QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday January 18, 2011 (Day 1)

1. (CA) Evaluate

$$\int_0^\infty \frac{x^2+1}{x^4+1}dx$$

Solution. We can consider the integration from $-\infty$ to ∞ instead. For $R > 1$, consider the contour that consists of the segment from $-R$ to R and the arc $\{Re^{i\theta} \mid \theta \in [0, \pi]\}$. Since $\frac{z^2+1}{z^4+1}$ decays as $|z|^{-2}$ on the complex plane as $|z| \to \infty$, this contour integral converges to twice of the original integral when $R \to \infty$.

The contour encloses two simple poles $e^{\pi i/4}$ and $e^{3\pi i/4}$ of the function. At $e^{\pi i/4}$ the residue of the function is $\frac{(e^{\pi i/4})^2+1}{\frac{d}{dz}(z^4+1)|_{z=e^{\pi i/4}}} = \frac{1+i}{4e^{3\pi i/4}} = \frac{-i}{2\sqrt{2}}$. Similarly the residue at $e^{3\pi i/4}$ is also $\frac{1-i}{4e^{9\pi i/4}} = \frac{-i}{2\sqrt{2}}$. The contour goes counter-clockwise, and thus the integration along the contour is $2\pi i \cdot \frac{-i}{\sqrt{2}} = \sqrt{2}\pi$, and thus the original integral is $\frac{\sqrt{2}}{2}\pi$.

2. (A) Let k be a field and V be a k-vector space of dimension n. Let $A \in End_k(V)$. Show that the following are equivalent:

(a) The minimal polynomial of A is the same as the characteristic polynomial of A.

(b) There exists a vector $v \in V$ such that $v, Av, A^2v, ..., A^{n-1}v$ is a basis of V.

Solution. The theorem on the existence of rational canonical form says that we have $V = \bigoplus_{i=1}^r V_i$ such that each V_i is invariant under T, that $T|_{V_i}$ satisfies both (a) and (b) above, and that if we denote by $p_i(x)$ the characteristic polynomial, we have $p_j(x)|p_i(x)$ for any $j > i$. One sees that the characteristic polynomial of T is then the product of that of its blocks, namely $\prod_{i=1}^r p_i(x)$, while the minimal polynomial of T is $p_1(x)$. This shows that (a) implies $r = 1$, and thus the result of (b). On the other hand, if (b) holds, then the minimal polynomial has degree equal to $\dim V$ and thus must be equal to the characteristic polynomial.

3. (T) Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

Solution. The Kunneth formula shows the homology groups of $S^1 \times S^1$ are $\mathbb{Z}$, $\mathbb{Z}^2$ and $\mathbb{Z}$ in dimension $0, 1$ and 2, respectively. Note that since the homology groups of S^1 are free, there is no contribution from the torsion part. The reduced homology groups of $S^1 \vee S^1 \vee S^2$ are the direct sums of that of two S^1 and S^2 in corresponding dimension, hence the same result.

The universal covering of $S^1 \times S^1$ is $\mathbb{R}^2$, which is contractible and thus have trivial reduced homology groups. On the other hand the universal covering space of $S^1 \vee S^1 \vee S^2$ is the universal covering space of $S^1 \vee S^1$ with each vertex attached an S^2. The second homology group H_2 of it is therefore an infinite direct sum of $\mathbb{Z}$.

4. (RA)

(a) Prove that any countable subset of the interval $[0,1] \subset \mathbb{R}$ is Lebesgue measurable, and has Lebesgue measure 0.

(b) Let $\Phi \subset [0,1]$ be the set of real numbers x that, when written as a decimal $x = 0.a_1a_2a_3...$, satisfy the rule $a_{n+2} \notin \{a_n, a_{n+1}\}$ for all $n \geq 1$. What is the Lebesgue measure of Φ?

Solution.

(a) Suppose the countable subset is $\{y_1, y_2, ..., y_n, ...\}$. For each k take the open interval $U_k = (y_k - \frac{\delta}{2^k}, y_k + \frac{\delta}{2^k})$, where $\delta > 0$ is fixed. Then U_k has length $\frac{\delta}{2^{k-1}}$ and $\bigcup U_k$ has the sum of lengths δ. As $\delta \to 0$, this shows $\{y_1, y_2, ..., y_n, ...\}$ has measure zero and in particular is Lebesgue measurable.

