lemma.

4. (AG) Let $\mathbb{P}^{n^2-1}$ be the variety of nonzero $n \times n$ complex matrices modulo scalars. Consider the set

$$X := \left\{ [A] \in \mathbb{P}^{n^2-1} \mid A \text{ is nilpotent} \right\}.$$

(a) Show that X is a closed subvariety of $\mathbb{P}^{n^2-1}$.

(b) Show that X is irreducible, and find its dimension.

Solution: For the first part, observe that for any n-by-n matrix A, the characteristic polynomial of A is

$$\mathrm{char}_A(T) = T^n + p_1(A)T^{n-1} + \ldots + p_{n-1}(A)T + p_n(A),$$

where the coefficients $p_1(A), \ldots, p_n(A)$ are homogeneous polynomials in the entries of A. The matrix A is nilpotent if and only if $A^n = 0$. In other words, the variety X is cut out by the vanishing of the homogeneous polynomials $p_1, \ldots, p_n$.

For the second, let $\mathcal{F}$ be the variety of complete flags in $\mathbb{C}^n$—that is, let $\mathrm{Gr}(k, n)$ be the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$ and let

$$\mathcal{F} := \{V_\bullet = (V_0, V_1, \ldots, V_n) \mid V_k \in \mathrm{Gr}(k, n) \text{ and } V_k \subset V_{k+1}\}.$$

Note that

$$\dim \mathcal{F} = \frac{n(n-1)}{2}.$$

Define an incidence variety

$$\Lambda := \{(A, V_\bullet) \in X \times \mathcal{F} \mid A \cdot V_\bullet \subset V_\bullet\}$$

which consists of pairs of a nilpotent element A and a flag $V_\bullet$ such that A preserves V. The fiber over the standard flag $E_\bullet$ defined by

$$E_k = \{(x_1, \ldots, x_k, 0, \ldots, 0) \in \mathbb{C}^n\}$$

consists exactly of the upper-triangular nilpotent matrices. Since any complete flag is conjugate to the standard flag, it follows that Λ fibers over $\mathcal{F}$ with fiber the projective space of dimension

$$\frac{n(n-1)}{2} - 1.$$

Therefore Λ is irreducible of dimension $n^2 - n - 1$.

The projection onto the first component

$$\pi : \Lambda \longrightarrow X$$

is surjective, because any nilpotent matrix is conjugate to an upper-triangular one and therefore stabilizes at least one flag. This implies that X is irreducible.

Moreover, recall that any nilpotent matrix of rank $n-1$ is conjugate to the maximal nilpotent Jordan block, which stabilizes only the standard flag $E_\bullet$. Therefore π is generically one-to-one, and it follows that

$$\dim X = n^2 - n - 1.$$

5. (AT) Let M be a connected closed 4-manifold such that $\pi_1(M)$ is perfect; that is, does not have any non-trivial abelian quotients. Determine the possible cohomology groups $H^*(M, \mathbb{Z})$.

Solution: We first claim that M is orientable. Let $p : \widetilde{M} \to M$ be the orientation cover of M. As M is connected, this is completely classified as a covering by any fibre $p^{-1}(m)$ together with the action of $\pi_1(M, m)$.

As the orientation covering is 2-fold, this is the same as a homomorphism $\pi_1(M, m) \to \Sigma_2 \simeq \mathbb{Z}/2$. It this was non-trivial, then it would be surjective, which is impossible since $\pi_1(M)$ is assumed to be perfect. Thus, we deduce that the orientation covering is the trivial 2-fold covering, so that M is orientable.

We will now determine the possible homology groups. Orientability tells us that $H^4(M, \mathbb{Z}) \simeq \mathbb{Z}$ and similarly $H_4(M, \mathbb{Z}) \simeq \mathbb{Z}$.

By Hurewicz theorem, we have $H_1(M, \mathbb{Z}) \simeq \pi_1(M)^{ab}$. By the perfectness assumption, the latter vanishes, and hence so does the former.

