1. (AG) Let $X$ be a smooth projective curve of genus $g$, and let $p \in X$ be a point. Show that there exists a nonconstant rational function $f$ which is regular everywhere except for a pole of order $\leq g + 1$ at $p$.

2. (CA) Let $U \subset \mathbb{C}$ be an open set containing the closed unit disc $\Delta = \{z \in \mathbb{C} : |z| \leq 1\}$, and suppose that $f$ is a function on $U$ holomorphic except for a simple pole at $z_0$ with $|z_0| = 1$. Show that if

$$\sum_{n=0}^{\infty} a_n z^n$$

denotes the power series expansion of $f$ in the open unit disk, then

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = z_0.$$ 

3. (RA) Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers that converges to some $A \in \mathbb{R}$. Prove that $(1 - x) \sum_{n=0}^{\infty} a_n x^n \to A$ as $x$ approaches 1 from below.

4. (A) Prove that every finite group of order $72 = 2^3 \cdot 3^2$ is not a simple group.

5. (AT) Let $X$ be a topological space and $A \subset X$ a subset with the induced topology. Recall that a retraction of $X$ onto $A$ is a continuous map $f : X \to A$ such that $f(a) = a$ for all $a \in A$.

Let $I = [0,1] \subset \mathbb{R}$ be the closed unit interval, and

$$M = I \times I/(0,y) \sim (1,1-y) \forall y \in I$$

the closed Möbius strip; by the boundary of the Möbius strip we will mean the image of $I \times \{0,1\}$ in $M$. Show that there does not exist a retraction of the Möbius strip onto its boundary.
6. (DG) Let $S$ be a surface of revolution

$$\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)) = (v \cos u, v \sin u, f(v))$$

where $0 < v < \infty$ and $0 \leq u \leq 2\pi$ and $f(v)$ is a $C^\infty$ function on $(0, \infty)$. Determine the set of all $0 \leq \alpha \leq 2\pi$ such that the curve $u = \alpha$ (called a meridian) is a geodesic of $S$, and determine the set of all $\beta > 0$ such that the curve $v = \beta$ (called a parallel) is a geodesic of $S$.

*Hint:* To determine whether a meridian or a parallel is a geodesic, parametrize it by its arc-length and use the arc-length equation besides the two second-order ordinary differential equations for a geodesic. For your convenience the formulas for the Christoffel symbols in terms of the first fundamental form $Edu^2 + 2Fdu dv + Gdv^2$ are listed below.

$$
\Gamma^1_{11} = \frac{GE_u - 2F_u + FE_v}{2(EG - F^2)}, \quad \Gamma^2_{11} = \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)},
$$
$$
\Gamma^1_{12} = \frac{GE_v - FG_u}{2(EG - F^2)}, \quad \Gamma^2_{12} = \frac{EG_u - FE_v}{2(EG - F^2)},
$$
$$
\Gamma^1_{22} = \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)}, \quad \Gamma^2_{22} = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}
$$

where the subscript $u$ or $v$ for the function $E$, $F$, or $G$ means partial differentiation of the function with respect to $u$ or $v$. 
1. (CA) Evaluate the integral
\[ \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 1} \, dx. \]
You need to prove that the error terms vanish in the residue calculation.

2. (AG) Let \( X \subset \mathbb{P}^n \) be an irreducible projective variety of dimension \( k \). Let \( \mathbb{G}(\ell, n) \) be the Grassmannian of \( \ell \)-planes in \( \mathbb{P}^n \) for some \( \ell < n - k \), and let \( C(X) \subset \mathbb{G}(\ell, n) \) the algebraic variety of \( \ell \)-planes meeting \( X \). Prove that \( C(X) \) is irreducible, and find its dimension.

3. (RA) Let \( \{f_n\} \) be a sequence of functions on \( X = (0, 1) \subset \mathbb{R} \), converging almost everywhere to \( f \). Suppose moreover that \( \sup_n \|f_n\|_{L^2(X)} \leq M \) for some \( M \) fixed. Under these conditions, answer the following questions by giving a counterexample or proving your answer.

   (a) Do we know \( \|f\|_{L^2(X)} < \infty \)?
   (b) Do we know \( \lim_{n \to \infty} \|f_n - f\|_{L^2(X)} = 0 \)? Do we know that \( \lim_{n \to \infty} \|f_n - f\|_{L^p(X)} = 0 \) for \( 1 < p < 2 \)?
   (c) If we assume, in addition, that \( \lim_{n \to \infty} \|f_n\|_{L^2(X)} = \|f\|_{L^2(X)} < \infty \), do we know that \( \lim_{n \to \infty} \|f_n - f\|_{L^2(X)} = 0 \)?

4. (A) Let \( R \) be a commutative ring with 1. Show that if every proper ideal of \( R \) is a prime ideal, then \( R \) is a field.