(b) Let E_k be the set of real numbers $x = 0.a_1a_2a_3...$ such that $a_{k+1} \neq a_{k+2}$. Let $F_k = E_k - E_k \cap (\bigcup_{i<k} E_i)$. One easily sees that F_k has measure $\frac{1}{10} \cdot \left(\frac{9}{10}\right)^{k-1}$ and they are disjoint. Thus $\bigsqcup F_k \subset [0,1]$ has measure 1. As $\Phi \subset [0,1] - \bigsqcup F_k$, Φ has measure zero.

5. (DG) Let $B \subset \mathbb{R}^4$ be the closed ball of radius 2 centered at the origin, with the metric induced from the euclidean metric on $\mathbb{R}^4$. Give an example of a smooth vector field v on B with the property that for any L there exists an integral curve of v with both endpoints on the boundary ∂B and length greater than L.

Solution. Take the vector field $(1 - x^2 - y^2)\frac{\partial}{\partial z} + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$. The function $x^2 + y^2$ is invariant on any integral curve. When $x^2 + y^2$ is close to 1, the "vertical" part of the vector field is no more than $2(1 - x^2 - y^2)$ times the horizontal part of the vector field. The vertical length such an integral curve has to travel is $2\sqrt{1 - x^2 - y^2}$, and thus the total length of the integral curve is no less than $\frac{2\sqrt{1-x^2-y^2}}{2(1-x^2-y^2)} = \sqrt{1 - x^2 - y^2}^{-1/2}$. For $x^2 + y^2$ arbitrarily close to 1 we get arbitrarily long integral curves.

6. (AG) Let $C \subset \mathbb{P}^3$ be a smooth, irreducible, non-degenerate curve of degree d.

(a) Show that $d \geq 3$.

(b) Show that every point $p \in \mathbb{P}^3$ lies on a secant and tangent line to C.

(c) If $d = 3$ show that every point of $\mathbb{P}^3 \backslash C$ lies on a unique secant or tangent line to C.

Solution.

(a) We may assume C is not a line. Take any three point on C which are not collinear and intersect C with the plane passing through these three points. As C is non-degenerate, C is not contained in this plane and thus the intersection number of C and this plane is 3 or more (when they intersect at more points or intersect at these three points with higher multiplicity). This says $d \geq 3$.

(b) Consider $X = \{(x, y, z) \mid x, y \in C,\ z \text{ is on the secant passing through } x, y\}$. Here z should be on the tangent line if $x = y$. Then X is closed in $C \times C \times \mathbb{P}^3$ and X projects to $C \times C$ with fiber $\mathbb{P}^1$. Hence X is irreducible with dimension 3. Consider the projection map from $C \times C \times \mathbb{P}^3$ to $\mathbb{P}^3$ and denote by S the image of X. S is the so-called secant variety, and we see that it is an irreducible closed subvariety of $\mathbb{P}^3$.

The statement to be proved is then that S is the whole $\mathbb{P}^3$. If this is not the case, then the map from X to S has every fiber at least dimension 1 (by semi-continuity of the dimension of fibers). Take any four non-planar points $x_1, ..., x_4$ on C. All lines through x_i to all points on C is a 2-dimensional closed subvariety of $\mathbb{P}^3$, and thus must be S itself. This says S is a cone over x_i; if we write A_S the subvariety in $\mathbb{A}^4$ whose projectivization is S, and $V_i \subset \mathbb{A}^4$ the line through the origin corresponding to x_i, then $A_S = A_S + V_i$. But this implies $A_S = \mathbb{A}^4$ and thus $S = \mathbb{P}^3$.

(c) Suppose otherwise x lies on two tangent or secant lines. Consider the plane that contains x and this two lines. Then this plane intersect C at least four times as it intersects C twice on each line. This contradicts with that $d = 3$.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Wednesday January 19, 2011 (Day 2)

1. (T) Find all the connected 2-sheeted and 3-sheeted covering spaces of $S^1 \vee S^1$ up to isomorphism of covering spaces without basepoints. Indicate which covering spaces are normal.

