QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday January 19, 2010 (Day 1)

1. Let (X, μ) be a measure space with $\mu(X) < \infty$. For $q > 0$, let $L^q = L^q(X, \mu)$ denote the Banach space completion of the space of bounded functions on X with the norm

$$||f||_q = \left(\int_X |f|^q \mu \right)^{\frac{1}{q}}.$$

Now suppose that $0 < p \leq q$. Prove that all functions in L^q are in L^p, and that the inclusion map $L^q \hookrightarrow L^p$ is continuous.

Solution. By Hölder's inequality

$$\|f\|_p^p = \left(\int_X |f|^p \cdot 1 d\mu \right) \leq \left(\int_X (|f|^p)^{\frac{q}{p}} d\mu \right)^{\frac{p}{q}} \left(\int_X (1)^{\frac{q}{q-p}} d\mu \right)^{1-\frac{p}{q}}$$

so

$$\|f\|_p^p \leq \|f\|_q^p \cdot \mu(X)^{1-\frac{p}{q}}$$

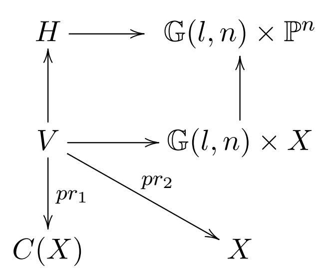
so

$$\|f\|_p \leq \|f\|_q \cdot \mu(X)^{\frac{1}{p}-\frac{1}{q}}.$$

Hence if f is in L^q, the left-hand side is finite hence so is the right-hand side, so f is in L^p. Also, the inequality shows that if $\|f\|_p$ is small then $\|f\|_q$ is also small, hence the inclusion $L^q \hookrightarrow L^p$ is continuous

2. Let $X \subset \mathbb{P}^n$ be an irreducible projective variety of dimension k, $\mathbb{G}(\ell, n)$ the Grassmannian of ℓ-planes in $\mathbb{P}^n$ for some $\ell < n - k$, and $C(X) \subset \mathbb{G}(\ell, n)$ the variety of ℓ-planes meeting X. Prove that $C(X)$ is irreducible, and find its dimension.

Solution. We have the diagram



Here H is the universal l-plane $\{(h, x) : x \in \mathbb{P}^n, h \in \mathbb{G}(l, n), x \in h\}$, and $V = H \cap (\mathbb{G}(l, n) \times X)$ For $x \in X$, the fiber over x is $V_x = \{h \in \mathbb{G}(l, n), x \in h\}$.

This can be identified with the subspace of the Grassmannian of $(l+1)$-dimensional subspaces in an $(n+1)$- dimensional vector space containing a fixed line, hence is isomorphic to the Grassmannian of l-dimensional subspaces of an n-dimensional vector space. It is therefore irreducible of dimension $l(n-l)>0$. But $pr_2 : V \twoheadrightarrow X$ is a surjective morphism with irreducible base and irreducible fibers of constant dimension $l(n-l)$, so V is also irreducible and has dimension dim V= dim $X + l(n-l)$= $k + l(n-l)$.

By definition, $C(X) = pr_1(X) \hookrightarrow \mathbb{G}(l,n)$, and hence is irreducible. If $h \in C(X)$, the fiber of V over h is $V_h = \{(h,x) \mid x \in h \cap X\} \simeq h \cap X$. Because $k+l<n$, we can find an $(n-k)$-plane that meets X at finitely many points. Then any l-plane h in this $(n-k)$-plane going through one of the intersection point will meet X at finitely many points. Hence for such h, V_h will be a finite set of points, so has dimension 0. By upper-semicontinuity of fiber dimension for the proper morphism pr_1, there is a dense open set where the fiber has dimension 0, and hence dim $C(X)$= dim V= $k + l(n-l)$.

3. Let λ be real number greater than 1. Show that the equation $ze^{\lambda - z} = 1$ has exactly one solution z with $|z| < 1$, and that this solution z is real. (Hint: use Rouché's theorem.)

Solution. On $|z| = 1$ w have

$$|z \cdot e^{\lambda - z}| = e^{\lambda - Re(z)} \geq e^{\lambda - 1} > 1$$

because $\lambda > 1$. Hence by Rouché's theorem $1 - ze^{\lambda - z}$ has the same number of zeroes counted with multiplicity as $ze^{\lambda - z}$ inside $|z| = 1$, hence has exactly one zero. Observe if z is a zero of $1 - ze^{\lambda - z}$ then so is $\bar{z}$ because λ is real, hence by uniqueness the unique zero must be real.

4. Let k be a finite field, with algebraic closure $\overline{k}$.

(a) For each integer $n \geq 1$, show that there is a unique subfield $k_n \subset \overline{k}$ containing k and having degree n over k.

(b) Show that k_n is a Galois extension of k, with cyclic Galois group.

(c) Show that the norm map $k_n^\times \to k^\times$ (sending a nonzero element of k_n to the product of its Galois conjugates) is a surjective homomorphism.

Solution. Let the cardinality of k be q, a prime power.

