QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Tuesday September 4, 2018 (Day 1)

1. (AT)

(a) Let X and Y be compact, oriented manifolds of the same dimension n. Define the *degree* of a continuous map $f : X \to Y$.

(b) Let $f : \mathbb{CP}^3 \to \mathbb{CP}^3$ be any continuous map. Show that the degree of f is of the form m^3 for some integer m.

(c) Show that conversely for any $m \in \mathbb{Z}$ there is a continuous map $f : \mathbb{CP}^3 \to \mathbb{CP}^3$ of degree m^3.

Solution: For the first part, the induced map $f^* : H^n(Y, \mathbb{Z}) \cong \mathbb{Z} \to H^n(X, \mathbb{Z}) \cong \mathbb{Z}$ (where the isomorphisms with $\mathbb{Z}$ are given by the orientation) is multiplication by some integer d; this is the degree of f.

For the second part, note that $H^*(\mathbb{CP}^3, \mathbb{Z}) \cong \mathbb{Z}[\zeta]/(\zeta^4)$ and that f^* is a ring homomorphism. If $f^*(\zeta) = m\zeta$, then $f^*(\zeta^3) = m^3\zeta^3$ and so the degree must be a cube. To see that all cubes occur, just consider the map $[X, Y, Z, W] \mapsto [X^m, Y^m, Z^m, W^m]$ for positive $d = m^3$; take complex conjugates to exhibit maps with negative degrees.

2. (A) Let G be a group.

(a) Prove that, if V and W are irreducible G-representations defined over a field $\mathbb{F}$, then a G-homomorphism $f : V \to W$ is either zero or an isomorphism.

(b) Let $G = D_8$ be the dihedral group with 8 elements. What are the dimensions of its irreducible representations over $\mathbb{C}$?

Solution: (a) If f is nonzero, then its image in W is a nontrivial subrepresentation of W and hence W itself by irreducibility; therefore, f is surjective. Similarly, the kernel of f is a subrepresentation of V, which, if nontrivial, must be V itself, contradicting the assumption that f is nonzero; therefore f is injective. (b) There are five irreducible representations of D_8, four one-dimensional ones coming from characters of the quotient $\mathbb{Z}/2 \times \mathbb{Z}/2$ of D_8 by its center, and one two-dimensional representation corresponding to the realization of D_8 as the group of automorphisms of the plane preserving a square of D_8. The fact that the irreducible representations have dimensions 1, 1, 1, 1 and 2 can also be seen by arguing that the number of irreducible representations is the same as the number of conjugacy classes in D_8, which is 5, and that the sum of the squares of their dimensions must be 8.

3. (CA) Let f_n be a sequence of analytic functions on the unit disk $\Delta \subset \mathbb{C}$ such that $f_n \to f$ uniformly on compact sets and such that f is not identically zero. Prove that $f(0) = 0$ if and only if there is a sequence $z_n \to 0$ such that $f_n(z_n) = 0$ for n large enough.

Solution: $\Longleftarrow$ We begin by observing that there must be some ϵ such that for n large enough, f_n is not zero in a neighborhood of the circle. By uniform convergence, since $f_n \to f$, it must also be that $f_n' \to f'$. thus

$$\lim \int_{C_\epsilon} \frac{f_n'}{f_n'} dz = \int \frac{f'}{f} dz$$

The right handside must, eventually, be larger than 1, so the left hand side must be as well. As this holds for every ϵ, we see that f has a zero in $B_\epsilon(0)$ for every ϵ. Since f is not identically zero it must be the only zero.

$\Longrightarrow$ By the argument principle,

$$\frac{1}{2\pi} \int_{C_\epsilon} \frac{f'}{f} dz = 1$$

where C_ϵ is the circle of radius ϵ around zero for some ϵ suffiently small. On the otherhand,

$$\lim \int_{C_\epsilon} \frac{f_n'}{f_n'} dz = \int \frac{f'}{f} dz \geq 1.$$

so by the argument principle again, then for every ϵ, there is an N large enoug that $n \geq N$ yields f_n has a at least one zero. applying this for each $\epsilon \to 0$ yields the result.