Solution. A connected 2-sheeted covering is given by a homomorphism $\pi_1(S^1 \vee S^1) \to S_2$ such that the image acts transitively on the set of two elements; this is given by how the path that lifts the loop interchanges different fibers. Since $\pi_1(S^1 \vee S^1)$ is freely generated by two generators, this is the same as giving two elements in S_3 so that they generates a subgroup of S_2 that acts transitively. We thus have 3 choices: $(e,(12)),((12),(12)),((12),e)$. They are all normal.

For 3-sheeted covering we are giving two elements in S_3 so that they generates a subgroup of S_2 that acts transitively, but *up to conjugacy of* S_3. This is because a renumbering of the underlying set of S_3 corresponds to a renumbering of fibers, which always give an isomorphism of covering spaces. There are therefore 7 possibilities: $(e,(123))$, $((12),(13))$, $((12),(123))$, $((123),e)$, $((123),(12))$, $((123),(123))$ and $((123),(132))$.

Among these possibilities 4 are normal; since a group of order 3 is always abelian, such coverings are normal iff they are abelian. A covering space described this way is abelian iff the two elements are abelian. This leaves $(e,(123))$, $((123),e)$, $((123),(123))$ and $((123),(132))$.

2. (RA) Let g be a differentiable function on $\mathbb{R}$ that is non-negative and has compact support.

(a) Prove that the Fourier transform $\hat{g}$ of g does not have compact support unless $g = 0$.

(b) Prove that there exist constants A and c such that for all $k \in \mathbb{N}$ the k-th derivative of $\hat{g}$ is bounded by cA^k.

Solution.

(a) By scaling if necessary we may assume the support of g lies inside $[-3,3]$. If the Fourier transform of $\hat{g}$ also has support in $(-N,N)$, then this implies in particular that, if we think of g as defined on $[-\pi,\pi]$ and consider its Fourier series, then all terms after the N-th term are zero. In particular g has a finite Fourier series and therefore must be analytic on $(-\pi,\pi)$. This contradicts with that supp $g \subset [-3,3]$ unless $g \equiv 0$.

(b) Since g is smooth with compact support, say supp $g \subset [-A, A]$, we have

$$|\frac{d^k}{dy^k}\int_{\mathbb{R}} e^{ixy} f(x)dx| = |\int_{\mathbb{R}} (ix)^k e^{ixy} f(x)dx| \leq A^k \cdot \sup_{x\in[-A,A]} f(x) \cdot 2A.$$

3. (DG) Let $S^2 \subset \mathbb{R}^3$ be the sphere of radius 1 centered at the origin, with the metric induced from the euclidean metric on $\mathbb{R}^3$. Introduce spherical coordinates $(\theta, \phi) \in [0, \pi] \times \mathbb{R}/(2\pi\mathbb{Z})$ on the complement of the north and south poles, where

$$(x, y, z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

The metric in these coordinates is given by the section

$$d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi$$

of the second symmetric power of the cotangent bundle T^*S^2; it has constant scalar curvature 1.

Now let u be a smooth function on S^2 depending only on the coordinate θ, and consider the metric given by the section

$$e^u(d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi).$$

(a) Compute the scalar curvature of this new metric in terms of u and its derivative.

(b) Prove that the integral over S^2 of the function you computed in Part (a) is equal to 4π.

Solution. *For our convenience, we replace u by $2u$ throughout the solution.*

(a) We compute using the method of orthonormal frame. We have $e^\theta = e^u d\theta$ and $e^\phi = e^u \sin\theta d\phi$ is an orthonormal coframe (orthonormal basis for cotangent bundle). Write ω_j^k to be the matrix 1-form so that $\nabla(e_j) = \omega_j^k e_k$, where ∇ is the Levi-Civita connection and $e_\theta = e^{-u}\frac{\partial}{\partial\theta}$ and $e_\phi = e^{-u}\csc\theta\frac{\partial}{\partial\phi}$ is the dual frame. The torsion-free condition of the connection gives the Cartan's structure equation (using Einstein summation convention)