(a) If a subfield $k_n \subset \bar{k}$ is of degree n over k, it has cardinality q^n. The multiplicative group $k_n^\times$ is a finite subgroup of the multiplicative group of a field, hence is cyclic. It has order $q^n - 1$. Hence $x^{q^n} - x = 0$ for all $x \in k_n$. On the other hand $x^{q^n} - x = 0$ has precisely q^n distinct solution in $\bar{k}$ (note $\frac{d}{dx}(x^{q^n} - x) = -1$ has no common zero with $x^{q^n} - x$), hence this forces k_n to be the set of zeroes of $x^{q^n} - x$ in $\bar{k}$. Note that in particular k is the set of zeroes of $x^q - x$. This shows there is at most one k_n. To show it exists, we must check that the zeroes of $x^{q^n} - x$ form a subfield of $\bar{k}$. But observe that it is the set of fixed point of the map $x \mapsto x^{q^n}$ defined on $\bar{k}$, which is a field endomorphism, so the set of fixed point is a subfield.

(b) We have $[k_n : k] = n$ by definition. Denote by F the map $x \mapsto x^q$ in $\bar{k}$. The characteristic of $\bar{k}$ divides q so this is an injective field endomorphism. The description of k_n above shows that it is stable under F, and F fixes k pointwise. Thus F is an automorphism of k_n over k. For any $k < n$, $x^{q^k} - x =$ has q^k solution in $\bar{k}$, so the solution set can not contain all of k_n. Hence $F^k \neq id$ on k_n, but $F^n = id$ on k_n. This shows that F generates a cyclic subgroup of $Aut(k_n/k)$. But the latter has size at most n, hence equality occurs, so k_n/k is Galois and has Galois group a cyclic group of order n, generated by F.

(c) Explicitly the norm map $k_n^\times \to k^\times$ is $x \mapsto x^{1+q+..+q^{n-1}} = x^{\frac{q^n-1}{q-1}}$. Because $\bar{k}$ is algebraically closed, for any $a \in k^\times$ there is $x \in \bar{k}^\times$ such that $x^{\frac{q^n-1}{q-1}} = a$. But then for such x we have $x^{q^n-1} = a^{q-1} = a$ because $a \in k$. Thus $x^{q^n} = x$ and $x \neq 0$, so $x \in k_n^\times$.

5. Suppose ω is a closed 2-form on a manifold M. For every point $p \in M$, let

$$R_p(\omega) = \{v \in T_pM \ : \ \omega_p(v, u) = 0 \text{ for all } u \in T_pM\}.$$

Suppose that the dimension of R_p is the same for all p. Show that the assignment $p \mapsto R_p$ as p varies in M defines an integrable subbundle of the tangent bundle TM.

Solution. We have the following identity for vector fields X, Y, Z:

$$d\omega(X, Y, Z) = -X\omega(Y, Z) + Y\omega(X, Z) - Z\omega(X, Y) + \omega([X, Y], Z) +$$

$$+\omega([Y, Z], X) - \omega([X, Z], Y) \quad (*)$$

To see this, observe that X acts derivations on $C^\infty(M)$ and that $[f \cdot X, Y]g = f \cdot [X, Y]g - Yf \cdot Xg$ for $f, g \in C^\infty(M)$, hence $[f \cdot X, Y] = [X, Y] - Yf \cdot X$. Hence

$$Y\omega(f \cdot X, Z) + \omega([f \cdot X, Y], Z)) = f \cdot Y\omega(X, Z) + f \cdot \omega([X, Y], Z) +$$

$$+Yf \cdot \omega(X, Z) - Yf \cdot \omega([X, Y], Z) = f \cdot (Y\omega(X, Z) + \omega([X, Y], Z)).$$

This and similar identities for Y, Z shows that the right-hand side of (*) is $C^\infty(M)$ linear, as is the left-hand side. Also note that both sides have the same variance under the action of S_3 via permuting X, Y, Z, and are both $\mathbb{C}$-linear in ω. To check the identity is also a local question. This reduces us to the case X, Y, Z are $\partial/\partial x_i$, $\partial/\partial x_j$, $\partial/\partial x_k$ and $\omega = f \cdot dx_1 \wedge dx_2 \wedge dx_3$ for some function f and local coordinates $x_1, ...x_n$, which quickly follows by

direct inspection (note in this case all the Lie brackets vanish).
We now show that the distribution defined by $R_p \subset T_pM$ is integrable (it is a subbundle of TM since the dimension of the fibers are constant). Indeed suppose X, Y are two vector fields belonging to it. Pick a point $p \in M$ and let Z_p be an arbitrary vector in T_pM. We can then find a global vector field Z which agrees with Z_p at p. Looking at the identity (*) at the point p, we then have (noting $d\omega = 0$)

$$0 = d\omega_p(X_p, Y_p, Z_p) = -(X\omega(Y,Z))_p + (Y\omega(X,Z))_p - (Z\omega(X,Y))_p +$$

$$+\omega_p([X,Y]_p, Z_p) + \omega_p([Y,Z]_p, X_p) - \omega_p([X,Z]_p, Y_p).$$

But $\omega(X,Z) = \omega(Y,Z) = \omega(X,Y) = \omega([X,Z],Y) = \omega([Y,Z],X) = 0$ by assumption on X, Y, hence $\omega_p([X,Y]_p, Z_p) = 0$. Since p and Z_p can be chosen arbitrarily, it follows that $[X,Y]$ also belongs to the distribution

6. Let X be a topological space. We say that two covering spaces $f : Y \to X$ and $g : Z \to X$ are isomorphic if there exists a homeomorphism $h : Y \to Z$ such that $g \circ h = f$. If X is a compact oriented surface of genus g (that is, a g-holed torus), how many connected 2-sheeted covering spaces does X have, up to isomorphism?