4. (AG) Let K be an algebraically closed field of characteristic 0, and let $\mathbb{P}^n$ be the projective space of homogeneous polynomials of degree n in two variables over K. Let $X \subset \mathbb{P}^n$ be the locus of n^{th} powers of linear forms, and let $Y \subset \mathbb{P}^n$ be the locus of polynomials with a multiple root (that is, a repeated factor).

(a) Show that X and $Y \subset \mathbb{P}^n$ are closed subvarieties.

(b) What is the degree of X?

(c) What is the degree of Y?

Solution: First, X is the image of the map $\mathbb{P}^1 \to \mathbb{P}^n$ sending $[a, b] \in \mathbb{P}^1$ to $(ax + by)^n \in \mathbb{P}^n$. This is projectively equivalent (in characteristic 0!) to the degree n Veronese map, whose image is a closed curve of degree n. Second, Y is the zero locus of the discriminant, which is a polynomial of degree $2n - 2$ in the coefficients of a polynomial of degree n (this number can be deduced from the Riemann-Hurwitz formula, which says that a degree n map from $\mathbb{P}^1$ to $\mathbb{P}^1$ has $2n-2$ branch points; that is, a general line in $\mathbb{P}^n$ meets Y in $2n-2$ points). Thus $Y \subset \mathbb{P}^n$ is a hypersurface of degree $2n - 2$.

5. (DG) Given a smooth function $f : \mathbb{R}^{n-1} \to \mathbb{R}$, define $F : \mathbb{R}^n \to \mathbb{R}$ by

$$F(x_1, \dots, x_n) := f(x_1, \dots, x_{n-1}) - x_n$$

and consider the preimage $X_f = F^{-1}(0) \subset \mathbb{R}^n$.

(a) Prove that X_f is a smooth manifold which is diffeomorphic to $\mathbb{R}^{n-1}$.

(b) Consider the two examples X_f and $X_g \subset \mathbb{R}^3$ with $f(x_1, x_2) = x_1^2 + x_2^2$ and $g(x_1, x_2) = x_1^2 - x_2^2$. Compute their normal vectors at every point $(x_1, x_2, x_3) \in X_f$ and $(x_1, x_2, x_3) \in X_g$.

Solution.
Part (a). The last row of the Jacobian of $F : \mathbb{R}^n \to \mathbb{R}$ is $(0, \dots, 0, 1)$ and so the Jacobian has rank 1 everywhere. This implies that $F^{-1}(0)$ is a smooth manifold. It has the global chart $\psi : F^{-1}(0) \to \mathbb{R}^{n-1}$ defined by

$$\psi(x_1, \dots, x_n) := (x^1, \dots, x_{n-1})$$

and is therefore diffeomorphic to $\mathbb{R}^{n-1}$.

Part (b). The first example is a paraboloid. Its normal vector is

$$\frac{(-2x, -2y, 1)}{\sqrt{1 + 4x^2 + 4y^2}}.$$

The second example is a hyperbolic paraboloid or "saddle surface". Its normal vector is

$$\frac{(-2x, 2y, 1)}{\sqrt{1 + 4x^2 + 4y^2}}.$$

6. (RA) Let $K \subset \mathbb{R}^n$ be a compact set. Show that for any measurable function $f : K \to \mathbb{C}$, it holds that

$$\lim_{p\to\infty} \|f\|_{L^p(K)} = \|f\|_{L^\infty(K)}.$$

(Recall that $\|f\|_{L^p(K)} = \left(\int_K |f|^p \mathrm{d}x\right)^{1/p}$ and that $\|f\|_{L^\infty(K)}$ is the essential supremum of f, i.e., the smallest upper bound if the behavior of f on null sets is ignored.)

Solution.
Let $p > 1$. Since $|f| \le \|f\|_{L^\infty(K)}$ holds almost everywhere, we have

$$\|f\|_{L^p(K)} = \left(\int_K |f|^p \mathrm{d}x\right)^{1/p} \le |K|^{1/p} \|f\|_{L^\infty(K)} \overset{p\to\infty}{\to} \|f\|_{L^\infty(K)}.$$

It remains to prove the lower bound. Let $\epsilon > 0$. By definition of the essential supremum, there exists a set $A \subset K$ of Lebesgue measure $|A| > 0$, such that $|f| \ge (1-\epsilon)\|f\|_{L^\infty(K)}$ holds on A. Hence,

$$\|f\|_{L^p(K)} \ge \left(\int_A |f|^p \mathrm{d}x\right)^{1/p} \ge (1-\epsilon)\|f\|_{L^\infty(K)} |A|^{1/p} \overset{p\to\infty}{\to} (1-\epsilon)\|f\|_{L^\infty(K)}.$$

Sending $\epsilon \to 0$ proves the claim.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Wednesday September 5, 2018 (Day 2)

1. (AG) Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree d.

