Qualifying Exams I, 2014 Spring

1. (Algebra) Let $k = \mathbb{F}_q$ be a finite field with q elements. Count the number of monic irreducible polynomials of degree 12 over k.

2. (Algebraic Geometry) (a) Show that the set of lines $L \subset \mathbb{P}^3_{\mathbb{C}}$ may be identified with a quadric hypersurface in $\mathbb{P}^5_{\mathbb{C}}$.

(b) Let $L_0 \subset \mathbb{P}^3_{\mathbb{C}}$ be a given line. Show that the set of lines not meeting L_0 is isomorphic to the affine space $\mathbb{A}^4_{\mathbb{C}}$.

3. (Complex Analysis) (a) Compute

$$\int_{|z|=1} \frac{z^{31}}{(2\bar{z}^{\,2}+3)^2\,(\bar{z}^{\,4}+2)^3}\,dz$$

Note that the integrand is not a meromorphic function.

(b) Evaluate the integral

$$\int_{x=0}^{\infty} \left(\frac{\sin x}{x}\right)^3 dx$$

by using the theory of residues. Justify carefully all the limiting processes in your computation.

4. (Algebraic Topology) Suppose that X is a finite connected CW complex such that $\pi_1(X)$ is finite and nontrivial. Prove that the universal covering $\tilde{X}$ of X cannot be contractible. (*Hint:* Lefschetz fixed point theorem.)

5. (Differential Geometry) Let $\mathbb{P}^2 = (\mathbb{C}^3 - \{0\})/\mathbb{C}^\times$, which is called the complex projective plane.

1. Show that $\mathbb{P}^2$ is a complex manifold by writing down its local coordinate charts and transitions.

2. Define $L \subset \mathbb{P}^2 \times \mathbb{C}^3$ to be the subset containing elements of the form $([x], \lambda x)$, where $x \in \mathbb{C}^3 - \{0\}$ and $\lambda \in \mathbb{C}$. Show that L is the total space of a holomorphic line bundle over $\mathbb{P}^2$ by writing down its local trivializations and transitions. It is called the tautological line bundle.

3. Using the standard Hermitian metric on $\mathbb{C}^3$ or otherwise, construct a Hermitian metric on the tautological line bundle. Express the metric in terms of local trivializations.

6. (Real Analysis) *(Schwartz's Theorem on Perturbation of Surjective Maps by Compact Maps Between Hilbert Spaces).* Let E, F be Hilbert spaces over $\mathbb{C}$, $S : E \to F$ be a compact $\mathbb{C}$-linear map, and $T : E \to F$ be a continuous surjective $\mathbb{C}$-linear map. Prove that the cokernel of $S+T : E \to F$ is finite-dimensional and the image of $S+T : E \to F$ is a closed subspace of F.

Here the compactness of the $\mathbb{C}$-linear map $S : E \to F$ means that for any sequence $\{x_n\}_{n=1}^{\infty}$ in E with $\|x_n\|_E \leq 1$ for all $n \in \mathbb{N}$ there exists some subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $S(x_{n_k})$ converges in F to some element of F as $k \to \infty$.

Hint: Verify first that the conclusion is equivalent to the following equivalent statement for the adjoints $T^*, S^* : F \mapsto E$ for T, S. The kernel of $T^* + S^*$ is finite-dimensional and the image of $T^* + S^*$ is closed. Then prove the equivalent statement.

Qualifying Exams II, 2014 Spring

1. (Algebra) Let A be a finite group of order n, and let $V_1, \cdots, V_k$ be its irreducible representations.

(a) Show that the dimensions of the vector spaces V_i satisfy the equality $\sum_{i=1}^{k}(\dim V_i)^2 = n$.

(b) What are the dimensions of the irreducible representations of the symmetric group S_6 of six elements.

2. (Algebraic Geometry) Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree ≥ 3.

(a) Show that C admits a regular map $f : C \to \mathbb{P}^1$ of degree $d - 1$.

(b) Show that C does not admit a regular map $f : C \to \mathbb{P}^1$ of degree e with $0 < e < d - 1$.

3. (Complex Analysis) Suppose that f is holomorphic in an open set containing the closed unit disk $\{|z| \leq 1\}$ in $\mathbb{C}$, except for a pole at z_0 on the unit circle $\{|z| = 1\}$. Show that if

$$\sum_{n=0}^{\infty} a_n z^n$$

denote the power series expansion of f in the open unit disk $\{|z| < 1\}$, then

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0.$$

4. (Algebraic Topology) Show that if $n > 1$, then every map from the real projective space $\mathbb{RP}^n$ to the n-torus T^n is null-homotopic.