Solution. By covering space theory, there is a bijection between connected 2-sheeted coverings of X up to isomorphism and conjugacy classes of index 2 subgroups of $\pi_1(X)$. As any index 2 subgroup is normal, this set is in bijection with the set of index 2 subgroups of $\pi_1(X)$, which is the same as the set of surjective group homomorphisms from $\pi_1(X)$ to $\mathbb{Z}/2\mathbb{Z}$. Because $\mathbb{Z}/2\mathbb{Z}$ is commutative, all such homomorphisms factor through the abelization $\pi_1(X)^{ab}$

Now for X the compact oriented surface of genus g, $\pi_1(X)$ has a presentation $\langle a_1, ...a_g, b_1, ...b_g \mid [a_1,b_1][a_2,b_2]...[a_g,b_g] = 1\rangle$, here the bracket $[a,b] = aba^{-1}b^{-1}$ is the commutator. Hence its abelization is the free abelian group on $2g$ generators $\mathbb{Z}^{2g}$. Thus specifying a surjective homomorphism from $\pi_1(X)$ to $\mathbb{Z}/2\mathbb{Z}$ is the same thing as specifying where each generator goes to, such that not all go to the identity. The number of such homomorphisms is thus $2^{2g} - 1$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday January 20, 2010 (Day 2)

1. Let a be an arbitrary real number and b a positive real number. Evaluate the integral

$$\int_0^\infty \frac{\cos(ax)}{\cosh(bx)} dx$$

(Recall that $\cosh(x) = \cos(ix) = \frac{1}{2}(e^x + e^{-x})$ is the hyperbolic cosine.)

Solution. We will use the residue theorem for the rectangular contour bounded by the real axis, the line $\Im z = \frac{\pi i}{b}$, and the lines $\Re z = \pm A$, where A will be large, and for the function $f(z) = \frac{e^{iax}}{\cosh(bx)}$. The integrals over the horizontal edges of the contour are

$$\int_{-A}^{A} \frac{e^{iax}}{\cosh(bx)} dx$$

and

$$\int_{-A}^{A} \frac{e^{iax-\frac{\pi a}{b}}}{\cosh(bx)} dx$$

Hence the contribution of the horizontal edges are

$$(1 + e^{-\frac{\pi a}{b}}) \int_{-A}^{A} \frac{e^{iax}}{\cosh(bx)} dx$$

The contribution of the vertical sides are

$$\pm i \int_0^{\frac{\pi}{b}} \frac{e^{ia(\pm A+iy)}}{\cosh(b(\pm A + iy))} dy$$

Now note $|\cosh(b(\pm A + iy))| = |e^{bA\pm iy} + e^{-bA\mp iy}| \geq e^{bA} - 1$, hence each side integral has norm bounded by $C\frac{1}{e^{bA}-1}$ for some constant C that does not depend on A, and hence tends to 0 when $A \to \infty$. Now the residue theorem says that the sum of the integral over the sides are the sum of $2\pi i$ times the residues of $f(z)$ inside the contour. For any A, the function $f(z)$ only has a simple pole at $z = \frac{\pi}{2b}$. We have $\cosh(b(x+\frac{\pi}{2b})) = i(e^{bx} - e^{-bx}) = 2ibx + O(x^3)$, hence $\mathrm{Res}_{z=\frac{\pi}{2b}} f = \frac{e^{-\frac{\pi a}{2b}}}{2ibx}$.

Thus letting $A \to \infty$ gives

$$(1 + e^{-\frac{\pi a}{b}}) \int_{-\infty}^{+\infty} \frac{e^{iax}}{\cosh(bx)} dx = 2\pi i \cdot \mathrm{Res}_{z=\frac{\pi}{2b}} f = \frac{\pi e^{-\frac{\pi a}{2b}}}{bx}$$

Taking real parts gives

$$\int_0^{+\infty} \frac{\cos(ax)}{\cosh(bx)} dx = \frac{\pi}{2bxcosh(\frac{\pi a}{2b})}.$$

2. For any irreducible plane curve $C \subset \mathbb{P}^2$ of degree $d > 1$, we define the *Gauss map* $g : C \to \mathbb{P}^{2^*}$ to be the rational map sending a smooth point $p \in C$ to its tangent line; we define the *dual curve* $C^* \subset \mathbb{P}^{2^*}$ of C to be the image of g.

(a) Show that the dual of the dual of C is C itself.

(b) Show that two irreducible conic curves $C, C' \subset \mathbb{P}^2$ are tangent if and only if their duals are.

Solution. The ground field is assumed to have characteristic 0, as the result fails otherwise. We can assume the ground field is $\mathbb{C}$.