(a) Let K_C be the canonical bundle of C. For what integer n is it the case that $K_C \cong \mathcal{O}_C(n)$?

(b) Prove that if $d \geq 4$ then C is not hyperelliptic.

(c) Prove that if $d \geq 5$ then C is not trigonal (that is, expressible as a 3-sheeted cover of $\mathbb{P}^1$).

Solution: By the adjunction formula, the canonical divisor class is $K_C = \mathcal{O}_C(d-3)$, that is, plane curves of degree $d-3$ cut out canonical divisors on C.

Now, if C were hyperelliptic—meaning that there exists a degree 2 map $\pi : C \to \mathbb{P}^1$—a general fiber of π would consist of two points $p, q \in C$ moving in a pencil, that is, such that $h^0(\mathcal{O}_C(p+q)) = 2$. But if $d \geq 4$ then any two points $p, q \in C$ impose independent conditions on the canonical series $|K_C|$; that is, $h^0(K_C(-p-q)) = g-2$, so by Riemann-Roch $h^0(\mathcal{O}_C(p+q)) = 1$, and hence C is not hyperelliptic. Similarly, if $d \geq 5$ then any three points $p, q, r \in C$ impose independent conditions on the canonical series $|K_C|$; by Riemann-Roch it follows that $h^0(\mathcal{O}_C(p+q+r)) = 1$ so C is not trigonal.

2. (CA) (The 1/4 theorem). Let $\mathcal{S}$ denote the class of functions that are analytic on the disk and one-to-one with $f(0) = 0$ and $f'(0) = 1$.

(a) Prove that if $f \in \mathcal{S}$ and w is not in the range of f then

$$g(z) = \frac{wf(z)}{(w - f(z))}$$

is also in $\mathcal{S}$.

(b) Show that for any $f \in \mathcal{S}$, the image of f contains the ball of radius $1/4$ around the origin. You may use the elementary result (Bieberbach) that if $f(z) = z + \sum_{k \geq 2} a_k z^k$ in $\mathcal{S}$ then $|a_2| \leq 2$.

Solution: The proof of the first part is by checking. Since $w \notin R(f)$ it is analytic on the disk. now observe that map $h(z)$ which is

$$h(z) = \frac{wz}{(w-z)}$$

is one-to-one. thus $g(z) = h \circ f$ so it is one-to-one. Finally since

$$g'(z) = \frac{w^2 f'(z)}{(w - f(z))^2}$$

it follows that $g(0) = 0$ and $g'(0) = 1$ as desired.

For the second part, Suppose that w is not in the image of f. Then we may look at

$$g(z) = \frac{wf(z)}{(w - f(z))^2}.$$

Observe that

$$|g''(0)| = |a_2 + \frac{1}{w}| \leq 2$$

and $a_2 \leq 2$. From this it follows that

$$|1/w| \leq |a_2| + |a_2 + 1/w| \leq 4,$$

from which it follows that $|w| \geq 1/4$. Thus the set $|w| \leq 1/4$ is in the image of f as desired.

3. (A) Find a polynomial $f \in \mathbb{Q}[x]$ whose Galois group (over $\mathbb{Q}$) is D_8, the dihedral group of order 8.

Solution: There are lots of ways to find examples. Here is one: consider a quartic polynomial whose cubic resolvent has exactly one rational root and discriminant is nonsquare. Indeed, ordering the roots as α_1 through α_4, suppose $\alpha_1\alpha_2 + \alpha_3\alpha_4$ is rational so that the Galois group is contained in the dihedral group generated by $(1324), (13)(24)$. We want to ensure that the Galois group is no smaller: that the other roots of the resolvent are not rational ensures that the Galois group is not contained in the Klein subgroup generated by $(12)(34), (13)(24)$; equivalently, this is the restriction that the discriminant be nonsquare and so the Galois group not be contained in the alternating group. But now, if K represents the splitting field, we have the exact sequence $1 \to \mathrm{Gal}(K/\mathbb{Q}(\sqrt{D})) \to \mathrm{Gal}(K/\mathbb{Q}) \to \mathbb{Z}/2 \to 1$ and if $\mathrm{Gal}(K/\mathbb{Q})$ were any smaller than the D_8 in which it is already contained, $\mathrm{Gal}(K/\mathbb{Q}(\sqrt{D}))$ would have order at most 2 and hence the polynomial would have (multiple) roots over $\mathbb{Q}(\sqrt{D})$. Hence it suffices to find a quartic polynomial with a cubic resolvent with exactly one rational root that stays irreducible over

