QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Tuesday September 1, 2020 (Day 1)

1. (AG) Let X be a smooth projective curve of genus g, and let $p \in X$ be a point. Show that there exists a nonconstant rational function f which is regular everywhere except for a pole of order $\leq g+1$ at p.

2. (CA) Let $U \subset \mathbb{C}$ be an open set containing the closed unit disc $\overline{\Delta} = \{z \in \mathbb{C} : |z| \leq 1\}$, and suppose that f is a function on U holomorphic except for a simple pole at z_0 with $|z_0| = 1$. Show that if

$$\sum_{n=0}^{\infty} a_n z^n$$

denotes the power series expansion of f in the open unit disk, then

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = z_0.$$

3. (RA) Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers that converges to some $A \in \mathbb{R}$. Prove that $(1-x)\sum_{n=0}^{\infty} a_n x^n \to A$ as x approaches 1 from below.

4. (A) Prove that every finite group of order $72 = 2^3 \cdot 3^2$ is not a simple group.

5. (AT) Let X be a topological space and $A \subset X$ a subset with the induced topology. Recall that a *retraction* of X onto A is a continuous map $f : X \to A$ such that $f(a) = a$ for all $a \in A$.

Let $I = [0,1] \subset \mathbb{R}$ be the closed unit interval, and

$$M = I \times I/(0,y) \sim (1, 1-y)\ \forall\, y \in I$$

the closed Möbius strip; by the *boundary* of the Möbius strip we will mean the image of $I \times \{0,1\}$ in M. Show that there does not exist a retraction of the Möbius strip onto its boundary.

6. (DG) Let S be a surface of revolution

$$\mathbf{r}(u,v) = (x(u,v),\ y(u,v),\ z(u,v)) = (v\cos u, v\sin u,\ f(v))$$

where $0 < v < \infty$ and $0 \le u \le 2\pi$ and $f(v)$ is a C^∞ function on $(0,\infty)$. Determine the set of all $0 \le \alpha \le 2\pi$ such that the curve $u = \alpha$ (called a meridian) is a geodesic of S, and determine the set of all $\beta > 0$ such that the curve $v = \beta$ (called a parallel) is a geodesic of S.

Hint: To determine whether a meridian or a parallel is a geodesic, parametrize it by its arc-length and use the arc-length equation besides the two second-order ordinary differential equations for a geodesic. For your convenience the formulas for the Christoffel symbols in terms of the first fundamental form $Edu^2 + 2Fdudv + Gdv^2$ are listed below.

$$\Gamma^1_{11} = \frac{GE_u - 2F_u + FE_v}{2(EG-F^2)}. \qquad \Gamma^2_{11} = \frac{2EF_u - EE_v - FE_u}{2(EG-F^2)},$$

$$\Gamma^1_{12} = \frac{GE_v - FG_u}{2(EG-F^2)}, \qquad \Gamma^2_{12} = \frac{EG_u - FE_v}{2(EG-F^2)},$$

$$\Gamma^1_{22} = \frac{2GF_v - GG_u - FG_v}{2(EG-F^2)}, \quad \Gamma^2_{22} = \frac{EG_v - 2FF_v + FG_u}{2(EG-F^2)}$$

where the subscript u or v for the function E, F, or G means partial differentiation of the function with respect to u or v.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday September 2, 2020 (Day 2)

1. (CA) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1} \mathrm{d}x.$$

You need to prove that the error terms vanish in the residue calculation.

2. (AG) Let $X \subset \mathbb{P}^n$ be an irreducible projective variety of dimension k. Let $\mathbb{G}(\ell, n)$ be the Grassmannian of ℓ-planes in $\mathbb{P}^n$ for some $\ell < n-k$, and let $C(X) \subset \mathbb{G}(\ell, n)$ the algebraic variety of ℓ-planes meeting X. Prove that $C(X)$ is irreducible, and find its dimension.

3. (RA) Let $\{f_n\}$ be a sequence of functions on $X = (0,1) \subset \mathbb{R}$, converging almost everywhere to f. Suppose moreover that $\sup_n \|f_n\|_{L^2(X)} \leq M$ for some M fixed. Under these conditions, answer the following questions by giving a counterexample or proving your answer.

(a) Do we know $\|f\|_{L^2(X)} < \infty$?

(b) Do we know $\lim_{n\to\infty} \|f_n - f\|_{L^2(X)} = 0$? Do we know that

$$\lim_{n\to\infty} \|f_n - f\|_{L^p(X)} = 0 \quad \text{for} \quad 1 < p < 2?$$

(c) If we assume, in addition, that $\lim_{n\to\infty} \|f_n\|_{L^2(X)} = \|f\|_{L^2(X)} < \infty$, do we know that

$$\lim_{n\to\infty} \|f_n - f\|_{L^2(X)} = 0?$$

4. (A) Let R be a commutative ring with 1. Show that if every proper ideal of R is a prime ideal, then R is a field.