$$de^k + \omega_j^k \wedge e^j = 0.$$

Also that ∇ is compatible with metric gives $\omega_j^k = -\omega_k^j$. So $\omega_\theta^\theta = \omega_\phi^\phi = 0$. Next we have $\omega_\phi^\theta = -\omega_\theta^\phi$, $de^\theta = 0$ says ω_ϕ^θ at every point is a multiple of e^ϕ, and $de^\phi = (e^u \sin\theta)' d\theta \wedge d\phi = \frac{(e^u\sin\theta)'}{e^{2u}\sin\theta} e^\theta \wedge e^\phi$ gives $\omega_\theta^\phi = -\frac{(e^u\sin\theta)'}{e^{2u}\sin\theta} e^\phi = -e^{-u}(e^u\sin\theta)' d\phi$.

The curvature 2-form is $d\omega + \omega\wedge\omega$. Note that $\omega\wedge\omega$ is zero because each entry of ω has only $d\phi$ term. And $(d\omega)_\theta^\phi = -(e^{-u}(e^u\sin\theta)')' d\theta \wedge d\phi =$

$-\frac{(e^{-u}(e^u\sin\theta)')'}{e^{2u}\sin\theta}e^\theta\wedge e^\phi$. So the scalar curvature is (or twice of it, in the convention in Wikipedia for general dimension)

$$S=\iota_{e_\phi}\iota_{e_\theta}((d\omega)^\phi_\theta)-\frac{(e^{-u}(e^u\sin\theta)')'}{e^{2u}\sin\theta}.$$

(b) The volume form is $e^{2u}\sin\theta d\theta\wedge d\phi$. We thus have to integrate

$$\int_0^{2\pi}\int_0^\pi -(e^{-u}(e^u\sin\theta)')'d\theta d\phi.$$

The deduction that this gives 4π is straightforward.

4. (AG) Show that no two of the following rings are isomorphic:

1. $\mathbb{C}[x,y]/(y^2-x)$.
2. $\mathbb{C}[x,y]/(y^2-x^2)$.
3. $\mathbb{C}[x,y]/(y^2-x^3)$.
4. $\mathbb{C}[x,y]/(y^2-x^4)$.
5. $\mathbb{C}[x,y]/(y^2-x^5)$.
6. $\mathbb{C}[x,y]/(y^3-x^4)$.

Solution. These rings are reduced (no non-trivial nilpotent elements). Two rings are isomorphic if and only if their corresponding complex analytic space (variety over $\mathbb{C}$) are isomorphic. The first one is the only one that is non-singular, and therefore non-isomorphic with any others.

The second and the fourth varieties are reducible, i.e. the rings are not integral domains. To distinct them from each other, note that they both have two component, namely two minimal prime ideals, which is $(x+y)$, $(x-y)$ for the second ring and $(x+y^2)$, $(x-y^2)$ for the fourth ring. The union of this two ideals gives the maximal ideal (x,y) for the second ring, but gives the non-primary ideal (x,y^2) for the fourth ring. This reflects that fact that in one case the two component intersect transversally and in the other case they intersect twice. This shows the second ring and the fourth ring are not isomorphic.

It remains to check that the third, fifth and sixth rings are different. They all have a unique singularity at the origin, which can be resolved (the same holds for all curves) by taking normalization, i.e. integral closure of the ring. They have the same integral closure $\mathbb{C}[t]$, in which the three rings may be written as $\mathbb{C}[t^2,t^3]$, $\mathbb{C}[t^2,t^5]$ and $\mathbb{C}[t^3,t^4]$. Now note that $\dim_{\mathbb{C}}\mathbb{C}/\mathbb{C}[t^2,t^3]=1$, $\dim_{\mathbb{C}}\mathbb{C}/\mathbb{C}[t^2,t^5]=2$, $\dim_{\mathbb{C}}\mathbb{C}/\mathbb{C}[t^3,t^4]=3$, and thus these rings are non-isomorphic. This number is called the delta invariant and measure the loss of geometric genus at this singularity.