$\mathbb{Q}(\sqrt{D})$. After some experimentation, $f(x) = x^4 + 3x + 3$ with cubic resolvent $(x+3)(x^2 - 3x + 3)$, discriminant $3^3 \cdot 5^2 \cdot 7$, and which over $\mathbb{Q}(\sqrt{21})$ has no roots, as one can check manually: suppose $\alpha + \beta\sqrt{21}$ were a root. We know the ring of integers of $\mathbb{Q}(\sqrt{21})$ and will use that $\tilde{\alpha} = 2\alpha, \tilde{\beta} = 2\beta$ are (usual) integers. Expanding the equation, we find that $4\alpha(\alpha^2 + 21\beta^2) = -3$, or $\tilde{\alpha}(\tilde{\alpha}^2 + 21\tilde{\beta}^2) = -6$. This cannot happen without $\tilde{\beta} = 0$, which is impossible as f is irreducible over $\mathbb{Q}$.

4. (RA)

(a) Let $a_k \geq 0$ be a monotone increasing sequence with $a_k \to \infty$, and consider the ellipse,

$$E(a_k) = \{v \in \ell^2(\mathbb{Z}) : \sum a_k v_k^2 \leq 1\}.$$

Show that $E(a_n)$ is a compact subset of $\ell^2(\mathbb{Z})$.

(b) Let $\mathbb{T}$ denote the one-dimensional torus; that is, $\mathbb{R}/2\pi\mathbb{Z}$, or $[0, 2\pi]$ with the ends identified. Recall that the space $H^1(\mathbb{T})$ is the closure of $C^\infty(\mathbb{T})$ in the norm

$$\|f\|_{H^1(\mathbb{T})} = \sqrt{\|f\|_{L^2(\mathbb{T})} + \|\frac{d}{dx}f\|_{L^2(\mathbb{T})}}.$$

Use part (a) to conclude that the inclusion $i : H^1(\mathbb{T}) \hookrightarrow L^2(\mathbb{T})$ is a compact operator.

Solution: Part a. Firstly, since a_k is monotone,

$$\sum a_k v_k^2 \leq 1$$

implies that for any L,

$$\sum_{k \geq L} v_k^2 \leq \frac{1}{a_L}.$$

Thus E is is norm bounded.

Suppose that we have sequence $v^n \in E(a_k)$. Observe that

$$v_k^n \leq \frac{1}{\sqrt{a_k}}.$$

Thus we may diagonalize this sequence pointwise to obtain a sequence v_k with $v_k \in \ell^\infty$. Passing to this subsequence, we see that by fatou's lemma,

$$\sum v_k^2 \leq \frac{1}{a_1},$$

i.e., v is in ℓ^2. It remains to show that $v^n \to v$ in ℓ^2. To see this, observe that

$$\sum |v_k^n - v_k|^2 \le \sum_{k \ge L} |v_k^n - v_k|^2 + \sum_{k \le L} \dots$$
$$\le \frac{2}{a_L} + \sum_{k \le L} |v_k^n - v_k|^2.$$

Sending $n \to \infty$ and then $L \to \infty$ yields the result.

Part b. It suffices to show that the unit ball of $H^1(\mathbb{T})$ is a compact subset of $L^2(\mathbb{T})$. By Parseval's identity/the fourier transform, it follows that

$$\sum k^2 |\hat{f}_n(k)|^2 \le C.$$

for some positive constant. This is a compact subset of ℓ_2 by part a. Thus by fourier inversion, the ball of H^1 is as well.

5. (AT) Consider the following topological spaces:

$$A = S^1 \times S^1 \qquad\qquad B = S^1 \vee S^1 \vee S^2.$$

(a) Compute the fundamental group of each space.

