1. (AT)
   (a) Let $X$ and $Y$ be compact, oriented manifolds of the same dimension $n$. Define the degree of a continuous map $f : X \rightarrow Y$.
   (b) Let $f : \mathbb{CP}^3 \rightarrow \mathbb{CP}^3$ be any continuous map. Show that the degree of $f$ is of the form $m^3$ for some integer $m$.
   (c) Show that conversely for any $m \in \mathbb{Z}$ there is a continuous map $f : \mathbb{CP}^3 \rightarrow \mathbb{CP}^3$ of degree $m^3$.

   **Solution**: For the first part, the induced map $f^* : H^n(Y, \mathbb{Z}) \cong \mathbb{Z} \rightarrow H^n(X, \mathbb{Z}) \cong \mathbb{Z}$ (where the isomorphisms with $\mathbb{Z}$ are given by the orientation) is multiplication by some integer $d$; this is the degree of $f$.

   For the second part, note that $H^*(\mathbb{CP}^3, \mathbb{Z}) \cong \mathbb{Z}[\zeta]/(\zeta^4)$ and that $f^*$ is a ring homomorphism. If $f^*(\zeta) = m\zeta$, then $f^*(\zeta^3) = m^3\zeta^3$ and so the degree must be a cube. To see that all cubes occur, just consider the map $[X, Y, Z, W] \mapsto [X^m, Y^m, Z^m, W^m]$ for positive $d = m^3$; take complex conjugates to exhibit maps with negative degrees.

2. (A) Let $G$ be a group.
   (a) Prove that, if $V$ and $W$ are irreducible $G$-representations defined over a field $F$, then a $G$-homomorphism $f : V \rightarrow W$ is either zero or an isomorphism.
   (b) Let $G = D_8$ be the dihedral group with 8 elements. What are the dimensions of its irreducible representations over $\mathbb{C}$?

   **Solution**: (a) If $f$ is nonzero, then its image in $W$ is a nontrivial subrepresentation of $W$ and hence $W$ itself by irreducibility; therefore, $f$ is surjective. Similarly, the kernel of $f$ is a subrepresentation of $V$, which, if nontrivial, must be $V$ itself, contradicting the assumption that $f$ is nonzero; therefore $f$ is injective. (b) There are five irreducible representations of $D_8$, four one-dimensional ones coming from characters of the quotient $\mathbb{Z}/2 \times \mathbb{Z}/2$ of $D_8$ by its center, and one two-dimensional representation corresponding to the
realization of $D_8$ as the group of automorphisms of the plane preserving a square of $D_8$. The fact that the irreducible representations have dimensions 1, 1, 1, 1 and 2 can also be seen by arguing that the number of irreducible representations is the same as the number of conjugacy classes in $D_8$, which is 5, and that the sum of the squares of their dimensions must be 8.

3. (CA) Let $f_n$ be a sequence of analytic functions on the unit disk $\Delta \subset \mathbb{C}$ such that $f_n \to f$ uniformly on compact sets and such that $f$ is not identically zero. Prove that $f(0) = 0$ if and only if there is a sequence $z_n \to 0$ such that $f_n(z_n) = 0$ for $n$ large enough.

Solution: $\iff$ We begin by observing that there must be some $\epsilon$ such that for $n$ large enough, $f_n$ is not zero in a neighborhood of the circle. By uniform convergence, since $f_n \to f$, it must also be that $f'_n \to f'$. Thus

$$\lim \int_{C_\epsilon} \frac{f'_n}{f_n} dz = \int \frac{f'}{f} dz$$

The right handside must, eventually, be larger than 1, so the left hand side must be as well. As this holds for every $\epsilon$, we see that $f$ has a zero in $B_\epsilon(0)$ for every $\epsilon$. Since $f$ is not identically zero it must be the only zero.

$\implies$ By the argument principle,

$$\frac{1}{2\pi} \int_{C_\epsilon} \frac{f'}{f} dz = 1$$

where $C_\epsilon$ is the circle of radius $\epsilon$ around zero for some $\epsilon$ sufficiently small. On the other hand,

$$\lim \int_{C_\epsilon} \frac{f'_n}{f_n} dz = \int \frac{f'}{f} dz \geq 1.$$

so by the argument principle again, then for every $\epsilon$, there is an $N$ large enough that $n \geq N$ yields $f_n$ has a at least one zero. applying this for each $\epsilon \to 0$ yields the result.