5. (AT) Let \( D = \{z \in \mathbb{C} : |z| \leq 1\} \) be the closed unit disc in the complex plane, and let \( X \) be the space obtained from \( D \) by identifying points on the boundary differing by multiplication by powers of \( e^{2\pi i/5} \); that is, we let \( \sim \) be the equivalence relation on \( D \) given by
\[ z \sim w \text{ if } |z| = |w| = 1 \text{ and } (z/w)^5 = 1. \]
(a) Find the homology groups of $X$ with coefficients in $\mathbb{Z}$.
(b) Find the homology groups of $X$ with coefficients in $\mathbb{Z}/5$.

6. (DG) Suppose $G$ is a compact Lie group with Lie algebra $\mathfrak{g}$. Consider an element $g \in G$, and let $\mathfrak{c} \subset \mathfrak{g}$ be the subalgebra $\mathfrak{c} = \{X|\text{Ad}_g(X) = X\}$. Show there exists some $\epsilon > 0$ such that for all $X \in \mathfrak{g}$ with $|X| < \epsilon$, there exists $Y \in \mathfrak{c}$ such that $g \exp(X)$ is conjugate to $g \exp(Y)$. 
1. (AG) Let $C \subset \mathbb{P}^3$ be an algebraic curve (that is, an irreducible, one-dimensional subvariety of $\mathbb{P}^3$), and suppose that $p_C(m)$ and $h_C(m)$ are its Hilbert polynomial and Hilbert function respectively. Which of the following are possible?

1. $p_C(m) = 3m + 1$ and $h_C(1) = 3$;
2. $p_C(m) = 3m + 1$ and $h_C(1) = 4$.

2. (RA) The weak law of large numbers states that the following is correct: Let $X_1, X_2, \ldots, X_n$ be independent random variables such that $|\mu_j| = |E X_j| \leq 1$ and $E(X_j - \mu_j)^2 = V_j \leq 1$. Let $S_n = X_1 + \ldots + X_n$. Then for any $\varepsilon > 0$

$$\lim_{n \to \infty} \mathbb{P}\left(|\frac{S_n - \sum j \mu_j}{n}| > \varepsilon\right) = 0.$$  

Now suppose that we don’t know the independence of the sequence $X_1, X_2, \ldots, X_n$, but we know that there is a function $g : \{0\} \cup \mathbb{N} \to \mathbb{R}$ with $\lim_{k \to \infty} g(k) = 0$ such that for all $j \geq i$

$$E X_i X_j = g(j - i).$$

In other words, the correlation functions vanishing asymptotically. Do we know whether the conclusion (+) still holds? Give a counterexample or prove your answer.

3. (CA)

(a) Suppose that both $f$ and $g$ are analytic in a neighborhood of a disk $D$ with boundary circle $C$. If $|f(z)| > |g(z)|$ for all $z \in C$, prove that $f$ and $f + g$ have the same number of zeros inside $C$, counting multiplicity.

(b) How many roots of

$$p(z) = z^7 - 2z^5 + 6z^3 - z + 1 = 0$$

are there in the unit disc in $|z| < 1$, again counting multiplicity?

4. (AT) Let $S^1 = \mathbb{R}/\mathbb{Z}$ be a circle, and let $S^2$ be a two-dimensional sphere. Consider involutions on both, with an involution on $S^1$ defined by $x \mapsto -x$
for $x \in \mathbb{R}$, and with $j : S^2 \to S^2$ defined by reflection about an equator. Let $M$ be the space of maps that respects these involutions, i.e.

$$M = \{ f : S^1 \to S^2 \mid f(-x) = j(f(x)) \}.$$

Show $M$ is connected but not simply-connected.

5. (DG) Let $\mathbb{H}$ denote the upper half-plane; that is, $\mathbb{H} = \{ z \in \mathbb{C} : \text{Im } z > 0 \}$, with the metric $\frac{1}{y^2} dx dy$ for $z = x + iy$. Suppose $\Gamma$ is a group of isometries acting on $\mathbb{H}$ such that $\mathbb{H}/\Gamma$ is a smooth surface $S$, and you are given that a fundamental domain $D$ for the action of $\Gamma$ on $\mathbb{H}$ is given as follows:

$$D = \{ x + iy \in \mathbb{H} \mid -\frac{3}{2} \leq x \leq \frac{3}{2}, (x-c)^2 + y^2 \geq \frac{1}{9} \text{ for } c \in \{ \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3} \} \}.$$

Compute $\chi(S)$ using Gauss-Bonnet. You may use that the (Gaussian) curvature of $\mathbb{H}$ is identically equal to $-1$.

6. (A) Fix a prime $p$.

i) Suppose $F$ is a field of characteristic $p$, and $c \in F$ is not of the form $a^p - a$ for any $a \in F$. Prove that the polynomial $P(X) = X^p - X - c$ is irreducible and that if $x$ is any root of $P$ then $F(x)$ is a normal extension of $F$ with Galois group isomorphic with $\mathbb{Z}/p\mathbb{Z}$.

ii) Suppose $Q \in \mathbb{Z}[X]$ is a monic polynomial of degree $p$ such that $Q \equiv X^p - X - c \mod p$ for some integer $c \neq 0 \mod p$, and that $Q$ has exactly $p - 2$ real roots. Prove that the Galois group of $Q$ is the full symmetric group $S_p$. 