(b) Compute the integral cohomology ring of each space.

(c) Show that B is not homotopy equivalent to any compact orientable manifold.

Solution: (a) The fundamental group construction preserves products, so

$$\pi_1(A) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}.$$

By the Van Kampen theorem,

$$\pi_1(B) \cong \pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z},$$

where the first step uses that S^2 is simply connected. (b) By the Künneth theorem,

$$H^*(S^1 \times S^1) \cong H^*(S^1) \otimes H^*(S^1) \cong \Lambda[x, y], \quad |x| = |y| = 1.$$

Since the reduced cohomology ring construction takes wedges of spaces to products of (nonunital) rings,

$$H^*(S^1 \vee S^1 \vee S^2) \cong \frac{\Lambda[x, y, z]}{xy = yz = zx = 0}, \quad |x| = |y| = 1, \ |z| = 2.$$

(c) Suppose that B is homotopy equivalent to the compact orientable manifold M. Choosing a fundamental class $[M]$, Poincaré duality guarantees that the assignment

$$(a, b) \mapsto \langle a \smile b, [M] \rangle$$

defines a symplectic form on $H^1(M)$. Since $xy = 0$, this pairing is degenerate, a contradiction.

6. (DG) Consider the set

$$G := \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & x & y \\ 0 & 0 & 1 \end{pmatrix} : x \in \mathbb{R}_+,\ y \in \mathbb{R} \right\},$$

and equip it with a smooth structure via the global chart that sends $(x, y) \in \mathbb{R}_+ \times \mathbb{R}$ to the corresponding element of G.

(a) Show that G is a Lie subgroup of the Lie group $GL(\mathbb{R}, 3)$.

(b) Prove that the set

$$\left\{ x\frac{\partial}{\partial x}, x\frac{\partial}{\partial y} \right\}$$

forms a basis of left-invariant vector fields on G.

(c) Find the structure constants of the Lie algebra $\mathfrak{g}$ of G with respect to the basis in (b).

Solution.
Part (a). We consider the multiplication and inverse operations on G:

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 & 0 \\ 0 & x & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} ax & 0 & 0 \\ 0 & ax & ay + b \\ 0 & 0 & 1 \end{pmatrix} \in G$$

and

$$\begin{pmatrix} x & 0 & 0 \\ 0 & x & y \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/x & 0 & 0 \\ 0 & 1/x & -y/x \\ 0 & 0 & 1 \end{pmatrix} \in G.$$

We see that G is closed under these operations, and that they are smooth. This proves that G is a Lie group itself.

Moreover: (a) Since the inverse of any element in G exists, G is a subset of $GL(\mathbb{R}, 3)$. (b) The inclusion map $G \to GL(\mathbb{R}, 3)$ is trivially a group homomorphism. (c) The inclusion map $G \to GL(\mathbb{R}, 3)$ is an immersion. To see this, recall that the smooth structure on $GL(\mathbb{R},3)$ is that of $\mathbb{R}^9$ and note that the map $(x,y) \mapsto (x,0,0,0,x,y,0,0,1)$ has rank 2.

Together, (a)-(c) imply that G is a Lie subgroup of $GL(\mathbb{R},3)$.

Part (b). Linear independence follows from the linear independence of $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$. To prove left-invariance, let us identify a vector (x,y) with the corresponding matrix in G. The formula for the product $G \times G \to G$ shows that left translation in G is given by

$$L_{(a,b)}(x,y) = (ax, ay+b),$$

and so $L_{(a,b)*} = a\,\mathrm{Id}_{\mathbb{R}^2}$. This shows that

$$L_{(a,b)*}X_{(x,y)} = X_{(a,b)(x,y)}$$

holds for both vector fields X in (b), hence they are left-invariant.

Part (c). Let us call the two vector fields in (b) X_1, X_2, respectively. Explicit computation shows that

$$[X_1, X_2] = X_2,$$

and so the non-zero structure constants are $f_{12}^2 = -f_{21}^2 = 1$.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Thursday September 6, 2018 (Day 3)

1. (AT) Let $p : E \to B$ be a k-fold covering space, and suppose that the Euler characteristic $\chi(E)$ is defined, nonzero, and relatively prime to k. Show that any CW decomposition of B has infinitely many cells.