4. (AG) Let $K$ be an algebraically closed field of characteristic 0, and let $\mathbb{P}^n$ be the projective space of homogeneous polynomials of degree $n$ in two variables over $K$. Let $X \subset \mathbb{P}^n$ be the locus of $n^{th}$ powers of linear forms, and let $Y \subset \mathbb{P}^n$ be the locus of polynomials with a multiple root (that is, a repeated factor).

(a) Show that $X$ and $Y \subset \mathbb{P}^n$ are closed subvarieties.
(b) What is the degree of $X$?
(c) What is the degree of $Y$?
Solution: First, $X$ is the image of the map $\mathbb{P}^1 \to \mathbb{P}^n$ sending $[a, b] \in \mathbb{P}^1$ to $(ax + by)^n \in \mathbb{P}^n$. This is projectively equivalent (in characteristic 0!) to the degree $n$ Veronese map, whose image is a closed curve of degree $n$. Second, $Y$ is the zero locus of the discriminant, which is a polynomial of degree $2n - 2$ in the coefficients of a polynomial of degree $n$ (this number can be deduced from the Riemann-Hurwitz formula, which says that a degree $n$ map from $\mathbb{P}^1$ to $\mathbb{P}^1$ has $2n - 2$ branch points; that is, a general line in $\mathbb{P}^n$ meets $Y$ in $2n - 2$ points). Thus $Y \subset \mathbb{P}^n$ is a hypersurface of degree $2n - 2$.

5. (DG) Given a smooth function $f : \mathbb{R}^{n-1} \to \mathbb{R}$, define $F : \mathbb{R}^n \to \mathbb{R}$ by

$$F(x_1, \ldots, x_n) := f(x_1, \ldots, x_{n-1}) - x_n$$

and consider the preimage $X_f = F^{-1}(0) \subset \mathbb{R}^n$.

(a) Prove that $X_f$ is a smooth manifold which is diffeomorphic to $\mathbb{R}^{n-1}$.

(b) Consider the two examples $X_f$ and $X_g \subset \mathbb{R}^3$ with $f(x_1, x_2) = x_1^2 + x_2^2$ and $g(x_1, x_2) = x_1^2 - x_2^2$. Compute their normal vectors at every point $(x_1, x_2, x_3) \in X_f$ and $(x_1, x_2, x_3) \in X_g$.

Solution.

Part (a). The last row of the Jacobian of $F : \mathbb{R}^n \to \mathbb{R}$ is $(0, \ldots, 0, 1)$ and so the Jacobian has rank 1 everywhere. This implies that $F^{-1}(0)$ is a smooth manifold. It has the global chart $\psi : F^{-1}(0) \to \mathbb{R}^{n-1}$ defined by

$$\psi(x_1, \ldots, x_n) := (x_1, \ldots, x_{n-1})$$

and is therefore diffeomorphic to $\mathbb{R}^{n-1}$.

Part (b). The first example is a paraboloid. Its normal vector is

$$\frac{(-2x, -2y, 1)}{\sqrt{1 + 4x^2 + 4y^2}}.$$

The second example is a hyperbolic paraboloid or “saddle surface”. Its normal vector is

$$\frac{(-2x, 2y, 1)}{\sqrt{1 + 4x^2 + 4y^2}}.$$
Let $K \subset \mathbb{R}^n$ be a compact set. Show that for any measurable function $f : K \to \mathbb{C}$, it holds that

$$\lim_{p \to \infty} \|f\|_{L^p(K)} = \|f\|_{L^\infty(K)}.$$ 

(Recall that $\|f\|_{L^p(K)} = \left(\int_K |f|^p \, dx\right)^{1/p}$ and that $\|f\|_{L^\infty(K)}$ is the essential supremum of $f$, i.e., the smallest upper bound if the behavior of $f$ on null sets is ignored.)

**Solution.**

Let $p > 1$. Since $|f| \leq \|f\|_{L^\infty(K)}$ holds almost everywhere, we have

$$\|f\|_{L^p(K)} = \left(\int_K |f|^p \, dx\right)^{1/p} \leq |K|^{1/p} \|f\|_{L^\infty(K)} \xrightarrow{p \to \infty} \|f\|_{L^\infty(K)}.$$ 

It remains to prove the lower bound. Let $\epsilon > 0$. By definition of the essential supremum, there exists a set $A \subset K$ of Lebesgue measure $|A| > 0$, such that $|f| \geq (1 - \epsilon)\|f\|_{L^\infty(K)}$ holds on $A$. Hence,

$$\|f\|_{L^p(K)} \geq \left(\int_A |f|^p \, dx\right)^{1/p} \geq (1 - \epsilon)\|f\|_{L^\infty(K)} |A|^{1/p} \xrightarrow{p \to \infty} (1 - \epsilon)\|f\|_{L^\infty(K)}.$$ 

Sending $\epsilon \to 0$ proves the claim.
1. (AG) Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree $d$.

