Qualifying Examination

HARVARD UNIVERSITY
Department of Mathematics
Tuesday, January 19, 2016 (Day 1)

PROBLEM 1 (DG)

Let S denote the surface in $\mathbb{R}^3$ where the coordinates (x, y, z) obey $x^2+y^2 = 1+z^2$. This surface can be parametrized by coordinates $t \in \mathbb{R}$ and $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ by the map

$$(t, \theta) \to \psi(t,\theta) = (\sqrt{1+t^2}\cos\theta, \sqrt{1+t^2}\sin\theta, t).$$

a) Compute the induced inner product on the tangent space to S using these coordinates.
b) Compute the Gaussian curvature of the metric that you computed in Part a).
c) Compute the parallel transport around the circle in S where $z = 0$ for the Levi-Civita connection of the metric that you computed in Part a).

PROBLEM 2 (T)

Let X be path-connected and locally path-connected, and let Y be a finite Cartesian product of circles. Show that if $\pi_1(X)$ is finite, then every continuous map from X to Y is null-homotopic. (Hint: recall that there is a fiber bundle $\mathbb{Z} \to \mathbb{R} \to S^1$.)

PROBLEM 3 (AN)

Let K be the field $\mathbb{C}(z)$ of rational functions in an indeterminate z, and let $F \subset K$ be the subfield $\mathbb{C}(u)$ where $u = (z^6 + 1)/z^3$.

a) Show that the field extension K/F is normal, and determine its Galois group.
b) Find all fields E, other than F and K themselves, such that $F \subset E \subset K$. For each E, determine whether the extensions E/F and K/E are normal.

PROBLEM 4 (AG)

The nodal cubic is the curve in $\mathbb{CP}^2$ (denoted by X) given in homogeneous coordinates (x, y, z) by the locus $\{zy^2 = x^2(x+z)\}$.

a) Give a definition of a rational map between algebraic varieties.

b) Show that there is a birational map from X to $\mathbb{CP}^1$.

c) Explain how to resolve the singularity of X by blowing up a point in $\mathbb{CP}^2$.

PROBLEM 5 (RA)

Let $\mathbb{B}$ and $\mathbb{L}$ denote Banach spaces, and let $\|\cdot\|_{\mathbb{B}}$ and $\|\cdot\|_{\mathbb{L}}$ denote their norms.

a) Let $L: \mathbb{B} \to \mathbb{L}$ denote a continuous, invertible linear map and let $\mathfrak{m}: \mathbb{B} \otimes \mathbb{B} \to \mathbb{L}$ denote a linear map such that $\|\mathfrak{m}(\phi \otimes \psi)\|_{\mathbb{L}} \leq \|\phi\|_{\mathbb{B}} \|\psi\|_{\mathbb{B}}$ for all $\phi, \psi \in \mathbb{B}$. Prove the following assertions:

- *There exists a number* $\kappa > 1$ *depending only on* L *such that if* $a \in \mathbb{B}$ *has norm less than* κ^{-2}, *then there is a unique solution to the equation* $L\phi + \mathfrak{m}(\phi \otimes \phi) = a$ *with* $\|\phi\|_{\mathbb{B}} < \kappa^{-1}$.
- *The norm of the solution from the previous bullet is at most* $\kappa \|a\|_{\mathbb{L}}$.

b) Recall that a Banach space is defined to be a *complete*, normed vector space. Is the assertion of Part a) of the first bullet always true if $\mathbb{B}$ is normed but not complete? If not, explain where the assumption that $\mathbb{B}$ is complete enters your proof of Part a).

PROBLEM 6 (CA)

Fix $a \in \mathbb{C}$ and an integer $n \geq 2$. Show that the equation $az^n + z + 1 = 0$ for a complex number z necessarily has a solution with $|z| \leq 2$.

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Wednesday, January 20, 2016 (Day 2)

PROBLEM 1 (DG)

Let k denote a positive integer. A non-optimal version of the Whitney embedding theorem states that any k-dimensional manifold can be embedded into $\mathbb{R}^{2k+1}$. Using this, show that any k-dimensional manifold can be immersed in $\mathbb{R}^{2k}$. (Hint: Compose the embedding with a projection onto an appropriate subspace.)