(a) Let the plane curve C be given by $F = 0$, where F is an irreducible polynomial of degree d in X, Y, Z. We will choose coordinates on $\mathbb{P}^2$ and its dual in such a way that the dual pairing is given by $X \cdot U + Y \cdot V + Z \cdot W$. Because $d > 1$, the dual curve C^* is actually a plane curve, and hence is given by $G = 0$ for some homogenous polynomial. The Gauss map is given by $[X : Y : Z] \mapsto [\frac{\partial}{\partial X}F(X,Y,Z) : \frac{\partial}{\partial Y}F(X,Y,Z) : \frac{\partial}{\partial Z}F(X,Y,Z)]$. It follows that $G(\frac{\partial}{\partial X}F, \frac{\partial}{\partial Y}F, \frac{\partial}{\partial Z}F) = 0$ on $F = 0$. Because F is irreducible, there exists a polynomial H such that $G(\frac{\partial}{\partial X}F, \frac{\partial}{\partial Y}F, \frac{\partial}{\partial Z}F) = H \cdot F$.
Differentiating both sides with respect to X, Y, Z, we see that

$$\begin{pmatrix} \frac{\partial^2}{\partial X^2}F & \frac{\partial^2}{\partial X \partial Y}F & \frac{\partial^2}{\partial X \partial Z}F \\ \frac{\partial^2}{\partial X \partial Y}F & \frac{\partial^2}{\partial Y^2}F & \frac{\partial^2}{\partial Y \partial Z}F \\ \frac{\partial^2}{\partial X \partial Z}F & \frac{\partial^2}{\partial Z \partial Y}F & \frac{\partial^2}{\partial Z^2}F \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial X}G \\ \frac{\partial}{\partial Y}G \\ \frac{\partial}{\partial Z}G \end{pmatrix} = H \begin{pmatrix} \frac{\partial}{\partial X}F \\ \frac{\partial}{\partial Y}F \\ \frac{\partial}{\partial Z}F \end{pmatrix} + F \begin{pmatrix} \frac{\partial}{\partial X}G \\ \frac{\partial}{\partial Y}G \\ \frac{\partial}{\partial Z}G \end{pmatrix} \qquad (*)$$

We claim that the determinant of the Hessian of F cannot vanish identically along $F = 0$. Suppose this were the case. This means that all non-singular points of $F = 0$ are inflection points, that is points whose tangent to C intersects C with multiplicity ≥ 3. Choose an affine chart of $\mathbb{P}^2$ such that $(0, 0) \in C$ is a non-singular point. We can then choose an analytic parameterization of C near $(0, 0)$ given by $t \mapsto \gamma(t) = (v(t), w(t))$, for t in a small disk. Then $(v(t), w(t))$ being an inflection point of C implies $\dot\gamma(t) \neq 0$ and $\ddot\gamma(t)$ are proportional. Thus $\dot\gamma(t) \wedge \ddot\gamma(t) = 0$. Differentiating with respect to t gives $\dot\gamma(t) \wedge \gamma^{(3)}(t) = 0$, so that $\gamma^{(3)}(t)$ is also proportional to $\dot\gamma(t)$. Continuing inductively gives $\dot\gamma(t) \wedge \gamma^{(n)}(t) = 0$ for all $n > 0$. But now $\gamma(t) = \dot\gamma(0)t + \frac{1}{2!}\ddot\gamma(0)t^2 + ...$, hence $\dot\gamma(0) \wedge \gamma(t) = 0$. But this means C has infinitely many intersections with a line, hence is a line, contradiction. Thus the Hessian of F is invertible on a dense open subset of C.

Now if we evaluate (*) at a non-singular non-inflection point in $[x : y : z] \in C$, we see that $[\frac{\partial}{\partial X}G : \frac{\partial}{\partial Y}G : \frac{\partial}{\partial Z}G]$ at that point is uniquely determined by

$$\begin{pmatrix} \frac{\partial^2}{\partial X^2}F & \frac{\partial^2}{\partial X \partial Y}F & \frac{\partial^2}{\partial X \partial Z}F \\ \frac{\partial^2}{\partial X \partial Y}F & \frac{\partial^2}{\partial Y^2}F & \frac{\partial^2}{\partial Y \partial Z}F \\ \frac{\partial^2}{\partial X \partial Z}F & \frac{\partial^2}{\partial Z \partial Y}F & \frac{\partial^2}{\partial Z^2}F \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial X}G \\ \frac{\partial}{\partial Y}G \\ \frac{\partial}{\partial Z}G \end{pmatrix} = H(x, y, z) \begin{pmatrix} \frac{\partial}{\partial X}F \\ \frac{\partial}{\partial Y}F \\ \frac{\partial}{\partial Z}F \end{pmatrix}$$

But Euler's formula says that $X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + Z\frac{\partial}{\partial Z}$ is multiplication by d on the space of homogenous polynomials of degree d, hence

$$\begin{pmatrix} \frac{\partial^2}{\partial X^2}F & \frac{\partial^2}{\partial X \partial Y}F & \frac{\partial^2}{\partial X \partial Z}F \\ \frac{\partial^2}{\partial X \partial Y}F & \frac{\partial^2}{\partial Y^2}F & \frac{\partial^2}{\partial Y \partial Z}F \\ \frac{\partial^2}{\partial X \partial Z}F & \frac{\partial^2}{\partial Z \partial Y}F & \frac{\partial^2}{\partial Z^2}F \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = d(d-1) \begin{pmatrix} \frac{\partial}{\partial X}F \\ \frac{\partial}{\partial Y}F \\ \frac{\partial}{\partial Z}F \end{pmatrix}$$

It thus follows that $[x : y : z] = [\frac{\partial}{\partial X}G : \frac{\partial}{\partial Y}G : \frac{\partial}{\partial Z}G]$, hence the composition of Gauss maps $C \to C^* \to C^{**}$ is generically the identity, hence C^{**} is canonically identified with C.