Solution: Suppose that B has a finite CW decomposition; in particular, $\chi(B)$ is defined. Restricted to each of the cells of B, the covering p is trivial, and the connected components of the total spaces of these restricted covers form a CW decomposition of E. Counting cells, we find that $\chi(E) = k \cdot \chi(B)$. Since $\chi(E) \neq 0$, it follows that k divides $\chi(E)$, a contradiction.

2. (RA) Let W be Gumbel distributed, that is $P(W \leq x) = e^{-e^{-x}}$. Let X_i be independent and identically distributed Exponential random variables with mean 1; that is, X_i are independent, with $P(X_i \leq x) = \exp(-\max x, 0)$.

Let

$$M_n = \max_{i \leq n} X_i.$$

Show that there are deterministic sequences a_n, b_n such that

$$\frac{M_n - b_n}{a_n} \to W$$

in law; that is, such that for any continuous bounded function F,

$$\mathbb{E}F\left(\frac{M_n - b_n}{a_n}\right) \to \mathbb{E}F(W).$$

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Solution: let $b_n = \log n$ and $a_n = 1$. Then

$$P(M_n - b_n \leq x) = P(X_i \leq x + \log n)^n$$

since X_i are i.i.d. Now

$$\begin{aligned} P(X \leq x + \log n)^n &= (1 - P(X > x + \log n))^n \\ &= \left(1 - \int_{x+\log n}^{\infty} e^{-w} dw\right)^n \\ &= \left(1 - \frac{1}{n}\int_{x}^{\infty} e^{-w} dw\right)^n \to e^{-e^{-x}} \end{aligned}$$

As $e^{-e^{-x}}$ is continuous every where and

$$P(M_n - b_n \leq x) \to P(W \leq x),$$

we see that $M_n - b_n \to W$ in law by the Portmanteau lemma.

3. (DG) Consider $\mathbb{R}^2$ as a Riemannian manifold equipped with the metric

$$g = e^x \mathrm{d}x^2 + \mathrm{d}y^2.$$

(i) Compute the Christoffel symbols of the Levi-Civita connection for g.

(ii) Show that the geodesics are described by the curves $x(t) = 2\log(At+B)$ and $y(t) = Ct + D$, for appropriate constants A, B, C, D.

(iii) Let $\gamma : \mathbb{R}_+ \to \mathbb{R}^2$, $\gamma(t) = (t, t)$. Compute the parallel transport of the vector $(1, 2)$ along the curve γ.

(iv) Are there two vector fields X, Y parallel to the curve γ, such that $g(X(t), Y(t))$ is non-constant?

Solution:
Part (i). We can identify

$$g^{-1} = \begin{pmatrix} e^{-x} & 0 \\ 0 & 1 \end{pmatrix}.$$

Denoting $x^1 = x$, $x^2 = y$, the only non-vanishing Christoffel symbol is

$$\Gamma^1_{11} = \frac{1}{2} g^{-1}_{11} \partial_1 g_{11} = \frac{1}{2}.$$

Part (ii). Using part (i), the two ODE describing the geodesic $(x(t), y(t))$ are given by

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = 0, \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = 0.$$

The second ODE is solved by $y(t) = Ct + D$. For the first ODE, we introduce $u(t) := \frac{\mathrm{d}x}{\mathrm{d}t}$ and obtain

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{1}{2}u^2 = 0.$$

By separation of variables, this is solved by $u(t) = \frac{2}{t+C_1}$. We integrate this to get x and find

$$x(t) = 2\log(t + C_1) + C_2 = 2\log(At + B),$$

where we redefined the constants in the second step.

Part (iii). The equation for parallel transport $\nabla_{\gamma'}(a^1, a^2) = 0$, with $\gamma(t) = (t, t)$, becomes

$$\frac{\mathrm{d}a^1}{\mathrm{d}t} + \frac{1}{2}a^1 = 0, \qquad \frac{\mathrm{d}a^2}{\mathrm{d}t} = 0.$$

These are solved by $a^1(t) = Ae^{-t/2}$ and $a^2(t) = B$, respectively. To satisfy the initial condition $(a^1(0), a^2(0)) = (1, 2)$, we take $A = 1$ and $B = 2$. The solution is thus

$$(a^1(t), a^2(t)) = (e^{-t/2}, 2).$$

Part (iv). No. Since ∇ is the Levi-Civita connection, the scalar product of two vectors is preserved by parallel transport.