(a) Let $K_C$ be the canonical bundle of $C$. For what integer $n$ is it the case that $K_C \cong \mathcal{O}_C(n)$?

(b) Prove that if $d \geq 4$ then $C$ is not hyperelliptic.

(c) Prove that if $d \geq 5$ then $C$ is not trigonal (that is, expressible as a 3-sheeted cover of $\mathbb{P}^1$).

Solution: By the adjunction formula, the canonical divisor class is $K_C = \mathcal{O}_C(d-3)$, that is, plane curves of degree $d-3$ cut out canonical divisors on $C$.

Now, if $C$ were hyperelliptic—meaning that there exists a degree 2 map $\pi : C \to \mathbb{P}^1$—a general fiber of $\pi$ would consist of two points $p, q \in C$ moving in a pencil, that is, such that $h^0(\mathcal{O}_C(p+q)) = 2$. But if $d \geq 4$ then any two points $p, q \in C$ impose independent conditions on the canonical series $|K_C|$; that is, $h^0(K_C(-p-q)) = g-2$, so by Riemann-Roch $h^0(\mathcal{O}_C(p+q)) = 1$, and hence $C$ is not hyperelliptic. Similarly, if $d \geq 5$ then any three points $p, q, r \in C$ impose independent conditions on the canonical series $|K_C|$; by Riemann-Roch it follows that $h^0(\mathcal{O}_C(p+q+r)) = 1$ so $C$ is not trigonal.

2. (CA) (The 1/4 theorem). Let $\mathcal{S}$ denote the class of functions that are analytic on the disk and one-to-one with $f(0) = 0$ and $f'(0) = 1$.

(a) Prove that if $f \in \mathcal{S}$ and $w$ is not in the range of $f$ then

$$g(z) = \frac{wf(z)}{w - f(z)}$$

is also in $\mathcal{S}$.

(b) Show that for any $f \in \mathcal{S}$, the image of $f$ contains the ball of radius $1/4$ around the origin. You may use the elementary result (Bieberbach) that if $f(z) = z + \sum_{k \geq 2} a_k z^k$ in $\mathcal{S}$ then $|a_2| \leq 2$. 
Solution: The proof of the first part is by checking. Since \( w \notin R(f) \) it is analytic on the disk. Now observe that map \( h(z) \) which is
\[
h(z) = \frac{wz}{(w-z)}
\]
is one-to-one. thus \( g(z) = h \circ f \) so it is one-to-one. Finally since
\[
g'(z) = \frac{w^2 f'(z)}{(w-f(z))^2}
\]
it follows that \( g(0) = 0 \) and \( g'(0) = 1 \) as desired.
For the second part, Suppose that \( w \) is not in the image of \( f \). Then we may look at
\[
g(z) = \frac{w f(z)}{(w-f(z))^2}.
\]
Observe that
\[
|g''(0)| = |a_2 + \frac{1}{w}| \leq 2
\]
and \( a_2 \leq 2 \). From this it follows that
\[
|1/w| \leq |a_2| + |a_2 + 1/w| \leq 4,
\]
from which it follows that \( |w| \geq 1/4 \). Thus the set \( |w| \leq 1/4 \) is in the image of \( f \) as desired.

3. (A) Find a polynomial \( f \in \mathbb{Q}[x] \) whose Galois group (over \( \mathbb{Q} \)) is \( D_8 \), the dihedral group of order 8.