(b) For an irreducible conic C the Gauss map is a linear isomorphism. Suppose two such conics C, C' are tangent at a point p. Then the image of p under the Gauss map of C, C' is a point q. By the previous part, we know that q will get sent to p under both the Gauss map of C and C'. But this means that C, C' are tangent at q.

3. Let Λ_1 and $\Lambda_2 \subset \mathbb{R}^4$ be complementary 2-planes, and let $X = \mathbb{R}^4 \setminus (\Lambda_1 \cup \Lambda_2)$ be the complement of their union. Find the homology and cohomology groups of X with integer coefficients.

Solution. Let

$$U = \mathbb{R}^4 \setminus \Lambda_1 \simeq S^1 \times \mathbb{R}^2$$
$$V = \mathbb{R}^4 \setminus \Lambda_2 \simeq S^1 \times \mathbb{R}^2$$
$$U \cap V = \mathbb{R}^4 \setminus (\Lambda_1 \cup \Lambda_2) = X$$
$$U \cup V = \mathbb{R}^4 \setminus pt \simeq S^3.$$

Then from the Mayer-Vietoris sequene we get

$$0 \to H_4(X) \to H_4(U) \oplus H_4(V) \to H_4(S^3) \to H_3(X) \to H_3(U) \oplus H_3(V) \to H_3(S^3) \to H_2(X) \to H_2(U)$$

$$\to H_2(S^3) \to H_1(X) \to H_1(U) \oplus H_1(V) \to H_1(S^3) \to H_0(X) \to ...$$

Since X is connected, $H_0(X) = \mathbb{Z}$. Plugging in the values of $H_*(S^1)$ and $H_*(S^3)$ we get

$$0 \to H_4(X) \to 0$$

$$0 \to H_3(X) \to 0 \to \mathbb{Z} \to H_2(X) \to 0$$
$$0 \to H_1(X) \to \mathbb{Z}^2 \to 0.$$

Hence $H_2(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}^2$, $H_0(X) = \mathbb{Z}$ and all other homology groups vanish. Note that all homology groups are $\mathbb{Z}$-free, hence the cohomology groups are just their $\mathbb{Z}$-duals. Thus $H^2(X) = \mathbb{Z}$, $H^1(X) = \mathbb{Z}^2$, $H^0(X) = \mathbb{Z}$

4. Let $X = \{(x, y, z) : x^2 + y^2 = 1\} \subset \mathbb{R}^3$ be a cylinder. Show that the geodesics on X are *helices*, that is, curves such that the angle between their tangent lines and the vertical is constant.

Solution. We have a parameterization of the cylinder given by $(\theta, z) \mapsto (\cos\theta, \sin\theta, z)$, with $z \in \mathbb{R}$, $\theta \in [0.2\pi]$. Thus $T_{(\theta,z)}$ is spanned by $(-\sin\theta, \cos\theta, 0)$ and $(0, 0, 1)$.

Suppose $t \mapsto (\theta(t), z(t))$ is a geodesic. Put $\gamma(t) = (\cos\theta(t), \sin\theta(t), z(t))$. Being a geodesic means $\ddot{\gamma}$ is orthogonal to $T_{(\theta(t),z(t))}$ (the dot denotes differentiation with respect to t). We have

$$\dot{\gamma} = (-\sin\theta \cdot \dot{\theta}, \cos\theta \cdot \dot{\theta}, \dot{z})$$
$$\ddot{\gamma} = (-\cos\theta \cdot \dot{\theta} - \sin\theta \cdot \ddot{\theta}, -\sin\theta \cdot \dot{\theta} + \cos\theta \cdot \ddot{\theta}, \ddot{z})$$

Thus the geodesic equations say

$$\sin\theta\cos\theta \cdot \dot{\theta} + \sin^2\theta \cdot \ddot{\theta} + \cos^2\theta \cdot \ddot{\theta} - \cos\theta\sin\theta \cdot \dot{\theta} = 0$$
$$\ddot{z} = 0$$

Hence $\ddot{\theta} = 0$, $\ddot{z} = 0$, so $\theta(t) = at + c$, $z = bt + d$ for some constants a, b. But this is precisely the equation of a helix (the tangent line is $(-a\sin\theta, a\cos\theta, b)$, which makes a constant angle with the vertical).

5. (a) Show that if every closed and bounded subspace of a Hilbert space E is compact, then E is finite dimensional.

(b) Show that any decreasing sequence of nonempty, closed, convex, and bounded subsets of a Hilbert space has a nonempty intersection.

(c) Show that any closed, convex, and bounded subset of a Hilbert space is the intersection of the closed balls that contain it.

(d) Deduce that any closed, convex, and bounded subset of a Hilbert space is compact in the weak topology.

Solution.