5. (CA) Let $f : \mathbb{C} \to \mathbb{C}$ be a nonconstant holomorphic function. Prove that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Solution. Assume z_0 is not in the closure of the image of $f(\mathbb{C})$. $1/(f(\mathbb{C}) - z_0)$ would then be bounded, and thus constant, a contradiction.

6. (A) Let a be a positive integer, and consider the polynomial

$$f_a(x) = x^6 + 3ax^4 + 3x^3 + 3ax^2 + 1 \in \mathbb{Q}[x]$$

(a) Show that it is irreducible.

(b) Show that the Galois group of f_a is solvable.

Solution. For (b) it's the same as to prove that all roots can be written using successive radicals. Let $y = x + \frac{1}{x}$. We observe $f_a(x) = y^3 + (3a - 3)y + 3$. That $y^3 + (3a-3)y+3$ is of course solvable, and thus all roots of x are solvable. This proves (b). For part (a), by Gauss lemma, we know factorization in $\mathbb{Q}[x]$ is the same as that in $\mathbb{Z}[x]$. Note that $y^3 + (3a - 3)y + 3$ is irreducible by Eisenstein criterion modulo 3. This shows that the size of Galois orbits of the roots of $f_a(x)$ is at least three; if $f_a(x)$ is reducible, it can only be factorized into two cubic polynomials.

However, by reduction modulo 3 for $f_a(x)$ again, we see that $f_a(x) \equiv (x^2+1)^3 \pmod 3$. This says $f_a(x)$ can only be factorized into even degree polynomials. Two results combined imply $f_a(x)$ is irreducible.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday January 20, 2011 (Day 3)

1. (DG) Let $(\ \cdot\ )$ be the standard inner product on $\mathbb{R}^3$, and let

$$S^2 = \{\mathbf{x} = (x_1, x_2, x_3) : (\mathbf{x} \cdot \mathbf{x}) = 1\}$$

be the sphere of radius 1 centered at the origin; identify the tangent space $T_{\mathbf{x}}S^2$ at a point $\mathbf{x} \in S^2$ with the subspace

$$T_{\mathbf{x}}S^2 = \{v \in \mathbb{R}^3 : (\mathbf{x} \cdot v) = 0\} \subset \mathbb{R}^3,$$

where $(\ \cdot\ )$ is the standard inner product on $\mathbb{R}^3$. Let $e \in \mathbb{R}^3$ be any fixed vector, and let V be the vector field on S^2 given by

$$V(\mathbf{x}) = e - (\mathbf{x} \cdot e)\mathbf{x}.$$

(a) Compute the Lie derivative by V of the 1-form $x_1 dx_2$.

(b) Define a Riemannian metric on S^2 by setting the inner product of tangent vectors $v, v' \in T_x S^2$ equal to $(v \cdot v')$, (that is, take the metric induced on S^2 by the euclidean metric on $\mathbb{R}^3$). Use the associated Levi-Civita connection to define a covariant derivative on the space of 1-forms on S^2.

(c) Compute the covariant derivative of the 1-form $x_1 dx_2$ in the direction of the vector field V.

Solution.

(a) We'll write $\mathbf{x} = (x, y, z)$, $e = (a, b, c)$ and $x_1 dx_2 = xdy$. We have $V(\mathbf{x}) = (a - Sx, b - Sy, c - Sz)$, where $S = ax + by + cz$. Using Cartan's formula, the Lie derivative

$$\mathcal{L}_V(xdy) = \iota_V(dx \wedge dy) + d(\iota_V xdy) = (a - Sx)dy - (b - Sy)dx + d((b - Sy)x)$$

$$= -axydx + (a - 2Sx - bxy)dy - cxydz.$$

(b) The covariant derivative on 1-forms can be given as follows: For 1-form $\alpha \in \Omega^1(S^2)$, we compute the dual vector field $V_\alpha\ V \in C^\infty(TS^2)$. Then $\nabla\alpha$ is defined to be the dual of ∇V_α, which is the projection of the derivative of V_α in $C^\infty(T\mathbb{R}^3)$. This can be seen as follows: For any vector field $u, v \in C^\infty(TS^2)$, we have

$$(\nabla_u \alpha)(v) = \alpha(\nabla_u v) - u.(\alpha(v)) = (V_\alpha \cdot \nabla_u v) - u.(V_\alpha \cdot v) = (\nabla_u T_\alpha \cdot v).$$

(c) Using part (b), we shall first compute the dual of xdy on S^2. On R^3 its dual is $x\frac{\partial}{\partial y}$. We then project this vector to the tangent space of S^2. Note that $x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}$ is the unit normal of S^2, and $(x\frac{\partial}{\partial y}\cdot x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}) = xy$. Hence the dual of xdy is $x\frac{\partial}{\partial y} - xy(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})$. One then computes the derivative of this vector field, project it to the tangent plane of S^2, then take the dual 1-form.