5. (Differential Geometry) Let $\mathbb{S}^2 := \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ be the unit sphere in the Euclidean space. Let $C = \{(r\cos t, r\sin t, h) : t \in \mathbb{R}\}$ be a circle in $\mathbb{S}^2$, where $r, h > 0$ are constants with $r^2 + h^2 = 1$.

1. Compute the holonomy of the sphere $\mathbb{S}^2$ (with the standard induced metric) around the circle C.

2. By using Gauss-Bonnet theorem or otherwise, compute the total curvature

$$\int_D \kappa \, \mathrm{d}A$$

where $D = \mathbb{S}^2 \cap \{z \geq h\}$ is the disc bounded by the circle C, and $\mathrm{d}A$ is the area form of $\mathbb{S}^2$.

6. (Real Analysis) *(Commutation of Differentiation and Summation of Integrals).* Let Ω be an open subset of $\mathbb{R}^d$ and $a < b$ be real numbers. For any positive integer n let $f_n(x, y)$ be a complex-valued measurable function on $\Omega \times (a, b)$. Let $a < c < b$. Assume that the following three conditions are satisfied.

(i) For each n and for almost all $x \in \Omega$ the function $f_n(x, y)$ as a function of y is absolutely continuous in y for $y \in (a, b)$.

(ii) The function $\frac{\partial}{\partial y} f_n(x, y)$ is measurable on $\Omega \times (a, b)$ for each n and the function

$$\sum_{n=1}^{\infty} \left| \frac{\partial}{\partial y} f_n(x, y) \right|$$

is integrable on $\Omega \times (a, b)$.

(iii) The function $f_n(x, c)$ is measurable on Ω for each n and the function $\sum_{n=1}^{\infty} |f_n(x, c)|$ is integrable on Ω.

Prove that the function

$$y \mapsto \int_{x \in \Omega} \sum_{n=1}^{\infty} f_n(x, y) dx$$

is a well-defined function for almost all points y of (a, b) and that

$$\frac{d}{dy} \int_{x \in \Omega} \sum_{n=1}^{\infty} f_n(x, y) dx = \sum_{n=1}^{\infty} \int_{x \in \Omega} \left(\frac{\partial}{\partial y} f_n(x, y) \right) dx$$

for almost all $y \in (a, b)$.

Hint: Use Fubini's theorem to exchange the order of integration and use convergence theorems for integrals of sequences of functions to exchange the order of summation and integration.

Qualifying Exams III, 2014 Spring

1. (Algebra) Prove or disprove: There exists a prime number p such that the principal ideal (p) in the ring of integers $\mathcal{O}_K$ in $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ is a prime ideal

2. (Algebraic Geometry) Let $\Gamma = \{p_1, \cdots, p_5\} \subset \mathbb{P}^2$ be a configuration of 5 points in the plane.

(a) What is the smallest Hilbert function Γ can have?

(b) What is the largest Hilbert function Γ can have?

(c) Find all the Hilbert functions Γ can have.

3. (Complex Analysis) *(Cauchy's Integral Formula for Smooth Functions and Solution of $\bar{\partial}$ Equation).* (a) Let Ω be a bounded domain in $\mathbb{C}$ with smooth boundary $\partial\Omega$. Let f be a C^∞ complex-valued function on some open neighborhood U of the topological closure $\bar{\Omega}$ of Ω in $\mathbb{C}$.

(i) Show that for $a \in \Omega$,

$$f(a) = \frac{1}{2\pi i}\int_{z\in\partial\Omega} \frac{f(z)dz}{z-a} + \frac{1}{2\pi i}\int_{\Omega} \frac{\frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z}}{z-a},$$

where

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \sqrt{-1}\,\frac{\partial f}{\partial y}\right)$$

with $z = x + \sqrt{-1}\,y$ and x, y real.

(ii) Show that $a \in \Omega$,

$$f(a) = -\frac{1}{2\pi i}\int_{z\in\partial\Omega} \frac{f(z)d\bar{z}}{\bar{z}-\bar{a}} + \frac{1}{2\pi i}\int_{\Omega} \frac{\frac{\partial f}{\partial z} dz \wedge d\bar{z}}{\bar{z}-\bar{a}},$$

where

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \sqrt{-1}\,\frac{\partial f}{\partial y}\right).$$

(iii) For $z \in \Omega$ define

$$h(z) = \frac{1}{2\pi i}\int_{\zeta\in\Omega} \frac{f(\zeta)\, d\zeta \wedge d\bar{\zeta}}{\zeta - z}.$$

Show that $\frac{\partial h}{\partial \bar{z}}(z) = f(z)$ on Ω.