(a) Suppose every closed and bounded subset of E is compact. If E is infinite-dimensional, we can choose an infinite sequence of orthonormal vectors e_1, e_2,... Consider the set $\{e_1, e_2, ...\}$ Because $\|e_i - e_j\| = \sqrt{2}$ for $i \neq j$, this sequence does not contain any Cauchy subsequence. In particular it is closed and clearly bounded, and cannot contain a convergent subsequence, hence is not compact.

(b) It follows from (d) that a closed, convex, bounded subset of E is weakly compact. Since a decreasing sequence of closed, convex, bounded subset has the finite intersection property, it follows from compactness that the intersection of the family is non-empty. (The argument below makes no use of (b)).

(c) Let C be a closed, convex, bounded subset of E. Suppose $a \notin C$. Let $c_0 \in C$ be the point in C closest to a (such a point exists because C is closed), and let $d = \|a - c_0\| > 0$.
We claim this point is unique: If c_1 is another such point, then

$$2d^2 = \|a - c_0\|^2 + \|a - c_1\|^2 = 2\|a - \frac{c_0 + c_1}{2}\|^2 + 2\|\frac{c_0 - c_1}{2}\| \geq 2d^2.$$

Thus equalities occur, and $c_0 = c_1$. Now let H be the hyperplane through c_0 orthogonal to the segment $[a, c_0]$. For any $c \in C$, $0 \leq t \leq 1$, we have $|a-(tc+(1-t)c_0)\|^2 \geq \|a-c_0\|^2$ hence $-2t(a-c_0, c-c_0)+t^2(c-c_0, c-c_0) \geq 0$. Letting $t \to 0$ gives $(c_0 - a, c_0 - c) \leq 0$, hence H separates a and C. Now let L be the hyperplane going through $\frac{a+c_0}{2}$ and perpendicular to $a - c_0$. We choose a point u in the line through a and c_0 such that $\|u - \frac{a+c_0}{2}\| = R$, where R is to be determined. For any $c \in C$, let c' be the orthogonal projection of c onto L, put $h = \|c - c'|$, $t = \|c' - \frac{a+c_0}{2}\|$ We have $\|u - c\|^2 - R^2 = (R - h)^2 + t^2 - R^2 = h^2 - 2Rh + t^2$. The last expression is ≤ 0 iff $R - \sqrt{R^2 - t^2} \leq h \leq R + \sqrt{R^2 - t^2}$. Now observe that as C is bounded, t and h are bounded as c varies in C, and $h \geq \frac{d}{2}$. Hence if we choose R sufficiently large we can ensure that $\|u - c\|^2 - R^2 \leq 0$. Then the closed ball centered at u of radius R will contain C but not a.

(d) In view of (c), it suffices to prove that the closed unit ball B in E is weakly compact. Consider the map $B \to \prod_{v \in E}[-\|v\|, \|v\|]$ given by $b \mapsto (b, v)$. The target is compact by Tychonoff's theorem. The map is clearly injective and the product topology on the target induces the weak topology on B by definition. Thus it suffices to check that B is closed. But this is clear because B is precisely the set of $(x_v)_v$ such that $x_{\lambda \cdot v + \mu \cdot w} = \lambda \cdot v + \mu \cdot w$ (the condition implies that $v \mapsto x_v$ is a continuous functional on E with norm ≤ 1, hence is of form $v \mapsto (b, v)$ for a unique $b \in B$).

6. Let p be a prime, and let G be the group $\mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$.

(a) How many subgroups of order p does G have?

(b) How many subgroups of order p^2 does G have? How many of these are cyclic?

Solution.

(a) A subgroup of order p of G is a one-dimensional $\mathbb{F}_p$-subspace of the p-torsion $G[p] \simeq \mathbb{F}_p^2$. There are $p^2 - 1$ non-zero vectors in $G[p]$, and each of them generate a one-dimensional $\mathbb{F}_p$-subspace. Each such line contains exactly $p - 1$ non-zero vectors, hence the number of order p subgroup is $\frac{p^2-1}{p-1} = p + 1$.

(b) Since G is abelian, any subgroup of order p^2 of G must be isomorphic to $\mathbb{Z}/p \oplus \mathbb{Z}/p$ or $\mathbb{Z}^2$. The first possibility happens precisely for the p-torsion subgroup $G[p]$, hence there is only one such subgroup.
To count the number of cyclic subgroups of G of order p^2, we count the number of elements not of order p (note G is killed by p^2). G has precisely p^2 elements of order dividing p, hence $p^4 - p^2$ elements of order p^2. Each such element generate a cyclic subgroup of order p^2. Each such subgroup contains $p^2 - p$ elements of exact order p^2. Hence G has $\frac{p^4-p^2}{p^2-p} = p(p+1)$ cyclic subgroup of order p^2.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday January 21, 2010 (Day 3)

1. Consider the ring

$$A = \mathbb{Z}[x]/(f) \quad \text{where} \quad f = x^4 - x^3 + x^2 - 2x + 4.$$

Find all prime ideals of A that contain the ideal (3).

Solution. Prime ideals of A that contains (3) are in bijection with prime ideals of

$$A/3A \cong \mathbb{F}_3[x]/(f) = \mathbb{F}_3[x]/(x-1)(x+1)(x^2-x-1).$$

Note $x^2 - x - 1$ is irreducible in $\mathbb{F}_3[x]$ because it has no zeroes in $\mathbb{F}_3$. Hence $A/3A$ has precisely 3 prime ideals, namely those generated by $x-1$, $x+1$ and x^2-x-1. Hence the primes of A containing (3) are $(3, x-1)$, $(3, x+1)$ and $(3, x^2-x-1)$.