2. (T) Let D^2 be the closed unit disk in $\mathbb{R}^2$. Prove the Brouwer fixed point theorem for maps $f : D^2 \to D^2$ by applying degree theory to the $S^2 \to S^2$ that sends both the northern and southern hemisphere of S^2 to the southern hemisphere via f.

Solution. We have to prove that such an f has a fixed point. Denote by $g : S^2 \to S^2$ the map constructed in the statement of the problem. Since the image of f is in the southern hemisphere, g is homotopic to the map that sends all points on S^2 to the southern pole, and thus g has degree 0. On the other hand, if f has no fixed point, then g has no fixed point as well, and g is homotopic to the antipodal map that sends every point to the opposite point on S^2. This is an orientation-reversing homeomorphism and has degree -1 instead, a contradiction.

3. (CA) Prove that for every $\lambda > 1$, the equation $ze^{\lambda - z} = 1$ has exactly one root in the unit disk $\mathbb{D}$ and that this root is real.

Solution. On $|z| = 1$ w have

$$|z \cdot e^{\lambda - z}| = e^{\lambda - Re(z)} \geq e^{\lambda - 1} > 1$$

because $\lambda > 1$. Hence by Rouché's theorem $1 - ze^{\lambda - z}$ has the same number of zeroes counted with multiplicity as $ze^{\lambda - z}$ inside $|z| = 1$, hence has exactly one zero.

Observe if z is a zero of $1 - ze^{\lambda - z}$ then so is $\bar{z}$ because λ is real, hence by uniqueness the unique zero must be real.

4. (A) Let K be an algebraically closed field of characteristic 0, and let $f \in K[x]$ be any cubic polynomial. Show that exactly one of the following two statement is true:

1. $f = \alpha(x - \lambda)^3 + \beta(x - \lambda)^2$ for some $\alpha, \beta, \lambda \in K$; or
2. $f = \alpha(x - \lambda)^3 + \beta(x - \mu)^3$ for some $\alpha, \beta \neq 0 \in K$ and $\lambda \neq \mu \in K$.

In the second case, show that λ and μ are unique up to order.

Solution. The statement in this problem is incorrect. Take $f = x^3 - 1$. f has no repeated root therefore the first case doesn't happen. Suppose $x^3 - 1 = \alpha(x - \lambda)^3 + \beta(x - \mu)^3$. Looking at the x and x^2 terms gives $\alpha\lambda + \beta\mu = \alpha\lambda^2 + \beta\mu^2 = 0$. This is impossible when $\lambda \neq \mu$ and $\alpha, \beta \neq 0$.

5. (AG) Let $Q \subset \mathbb{P}^{2n+1}$ be a smooth quadric hypersurface in an odd-dimensional projective space over $\mathbb{C}$.

(a) What is the largest dimension of a linear subspace of $\mathbb{P}^{2n+1}$ contained in Q.

(b) What is the dimension of the family of such planes?

Solution. The quadric corresponds to a quadratic form $q(v)$ with $2(v,w) = q(v+w) - q(v) - q(w)$ on $\mathbb{C}^{2n+2}$. If q has non-trivial kernel, i.e. there is $v \neq 0 \in \mathbb{C}^{2n+2}$ s.t. $(v,w) = 0\ \forall w \in \mathbb{C}^{2n+2}$. Then $(\partial_w q)|_v = (w,v) = 0\ \forall w$ and v is a singular point on Q, contradicts to that Q is non-singular. So we'll begin with a non-degenerate quadratic form q.