Solution: There are lots of ways to find examples. Here is one: consider a quartic polynomial whose cubic resolvent has exactly one rational root and discriminant is nonsquare. Indeed, ordering the roots as \( \alpha_1 \) through \( \alpha_4 \), suppose \( \alpha_1 \alpha_2 + \alpha_3 \alpha_4 \) is rational so that the Galois group is contained in the dihedral group generated by \( (1324), (13)(24) \). We want to ensure that the Galois group is no smaller: that the other roots of the resolvent are not rational ensures that the Galois group is not contained in the Klein subgroup generated by \( (12)(34), (13)(24) \); equivalently, this is the restriction that the discriminant be nonsquare and so the Galois group not be contained in the alternating group. But now, if \( K \) represents the splitting field, we have the exact sequence \( 1 \to \text{Gal}(K/\mathbb{Q}(\sqrt{D})) \to \text{Gal}(K/\mathbb{Q}) \to \mathbb{Z}/2 \to 1 \) and if \( \text{Gal}(K/\mathbb{Q}) \) were any smaller than the \( D_8 \) in which it is already contained, \( \text{Gal}(K/\mathbb{Q}(\sqrt{D})) \) would have order at most 2 and hence the polynomial would have (multiple) roots over \( \mathbb{Q}(\sqrt{D}) \). Hence it suffices to find a quartic polynomial with a cubic resolvent with exactly one rational root that stays irreducible over
\( \mathbb{Q}(\sqrt{D}) \). After some experimentation, \( f(x) = x^4 + 3x + 3 \) with cubic resolvent \((x + 3)(x^2 - 3x + 3)\), discriminant \(3^3 \cdot 5^3 \cdot 7\), and which over \( \mathbb{Q}(\sqrt{21}) \) has no roots, as one can check manually: suppose \( \alpha + \beta \sqrt{21} \) were a root. We know the ring of integers of \( \mathbb{Q}(\sqrt{21}) \) and will use that \( \tilde{\alpha} = 2\alpha, \tilde{\beta} = 2\beta \) are (usual) integers. Expanding the equation, we find that \( 4\alpha(\alpha^2 + 21\beta^2) = -3 \), or \( \tilde{\alpha}(\tilde{\alpha}^2 + 21\tilde{\beta}^2) = -6 \). This cannot happen without \( \tilde{\beta} = 0 \), which is impossible as \( f \) is irreducible over \( \mathbb{Q} \).

4. (RA)

(a) Let \( a_k \geq 0 \) be a monotone increasing sequence with \( a_k \to \infty \), and consider the ellipse,

\[
E(a_k) = \{ v \in \ell^2(\mathbb{Z}) : \sum a_k v_k^2 \leq 1 \}.
\]

Show that \( E(a_n) \) is a compact subset of \( \ell^2(\mathbb{Z}) \).

(b) Let \( T \) denote the one-dimensional torus; that is, \( \mathbb{R}/2\pi\mathbb{Z} \), or \([0, 2\pi]\) with the ends identified. Recall that the space \( H^1(T) \) is the closure of \( C^\infty(T) \) in the norm

\[
\|f\|_{H^1(T)} = \sqrt{\|f\|_{L^2(T)} + \|\frac{d}{dx}f\|_{L^2(T)}}.
\]

Use part (a) to conclude that the inclusion \( i : H^1(T) \hookrightarrow L^2(T) \) is a compact operator.

Solution: Part a. Firstly, since \( a_k \) is monotone,

\[
\sum a_k v_k^2 \leq 1
\]

implies that for any \( L \),

\[
\sum_{k \geq L} v_k^2 \leq \frac{1}{a_L}.
\]

Thus \( E \) is is norm bounded.

Suppose that we have sequence \( v^n \in E(a_k) \). Observe that

\[
v_k^n \leq \frac{1}{\sqrt{a_k}}.
\]

Thus we may diagonalize this sequence pointwise to obtain a sequence \( v_k \) with \( v_k \in \ell^\infty \). Passing to this subsequence, we see that by fatou’s lemma,

\[
\sum v_k^2 \leq \frac{1}{a_1},
\]
i.e., \( v \) is in \( \ell^2 \). It remains to show that \( v^n \to v \) in \( \ell^2 \). To see this, observe that

\[
\sum |v^n_k - v_k|^2 \leq \sum_{k \geq L} |v^n_k - v_k|^2 + \sum_{k \leq L} ... \\
\leq \frac{2}{a_L} + \sum_{k \leq L} |v^n_k - v_k|^2.
\]

Sending \( n \to \infty \) and then \( L \to \infty \) yields the result.

Part b. It suffices to show that the unit ball of \( H^1(\mathbb{T}) \) is a compact subset of \( L^2(\mathbb{T}) \). By Parseval’s identity/the fourier transform, it follows that

\[
\sum k^2 |\hat{f}_n(k)|^2 \leq C.
\]

for some positive constant. This is a compact subset of \( \ell_2 \) by part a. Thus by fourier inversion, the ball of \( H^1 \) is as well.