4. (A) Let G be a group of order 78.

(a) Show that G contains a normal subgroup of index 6.

(b) Show by example that G may contain a subgroup of index 13 that is not normal.

Solution: (a) Sylow theory guarantees the existence of a 13-Sylow subgroup $H \leq G$, which has index 6. This Sylow subgroup is unique and hence normal; indeed, the number of such divides 6 and is congruent to 1 mod 13 by Sylow's theorems. (b) Take G to be the semidirect product $C_{13} \rtimes S_3$ of the cyclic group of order 13 and the symmetric group on 3 letters, where S_3 acts via the composite

$$S_3 \xrightarrow{\text{sgn}} C_2 \xrightarrow{\text{inv}} \mathrm{Aut}(C_{13})$$

of the sign homomorphism and the inversion homomorphism (we use that C_{13} is Abelian). We claim that the subgroup $S_3 \leq G$ is not normal. To see why this is so, let $\sigma \in S_3$ be an odd permutation and $\rho \in C_{13}$ a generator, and compute that

$$(\rho, e)(e, \sigma)(\rho^{-1}, e) = (\rho^2, \sigma) \notin S_3.$$

5. (AG) Let K be an algebraically closed field of characteristic 0, and consider the curve $C \subset \mathbb{A}^3$ over K given as the image of the map

$$\begin{aligned} \phi : & \mathbb{A}^1 \to \mathbb{A}^3 \\ & t \mapsto (t^3, t^4, t^5). \end{aligned}$$

Show that no neighborhood of the point $\phi(0) = (0,0,0) \in C$ can be embedded in $\mathbb{A}^2$.

Solution: Suppose $f(x,y,z)$ is any polynomial on $\mathbb{A}^3$ vanishing on C. The constant term of f must be zero, since f vanishes at $(0,0,0) \in C$, and the linear terms of f must also be zero, since the pullback to $\mathbb{A}^1$ of any monomial in x, y and z of degree 2 or more must vanish to order at least 6. In other words, the ideal $I(C)$ is contained in the square $(x,y,z)^2$ of the maximal ideal of the origin. In particular, the Zariski tangent space to C at $(0,0,0)$ is three dimensional, and hence no neighborhood of this point is embeddable in $\mathbb{A}^2$.

6. (CA) Let $f(z)$ be an entire function such that

a) $f(z)$ vanishes at all points $z = n$, $n \in \mathbb{Z}$;

b) $|f(z)| \leq e^{\pi |\operatorname{Im} z|}$ for all $z \in \mathbb{C}$.

Prove that $f(z) \; = \; c \sin \pi z$, with $c \in \mathbb{C}$, $|c| \leq 1$.

Solution: Define $h(z) = (\sin \pi z)^{-1} f(z)$. The hypotheses imply that $h(z)$ is entire. Then, for $\operatorname{Im} z > 0$,

$$|h(z)| \; = \; |\sin \pi z|^{-1} \, |f(z)| \; \leq \; |\sin \pi z|^{-1} \, e^{\pi |\operatorname{Im} z|} \; \leq \; 2(1 - e^{-2\pi \operatorname{Im} z})^{-1} \, .$$

Since the hypotheses are invariant under the substitution $z \mapsto -z$, we get the analogous bound for $\operatorname{Im} z < 0$. Thus $h(z)$ is uniformly bounded on $|\operatorname{Im} z| \geq \delta$, $\delta > 0$. On the vertical lines $\operatorname{Re} z = (n + 1/2)\pi$, $n \in \mathbb{Z}$, $|\sin \pi z|^{-1} e^{\pi |\operatorname{Im} z|} = 2(1 + e^{-2\pi |\operatorname{Im} z|})^{-1}$, which is bounded by 2. Applying the maximum principle to $h(z)$ on the rectangles with sides $\operatorname{Im} z = \pm 1$, $\operatorname{Re} z = (n \pm 1/2)\pi$, we find that $h(z)$ is a bounded entire function, hence $f(z) = c \sin \pi z$ with $c \in \mathbb{C}$. Evaluating the inequality $|f(z)| = c \, |\sin \pi z| \leq e^{\pi |\operatorname{Im} z|}$ at $z = 1/2$ leads to $|c| \leq 1$.