(a) q cannot have a isotropic subspace of dimension $n+2$ because such a space will have a dimension n orthogonal complement. Therefore Q cannot contains a $n+1$-dimensional linear subspace. We'll construct a n-dimensional linear subspace in Q in (b). So n is the largest dimension for linear subspaces in Q.

(b) Consider $P_k = \{l \in \mathbb{G}(k-1, 2n+1) \mid l \text{ lies in } Q\}$ of $(k-1)$-dimensional (projective) subspaces that lie in Q. Also we take P_0 to be a point. Consider the incidence correspondence $Q_k = \{l_1 \in P_k, l_2 \in P_{k+1} \mid l_1 \text{ lies in } l_2\}$. The fiber of the projection map from Q_k to P_k at $l_1 \in P_k$ can be given as follows: let V_1 be the affine subspace of $\mathbb{C}^{2n+2}$ that corresponds to l_1. Then to get l_2 one has to find a line in $V_1^{\perp}/V_1$ that is isotropic with respect to the induced non-degenerate quadratic form on $V_1^{\perp}/V_1$, namely a choice of a point on a quadric in $\mathbb{P}^{2n+1-2k}$. Therefore the fiber is non-empty and has constant dimension $2n-2k$.

Each fiber of the projection map from Q_k to P_{k+1} is isomorphic to $\mathbb{P}^k$. Thus $\dim P_{k+1} = \dim P_k + (2n-2k) - k$. One then computes $\dim P_{n+1} = 2n + (2n-3) + \ldots + (-n) = n(n+1)/2$.

6. (RA) Let $\mathbb{H}$ and $\mathbb{L}$ denote a pair of Banach spaces.

(a) Prove that a linear map from $\mathbb{H}$ to $\mathbb{L}$ is continuous if and only if it's bounded

(b) Define what is meant by a compact linear map from $\mathbb{H}$ to $\mathbb{L}$.

(c) Now let $\mathbb{H}$ and $\mathbb{L}$ be the Banach spaces obtained by completing the space $C_c^{\infty}([0,1])$ of compactly supported C^{∞} functions on $[0,1]$ using the norms with squares

$$||f||_{\mathbb{H}}^2 = \int_{[0,1]} |\frac{df}{ds}|^2 s^2 ds \text{ and } ||f||_{\mathbb{L}}^2 = \int_{[0,1]} |f|^2 ds.$$

The identity map $C_c^{\infty}([0,1])$ extends to a bounded linear map $\phi : \mathbb{H} \to \mathbb{L}$ (you don't need to prove this). Prove that ϕ is not compact.

Solution.

(a) If $\phi : \mathbb{H} \to \mathbb{L}$ is bounded with $||\phi|| = A$, then the preimage of the open ball of radius r centered at $\phi(x)$ contains the ball of radius r/A centered at x. This shows ϕ is continuous at x. On the other hand, if it is continuous, then the preimage of the open ball of radius 1 centered at $0 \in \mathbb{L}$ contains an open ball of radius B centered at $0 \in \mathbb{H}$. This says $||x|| < B$ implies $||\phi(x)|| < 1$ and by linearity $||x|| < r \Rightarrow ||\phi(x)|| < r/B$, i.e. $||\phi|| \leq 1/B$.

(b) A continuous linear map $\phi : \mathbb{H} \to \mathbb{L}$ is compact if for any sequence of vectors $\{x_n\}$ in a bounded subset of $\mathbb{H}$, $\{\phi(x_n)\}$ has a convergent subsequence.

(c) Take f to be any non-zero smooth function with support in $[1/2, 1]$. For any $\alpha > 1$, consider scaling $f_\alpha(x) := \alpha f(\alpha^2 x)$ (with the convention $f(x) = 0$ for $x > 1$). We observe that $||f||_{\mathbb{H}}^2 = ||f_\alpha||_{\mathbb{H}}^2$ and $||f||_{\mathbb{L}}^2 = ||f_\alpha||_{\mathbb{L}}^2$. Take $\alpha_i = 2, 4, 8, ...$ so that each f_α have disjoint support. So $\{f_{\alpha_i}\}$ is bounded but $\{\phi(f_{\alpha_i})\}$ has no convergent subsequence.