5. (AT) Let $D = \{z \in \mathbb{C} : |z| \leq 1\}$ be the closed unit disc in the complex plane, and let X be the space obtained from D by identifying points on the boundary differing by multiplication by powers of $e^{2\pi i/5}$; that is, we let $\sim$ be the equivalence relation on D given by

$$z \sim w \;\text{ if }\; |z| = |w| = 1 \;\text{ and }\; (z/w)^5 = 1.$$

(a) Find the homology groups of X with coefficients in $\mathbb{Z}$.

(b) Find the homology groups of X with coefficients in $\mathbb{Z}/5$.

6. (DG) Suppose G is a compact Lie group with Lie algebra $\mathfrak{g}$. Consider an element $g \in G$, and let $\mathfrak{c} \subset \mathfrak{g}$ be the subalgebra $\mathfrak{c} = \{X | \mathrm{Ad}_g(X) = X\}$. Show there exists some $\epsilon > 0$ such that for all $X \in \mathfrak{g}$ with $|X| < \epsilon$, there exists $Y \in \mathfrak{c}$ such that $g \exp(X)$ is conjugate to $g \exp(Y)$.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Thursday September 3, 2020 (Day 3)

1. (AG) Let $C \subset \mathbb{P}^3$ be an algebraic curve (that is, an irreducible, one-dimensional subvariety of $\mathbb{P}^3$), and suppose that $p_C(m)$ and $h_C(m)$ are its Hilbert polynomial and Hilbert function respectively. Which of the following are possible?

1. $p_C(m) = 3m + 1$ and $h_C(1) = 3$;
2. $p_C(m) = 3m + 1$ and $h_C(1) = 4$.

2. (RA) The weak law of large numbers states that the following is correct: Let $X_1, X_2, \ldots X_n$ be independent random variables such that $|\mu_j| = |\mathbb{E}X_j| \leq 1$ and $\mathbb{E}(X_j - \mu_j)^2 = V_j \leq 1$. Let $S_n = X_1 + \ldots + X_n$. Then for any $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\Big(|\frac{S_n - \sum_j \mu_j}{n}| > \varepsilon\Big) = 0. \qquad (+)$$

Now suppose that we don't know the independence of the sequence $X_1, X_2, \ldots X_n$, but we know that there is a function $g : \{0\} \cup \mathbb{N} \to \mathbb{R}$ with $\lim_{k\to\infty} g(k) = 0$ such that for all $j \geq i$

$$\mathbb{E}X_iX_j = g(j-i)$$

In other words, the correlation functions vanishing asymptotically. Do we know whether the conclusion (+) still holds? Give a counterexample or prove your answer.

3. (CA)

(a) Suppose that both f and g are analytic in a neighborhood of a disk D with boundary circle C. If $|f(z)| > |g(z)|$ for all $z \in C$, prove that f and $f + g$ have the same number of zeros inside C, counting multiplicity.

(b) How many roots of

$$p(z) = z^7 - 2z^5 + 6z^3 - z + 1 = 0$$

are there in the unit disc in $|z| < 1$, again counting multiplicity?

4. (AT) Let $S^1 = \mathbb{R}/\mathbb{Z}$ be a circle, and let S^2 be a two-dimensional sphere. Consider involutions on both, with an involution on S^1 defined by $x \mapsto -x$

for $x \in \mathbb{R}$, and with $j : S^2 \to S^2$ defined by reflection about an equator. Let M be the space of maps that respects these involutions, i.e.

$$M = \{f : S^1 \to S^2 \mid f(-x) = j(f(x))\}.$$

Show M is connected but not simply-connected.

5. (DG) Let $\mathbb{H}$ denote the upper half-plane; that is, $\mathbb{H} = \{z \in \mathbb{C} \ : \ \mathrm{Im}\, z > 0\}$, with the metric $\frac{1}{y^2} dxdy$ for $z = x + iy$. Suppose Γ is a group of isometries acting on $\mathbb{H}$ such that $\mathbb{H}/\Gamma$ is a smooth surface S, and you are given that a fundamental domain D for the action of Γ on $\mathbb{H}$ is given as follows:

$$D = \{x + iy \in \mathbb{H} \mid -\frac{3}{2} \le x \le \frac{3}{2}, (x-c)^2 + y^2 \ge \frac{1}{9} \text{ for } c \in \{\pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}\}\}.$$

Compute $\chi(S)$ using Gauss-Bonnet. You may use that the (Gaussian) curvature of $\mathbb{H}$ is identically equal to -1.

6. (A) Fix a prime p.
i) Suppose F is a field of characteristic p, and $c \in F$ is not of the form $a^p - a$ for any $a \in F$. Prove that the polynomial $P(X) = X^p - X - c$ is irreducible and that if x is any root of P then $F(x)$ is a normal extension of F with Galois group isomorphic with $\mathbf{Z}/p\mathbf{Z}$.
ii) Suppose $Q \in \mathbf{Z}[X]$ is a monic polynomial of degree p such that $Q \equiv X^p - X - c \bmod p$ for some integer $c \not\equiv 0 \bmod p$, and that Q has exactly $p-2$ real roots. Prove that the Galois group of Q is the full symmetric group S_p.