2. Let f be a holomorphic function on a domain containing the closed disc $\{z : |z| \leq 3\}$, and suppose that

$$f(1) = f(i) = f(-1) = f(-i) = 0.$$

Show that

$$|f(0)| \leq \frac{1}{80} \max_{|z|=3} |f(z)|$$

and find all such functions for which equality holds in this inequality.

Solution. The assumption on f implies $f(z) = (z^4-1)g(z)$ for an analytic function $g(z)$ with the same domain as f. We have $|f(0)| = |g(0)|$. By the maximum modulus principle, $|g(0)| \leq \max_{|z|=3} |g(z)|$. On $|z| = 3$, $|f(z)| = |(z^4-1)g(z)| \geq (3^4-1)|g(z)|$. Hence $|f(0)| \leq \frac{1}{80} \max_{|z|=3} |f(z)|$. For equality to appear, we must have $|g(0)| = \max_{|z|=3} |g(z)|$, hence g is constant. But then $f(z) = c(z^4-1)$ does not make the equality hold, because $\max_{|z|=3} |z^4-1| = 82$.

3. Let $f : \mathbb{R} \to \mathbb{R}^+$ be a differentiable, positive real function. Find the Gaussian curvature and mean curvature of the surface of revolution

$$S = \{(x, y, z) : y^2 + z^2 = f(x)\}.$$

Solution. The surface S has a parameterization $(x,\theta)\mapsto\Phi(x,\theta)=(x,\sqrt{f(x)}\cos\theta,\sqrt{f(x)}\sin\theta)$ for $x\in\mathbb{R}$, $\theta\in[0,2\pi)$. Hence (the lower index signifies the variable with respect to which we differentiate)

$$\Phi_x=(1,\frac{f'}{2\sqrt{f}}\cos\theta,\frac{f'}{2\sqrt{f}}\sin\theta)$$

$$\Phi_\theta=(0,-\sqrt{f}\sin\theta,\sqrt{f}\cos\theta)$$

$$\Phi_x\wedge\Phi_\theta=(\frac{f'}{2},-\sqrt{f}\cos\theta,-\sqrt{f}\sin\theta)$$

$$|\Phi_x\wedge\Phi_\theta|=\sqrt{f+\frac{f'^2}{4}}$$

$$\Phi_{xx}=(0,\frac{2ff''-f'^2}{4f\sqrt{f}}\cos\theta,\frac{2ff''-f'^2}{4f\sqrt{f}}\sin\theta)$$

$$\Phi_{x\theta}=(0,-\frac{f'}{2\sqrt{f}}\sin\theta,\frac{f'}{2\sqrt{f}}\cos\theta)$$

$$\Phi_{\theta\theta}=(0,-\sqrt{f}\cos\theta,-\sqrt{f}\sin\theta)$$

The second fundamental form is $Ldx^2+2Mdxd\theta+Nd\theta^2$, where

$$L=\frac{\Phi_x\wedge\Phi_\theta}{|\Phi_x\wedge\Phi_\theta|}\cdot\Phi_{xx}=-\frac{2ff''-f'^2}{4f\sqrt{f+\frac{f'^2}{4}}}$$

$$M=\frac{\Phi_x\wedge\Phi_\theta}{|\Phi_x\wedge\Phi_\theta|}\cdot\Phi_{x\theta}=0$$

$$N=\frac{\Phi_x\wedge\Phi_\theta}{|\Phi_x\wedge\Phi_\theta|}\cdot\Phi_{\theta\theta}=\frac{f}{\sqrt{f+\frac{f'^2}{4}}}$$

The Gaussian curvature is

$$K=L\cdot N-M^2=-\frac{2ff''-f'^2}{4f+f'^2}$$

The mean curvature is

$$H=L+N=\frac{4f^2-2ff''+f'^2}{4f\sqrt{f+\frac{f'^2}{4}}}$$

4. Show that for any given natural number n, there exists a finite Borel measure on the interval $[0,1]\subset\mathbb{R}$ such that for any real polynomial of degree at most n, we have

$$\int_0^1 p\,d\mu=p'(0).$$

Show, on the other hand, that there does *not* exist a finite Borel measure on the interval $[0,1] \subset \mathbb{R}$ such that for any real polynomial we have

$$\int_0^1 p\,d\mu = p'(0).$$

Solution. Note that $P_k(x) = (k+1)x^k$ with $k \leq n$ form basis for the space of polynomials of degree at most n. We will construct the desired measure μ as follows: on $[\frac{i}{n+1}, \frac{i+1}{n+1}]$ μ will be x_i times the Lebesgue measure. We show that there is a choice of x_i so that μ has the desired property. We want to have

$$\int_0^1 P_k d\mu = \sum_{i=0}^n ((\frac{i+1}{n+1})^{k+1} - (\frac{i}{n+1})^{k+1})x_i = P_k'(0).$$

This is a system of $n+1$ linearly independent linear equation in $n+1$ variables, hence has a solution. (To see the linear independence, note the matrix $(\frac{i+1}{n+1})^{k+1} - (\frac{i}{n+1})^{k+1})_{i,k}$ has the same determinant as a Van der Monde determinant, as can be seen by adding the first column to the second column, then add the second column to the third column and so on).