PROBLEM 2 (T)

Let X be a CW-complex with a single cell in each of the dimensions 0, 1, 2, 3, and 5 and no other cells.

a) What are the possible values of $H_*(X; \mathbb{Z})$? (Note: it is not sufficient to consider $H_n(X; \mathbb{Z})$ for each n independently. The value of $H_1(X; \mathbb{Z})$ may constrain the value of $H_2(X; \mathbb{Z})$, for instance.)

b) Now suppose in addition that X is its own universal cover. What extra information does this provide about $H_*(X; \mathbb{Z})$?

PROBLEM 3 (AN)

Let k be a finite field of characteristic p, and n a positive integer. Let G be the group of invertible linear transformations of the k-vector space k^n. Identify G with the group of invertible $n \times n$ matrices with entries in k (acting from the left on column vectors).

a) Prove that the order of G is $\prod_{m=0}^{n-1}(q^n - q^m)$ where q is the number of elements of k.

b) Let U be the subgroup of G consisting of upper-triangular matrices with all diagonal entries equal 1. Prove that U is a p-Sylow subgroup of G.

c) Suppose $H \subset G$ is a subgroup whose order is a power of p. Prove that there is a basis $(v_1, v_2, \ldots, v_n)$ of k^n such that for every $h \in H$ and every $m \in \{1, 2, 3, \ldots, n\}$, the vector $h(v_m) - v_m$ is in the span of $\{v_d: d < m\}$.

PROBLEM 4 (AG)

Let X be a complete intersection of surfaces of degrees a and b in $\mathbb{CP}^3$. Compute the Hilbert polynomial of X.

PROBLEM 5 (RA)

Let $\mathcal{C}^0$ denote the vector space of continuous functions on the interval $[0,1]$. Define a norm on $\mathcal{C}^0$ as follows: If $f \in \mathcal{C}^0$, then its norm (denoted by $\|f\|$) is

$$\|f\| = \sup_{t\in[0,1]} |f(t)| \ .$$

Let $\mathcal{C}^\infty$ denote the space of smooth functions on [0, 1]. View $\mathcal{C}^\infty$ as a normed, linear space with the norm defined as follows: If $f \in \mathcal{C}^\infty$, then its norm (denoted by $\|f\|_*$) is

$$\|f\|_* = \int_{[0,1]} \left(\left|\frac{d}{dt} f\right| + |f|\right) dt \ .$$

a) Prove that $\mathcal{C}^0$ is Banach space with respect to the norm $\|\cdot\|$. In particular, prove that it is complete.
b) Let ψ denote the 'forgetful' map from $\mathcal{C}^\infty$ to $\mathcal{C}^0$ that sends f to f. Prove that ψ is a bounded map from $\mathcal{C}^\infty$ to $\mathcal{C}^0$, but not a compact map from $\mathcal{C}^\infty$ to $\mathcal{C}^0$.

PROBLEM 6 (CA)

Let $\mathbb{D}$ denote the closed disk in $\mathbb{C}$ where $|z| \le 1$. Fix $R > 0$ and let $\varphi: \mathbb{D} \to \mathbb{C}$ denote a continuous map with the following properties:

i) φ is holomorphic on the interior of $\mathbb{D}$.
ii) $\varphi(0) = 0$ and its z-derivative, φ', obeys $\varphi'(0) = 1$.
iii) $|\varphi| \le R$ for all $z \in \mathbb{D}$.

Since $\varphi'(0) = 1$, there exists $\delta > 0$ such that φ maps the $|z| < \delta$ disk diffeomorphically onto its image. Prove the following:

a) There is a unique solution in [0, 1] to the equation $2R\delta = (1 - \delta)^3$.
b) Let δ_* denote the unique solution to this equation. If $0 < \delta < \delta_*$, then φ maps the $|z| < \delta$ disk diffeomorphically onto its image.