By universal coefficient theorem combined with vanishing of H_1, we deduce that

$$H^2(M, \mathbb{Z}) \simeq Hom(H_2(M, \mathbb{Z}), \mathbb{Z}).$$

The latter group is torsion-free, and we deduce that $H^2(M, \mathbb{Z})$ is a torsion free abelian group, hence finite free rank as M is compact. The Poincare duality isomorphism $H_2(M, \mathbb{Z}) \simeq H^2(M, \mathbb{Z})$ allows us to deduce that the second homology group is also free of finite rank.

Similarly, we have Poincare isomorphism $H_3(M, \mathbb{Z}) \simeq H^1(M, \mathbb{Z})$ and a universal coefficient isomorphism $H^1(M, \mathbb{Z}) \simeq Hom(H_1(M, \mathbb{Z}), \mathbb{Z})$. The last group vanishes, and we deduce the same is true for the third homology groups.

These shows that the possible homology groups of M are respectively

$$\mathbb{Z}, 0, A, 0, \mathbb{Z}$$

where A is free of finite rank. All of these can be realized by a connected sum of complex projective planes.

6. (CA) Let $a < b$ and $f(z)$ be a continuous function on the closed strip $\{a \leq x \leq b\}$ which is holomorphic on its interior $\{a < x < b\}$, where $z = x + iy$, such that $|f(z)| = O\left(e^{\varepsilon|y|}\right)$ on $\{a \leq x \leq b\}$ for every $\varepsilon > 0$ as $|y| \to \infty$. If $|f(z)| \leq M$ on the boundary $\{x = a \ \text{ or } \ x = b\}$ of the strip $\{a \leq x \leq b\}$ and on the interval $[a, b]$ for some positive number M, prove that $|f(z)| \leq M$ on the entire closed strip $\{a \leq x \leq b\}$.

Hint: Consider

$$g_\varepsilon(z) = e^{\varepsilon i z} f(z) \quad \text{and } h_\varepsilon(z) = e^{-\varepsilon i z} f(z).$$

Solution: Fix arbitrarily $\varepsilon > 0$. Let $C_\varepsilon > 0$ such that $|f(x+iy)| \leq C_\varepsilon e^{\frac{\varepsilon}{2}|y|}$ on $\{a \leq x \leq b\}$ for any $y \in \mathbb{R}$. Since

$$|g_\varepsilon(a+iy)| = e^{-\varepsilon y}\, |f(a+iy)| \leq e^{-\varepsilon y}\, C_\varepsilon e^{\frac{\varepsilon}{2} y} \leq M$$

and

$$|g_\varepsilon(b+iy)| = e^{-\varepsilon y}\, |f(b+iy)| \leq e^{-\varepsilon y}\, C_\varepsilon e^{\frac{\varepsilon}{2} y} \leq M$$

when $y \geq T_\varepsilon$ for some sufficiently large positive number T_ε. By the maximum modulus principle applied to $g_\varepsilon(z)$ on the rectangle with vertices

$$a, \ b, \ b + iT, \ a + iT$$

when $T \geq T_\varepsilon$, we conclude that $|g_\varepsilon(z)| \leq M$ on the half strip

$$\{a \leq x \leq b\} \cap \{y \geq 0\}.$$

Passing to limit as $\varepsilon \to 0^+$, we obtain $|f(z)| \leq M$ on the half strip

$$\{a \leq x \leq b\} \cap \{y \geq 0\}.$$

Repeat the same argument with $g_\varepsilon(z)$ replaced by $h_\varepsilon(z)$ and with the condition $y \geq T_\varepsilon$ replaced by $y \leq -S_\varepsilon$ for some sufficiently large positive number S_ε. Analogously we get the conclusion that $|f(z)| \leq M$ on the half strip

$$\{a \leq x \leq b\} \cap \{y \leq 0\}.$$