Now suppose μ is a finite Borel measure on $[0,1]$ such that

$$\int_0^1 p d\mu = p'(0)$$

for all real polynomials p. Let $f(x)$ be the characteristic function of the set $\{0\}$, and put $q_n(x) = (1-x)^n$. Then

$$|\int_0^1 (f - q_n) d\mu| \leq \mu([0.1])$$

so

$$n = q_n'(0) = \int_0^1 q_n d\mu \leq \int_0^1 f d\mu + \mu([0,1])$$

for all n, a contradiction.

5. Let $X = \mathbb{RP}^2 \times \mathbb{RP}^4$.

(a) Find the homology groups $H_*(X, \mathbb{Z}/2)$

(b) Find the homology groups $H_*(X, \mathbb{Z})$

(c) Find the cohomology groups $H^*(X, \mathbb{Z})$

Solution. The Künneth formula says (for coefficient ring R)

$$H_n(X \times Y, R) = \oplus_{i+j=n} H_i(X,R) \otimes H_j(Y,R) \oplus_{i+j=n-1} \mathrm{Tor}_R(H_i(X,R), H_j(Y,R)).$$

In particular for a field

$$H_n(X \times Y, R) = \oplus_{i+j=n} H_i(X,R) \otimes H_j(Y,R)$$

We have (the groups not shown are 0)

	0	1	2	3	4
$H_*(\mathbb{RP}^2, \mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	0
$H_*(\mathbb{RP}^4, \mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$H_*(\mathbb{RP}^2, \mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}/2$	0	0	0
$H_*(\mathbb{RP}^4, \mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0

So by the Künneth formula:

(a)

	0	1	2	3	4	5	6
$H_*(X, \mathbb{Z}/2)$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$

(b)

	0	1	2	3	4	5
$H_*(X, \mathbb{Z})$	$\mathbb{Z}$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

(c) By the universal coefficient theorem (for an abelian group G)

$$H^n(X, G) = \mathrm{Hom}(H_n(X, \mathbb{Z}), G) \oplus \mathrm{Ext}^1(H_{n-1}(X, \mathbb{Z}), G)$$

hence by the previous part

	0	1	2	3	4	5	6
$H^*(X, \mathbb{Z})$	$\mathbb{Z}$	0	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

6. By a *twisted cubic curve* we mean the image of the map $\mathbb{P}^1 \to \mathbb{P}^3$ given by

$$[X, Y] \mapsto \big[F_0(X, Y), F_1(X, Y), F_2(X, Y), F_3(X, Y)\big]$$

where $F_0, \dots, F_3$ form a basis for the space of homogeneous cubic polynomials in X and Y.

(a) Show that if $C \subset \mathbb{P}^3$ is a twisted cubic curve, then there is a 3-dimensional vector space of homogeneous quadratic polynomials on $\mathbb{P}^3$ vanishing on C.

(b) Show that C is the common zero locus of the homogeneous quadratic polynomials vanishing on it.

(c) Suppose now that $Q, Q' \subset \mathbb{P}^3$ are two distinct quadric surfaces containing C. Describe the intersection $Q \cap Q'$.

Solution.

(a) Up to a projective automorphism, the twisted cubic is isomorphic to the parametric curve $[X : Y] \mapsto [X^3 : X^2Y : XY^2 : Y^3]$. Given a homogenous quadratic polynomial Q, Q will vanish on this curve iff $Q(X^3, X^2Y, XY^2, Y^3)$ is the *zero* polynomial. This happens iff each coefficient of this degree 6 homogenous polynomial vanish. This gives 7 linear condition on the 10 coefficients of Q, which are linearly independent because each equation involve a distinct set of coefficients. It follows that the space of homogenous quadratic polynomial vanishing on a twisted cubic has dimension $10 - 7 = 3$.

(b) As above, we assume the twisted cubic is given by $[X : Y] \mapsto [X^3 : X^2Y : XY^2 : Y^3]$. In this case we see that it lies in the 3 quadrics $AD - BC = 0$, $B^2 - AC = 0$ and $C^2 - BD = 0$ (here $[A : B : C : D]$ are homogenous coordinates for $\mathbb{P}^3$). We claim that the intersection of these three quadrics is the twisted cubic. Indeed assume $[A : B : C : D]$ lies in the intersection. Without loss of generality we assume $A \neq 0$, and put $A = X^3$ for $X \neq 0$. Put $Y = B/X^2$. Then $C = B^2/A = XY^2$ and $D = BC/A = Y^3$, so $[A : B : C : D]$ lies in the twisted cubic.

(c) From the definition, the twisted cubic C does not lie in any hyperplane. Hence it cannot lie in any reducible quadric, so all quadrics that contain it are irreducible. For quadrics Q, Q' containing C, their intersection will have all component of dimension 1. The intersection has total multiplicity 4 by Bézout's theorem. The twisted cubic part contributes a multiple of 3. So the intersection contains another component with multiplicity 1 and is a curve of degree 1, i.e. a line. Hence $Q \cap Q'$ is the union of the twisted cubic and a line (all with multiplicity 1) .