Qualifying Examination

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Thursday, January 21, 2016 (Day 3)

PROBLEM 1 (DG)
Recall that a symplectic manifold is a pair (M, ω), where M is a smooth manifold and ω is a closed nondegenerate differential 2-form on M. (The 2-form ω is called the symplectic form.)

a) Show that if H: M $\to$ R is a smooth function, then there exists a unique vector field, to be denoted by X_H, satisfying $\iota_{X_H}\omega = dH$. (Here, ι denotes the contraction operation.)
b) Supposing that t > 0 is given, suppose in what follows that the flow of X_H is defined for time t, and let ϕ_t denote the resulting diffeomorphism of M. Show that $\phi_t{}^*\omega = \omega$.
c) Denote the Euclidean coordinates on $\mathbb{R}^4$ by (x_1, y_1, x_2, y_2) and use these to define the symplectic form $\omega_0 = dx^1 \wedge dy^1 + dx^2 \wedge dy^2$. Find a function H: $\mathbb{R}^4 \to \mathbb{R}$ such that the diffeomorphism $\phi_{t=1}$ that is defined by the time t = 1 flow of X_H fixes the half space where $x_1 \leq 0$ and moves each point in the half space where $x_1 \geq 1$ by 1 in the y_2 direction.

PROBLEM 2 (T)
Let X denote a finite CW complex and let f: X $\to$ X be a self-map of X. Recall that the Lefschetz trace of f, denoted by $\tau(f)$, is defined by the rule

$$\tau(f) = \sum_{n=0}^{\infty} (-1)^n \operatorname{tr}(f_n : H_n(X;\mathbb{Q}) \to H_n(X;\mathbb{Q}))$$

with f_n denoting the induced homomorphism. Use $\tau(\cdot)$ to answer the following:

a) Does there exist a continuous map from $\mathbb{RP}^2$ to itself with no fixed points? If so, give an example; and if not, give a proof.
b) Does there exist a continuous map from $\mathbb{RP}^3$ to itself with no fixed points? If so, give an example; and if not, give a proof.

PROBLEM 3 (AN)
Let A be the ring $\mathbb{Z}[\sqrt[5]{2016}\,] = \mathbb{Z}[X]/(X^5 - 2016)$. Given that 2017 is prime in $\mathbb{Z}$, determine the factorization of $2017\cdot A$ into prime ideals of A.

PROBLEM 4 (AG)

a) State a version of the Riemann–Roch theorem.

b) Apply this theorem to show that if X is a complete nonsingular curve and $P \in X$ is any point, there is a rational function on X which has a pole at P and is regular on $X-\{P\}$.

PROBLEM 5 (RA)

Let $\wp$ denote a probability measure for a real valued random variable with mean 0. Denote this random variable by x. Suppose that the random variable $|x|$ has mean equal to 2.

a) Given $R > 2$, state a non-trivial upper bound for event that $x \geq R$. (The trivial upper bound is 1.)

b) Give a non-zero lower bound for the standard deviation of x.

c) A function f on $\mathbb{R}$ is Lipshitz when there exists a number $c \geq 0$ such that

$$|f(p) - f(p')| \leq c|p - p'| \quad \text{for any pair } p, p' \in \mathbb{R}.$$

Let $\hat{\wp}$ denote the function on $\mathbb{R}$ whose value at a given $p \in \mathbb{R}$ is the expectation of the random variable e^{ipx}. (This is the *characteristic function of* $\wp$.) Give a rigorous proof that $\hat{\wp}$ is Lipshitz and give an upper bound for c in this case.

d) Suppose that the standard deviation of x is equal to 4. Let N denote an integer greater than 1, and let $\{x_1, \ldots, x_N\}$ denote a set of independent random variables each with probabilities given by $\wp$. Use S_N to denote the random variable $\frac{1}{N}(x_1 + \cdots + x_N)$. The central limit theorem gives an integral that approximates the probability of the event where $S_N \in [-1, 1]$ when N is large. Write this integral.

PROBLEM 6 (CA)

Let $H \subset \mathbb{C}$ denote the open right half plane, thus $H = \{z = x + iy: x > 0\}$. Suppose that $f: H \to \mathbb{C}$ is a bounded, analytic function such that $f(1/n) = 0$ for each positive integer n. Prove that $f(z) = 0$ for all z.

(Hint: Consider the behavior of the sequence of functions $\{h_N(z) = \prod_{n=1}^{N} \frac{z - 1/n}{z + 1/n}\}_{N=1,2,\ldots}$ on H and, in particular, on the positive real axis.}