Hint: For (i), apply Stokes's theorem to $d\left(f(z)\frac{dz}{z-a}\right)$ on Ω minus a closed disk of radius $\varepsilon > 0$ centered at a and then let $\varepsilon \to 0$.

For the proof of (iii), for any fixed $z \in \Omega$, apply Stokes's theorem to $d\left(f(\zeta)\log|\zeta - z|d\bar{\zeta}\right)$ (with variable ζ) on Ω minus a closed disk of radius $\varepsilon > 0$ centered at z and then let $\varepsilon \to 0$. Then apply $\frac{\partial}{\partial \bar{z}}$ and use (ii).

(b) Let $\mathbb{D}_r$ be the open disk of radius $r > 0$ in $\mathbb{C}$ centered at 0. Prove that for any C^∞ complex-valued function g on $\mathbb{D}_1$ there exists some C^∞ complex-valued function h on $\mathbb{D}_1$ such that $\frac{\partial h}{\partial \bar{z}} = g$ on $\mathbb{D}_1$.

Hint: First use (a)(iii) to show that for $0 < r < 1$ there exists some C^∞ complex-valued function h_r on $\mathbb{D}_1$ such that $\frac{\partial h_r}{\partial \bar{z}} = g$ on $\mathbb{D}_r$. Then use some approximation and limiting process to construct h.

4. (Algebraic Topology) Suppose that X is contractible and that some point a of X has a neighborhood homeomorphic to $\mathbb{R}^k$. Prove that $H_n(X \setminus \{a\}) \simeq H_n(S^{k-1})$ for all n.

5. (Differential Geometry) Let $U_+ = \mathbb{R}^2 - (\mathbb{R}_{\leq 0} \times \{0\})$, $U_- = \mathbb{R}^2 - (\mathbb{R}_{\geq 0} \times \{0\})$, and $U_0 = \mathbb{R}^2 - (\mathbb{R} \times \{0\})$. Let B be obtained by gluing U_+ and U_- over U_0 by the map $\psi : U_0 \to U_0$ defined by

$$\psi(x, y) = (x, y)$$

when $y < 0$, and

$$\psi(x, y) = (x + y, y)$$

when $y > 0$.

1. Show that B is a manifold.

2. Show that the trivial connections on the tangent bundles of U_+ and U_- glue together and give a global connection on the tangent bundle TB. Compute the curvature of this connection.

3. Compute the holonomy of the above connection around the loop $\gamma : [0, 2\pi] \to B$ determined by $\gamma|_{U_+}(\theta) = (\cos\theta, \sin\theta)$ for $\theta \in (0, 2\pi)$.

6. (Real Analysis) *(Bernstein's Theorem on Approximation of Continuous Functions by Polynomials).* Use the probabilistic argument outlined in the two steps below to prove the following theorem of Bernstein. Let f be a real-valued continuous function on $[0,1]$. For any positive integer n let

$$B_n(f;x) = \sum_{j=0}^{n} f\left(\frac{j}{n}\right)\binom{n}{j} x^j(1-x)^{n-j}$$

be the *Bernstein polynomial.* Then $B_n(f;x)$ converges to f uniformly on $[0,1]$ as $n \to \infty$.

Step One. For $0 < x < 1$ consider the binomial distribution

$$b(n,x,j) = \binom{n}{j} x^j(1-x)^{n-j}$$

for $0 \leq j \leq n$, which is the probability of getting j heads and $n-j$ tails in tossing a coin n times if the probability of getting a head is x. Verify that the mean μ of this probability distribution is nx and its standard deviation σ is $\sqrt{nx(1-x)}$.

Step Two. Let X be the random variable which assumes the value j with probability $b(n,x,j)$ for $0 \leq j \leq n$. Consider the random variable $Y = \left|f(x) - f\left(\frac{X}{n}\right)\right|$ which assumes the value $\left|f(x) - f\left(\frac{j}{n}\right)\right|$ with probability $b(n,x,j)$ for $0 \leq j \leq n$. Prove Bernstein's theorem by bounding, for an arbitrary positive number ε, the sum which defines the expected value $E(Y)$ of the random variable Y, after breaking the sum up into two parts defined respectively by $|j-\mu| \geq \eta\sigma$ and $|j-\mu| < \eta\sigma$ for some appropriate positive number η depending on ε and the uniform bound of f.