5. (AT) Consider the following topological spaces:

\[ A = S^1 \times S^1 \quad B = S^1 \vee S^1 \vee S^2. \]

(a) Compute the fundamental group of each space.
(b) Compute the integral cohomology ring of each space.
(c) Show that \( B \) is not homotopy equivalent to any compact orientable manifold.

\[ \text{Solution:} \quad (a) \quad \text{The fundamental group construction preserves products, so} \]

\[ \pi_1(A) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}. \]

By the Van Kampen theorem,

\[ \pi_1(B) \cong \pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}, \]

where the first step uses that \( S^2 \) is simply connected. (b) By the K"unneth theorem,

\[ H^*(S^1 \times S^1) \cong H^*(S^1) \otimes H^*(S^1) \cong \Lambda[x,y], \quad |x| = |y| = 1. \]

Since the reduced cohomology ring construction takes wedges of spaces to products of (nonunital) rings,

\[ H^*(S^1 \vee S^1 \vee S^2) \cong \frac{\Lambda[x,y,z]}{xy = yz = zx = 0}, \quad |x| = |y| = 1, |z| = 2. \]
(c) Suppose that \( B \) is homotopy equivalent to the compact orientable manifold \( M \). Choosing a fundamental class \([M]\), Poincaré duality guarantees that the assignment

\[
(a, b) \mapsto \langle a \sim b, [M] \rangle
\]

defines a symplectic form on \( H^1(M) \). Since \( xy = 0 \), this pairing is degenerate, a contradiction.

6. (DG) Consider the set

\[
G := \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & x & y \\ 0 & 0 & 1 \end{pmatrix} : x \in \mathbb{R}^+, \ y \in \mathbb{R} \right\},
\]

and equip it with a smooth structure via the global chart that sends \((x, y) \in \mathbb{R}^+ \times \mathbb{R}\) to the corresponding element of \( G \).

(a) Show that \( G \) is a Lie subgroup of the Lie group \( GL(\mathbb{R}, 3) \).

(b) Prove that the set

\[
\left\{ x \frac{\partial}{\partial x}, x \frac{\partial}{\partial y} \right\}
\]

forms a basis of left-invariant vector fields on \( G \).

(c) Find the structure constants of the Lie algebra \( \mathfrak{g} \) of \( G \) with respect to the basis in (b).

\textbf{Solution.}

Part (a). We consider the multiplication and inverse operations on \( G \):

\[
\begin{pmatrix} a & 0 & 0 \\ 0 & a & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 & 0 \\ 0 & x & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} ax & 0 & 0 \\ 0 & ax & ay + b \\ 0 & 0 & 1 \end{pmatrix} \in G
\]

and

\[
\begin{pmatrix} x & 0 & 0 \\ 0 & x & y \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/x & 0 & 0 \\ 0 & 1/x & -y/x \\ 0 & 0 & 1 \end{pmatrix} \in G.
\]

We see that \( G \) is closed under these operations, and that they are smooth. This proves that \( G \) is a Lie group itself.

Moreover: (a) Since the inverse of any element in \( G \) exists, \( G \) is a subset of \( GL(\mathbb{R}, 3) \). (b) The inclusion map \( G \to GL(\mathbb{R}, 3) \) is trivially a group homomorphism. (c) The inclusion map \( G \to GL(\mathbb{R}, 3) \) is an immersion. To see this,
recall that the smooth structure on $GL(\mathbb{R}, 3)$ is that of $\mathbb{R}^9$ and note that the map $(x, y) \mapsto (x, 0, 0, 0, x, y, 0, 0, 1)$ has rank 2.

Together, (a)-(c) imply that $G$ is a Lie subgroup of $GL(\mathbb{R}, 3)$.

Part (b). Linear independence follows from the linear independence of $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$.

To prove left-invariance, let us identify a vector $(x, y)$ with the corresponding matrix in $G$. The formula for the product $G \times G \rightarrow G$ shows that left translation in $G$ is given by

$$L_{(a,b)}(x, y) = (ax, ay + b),$$

and so $L_{(a,b)*} = a \text{Id}_{\mathbb{R}^2}$. This shows that

$$L_{(a,b)*}X(x,y) = X_{(a,b)}(x, y)$$

holds for both vector fields $X$ in (b), hence they are left-invariant.

Part (c). Let us call the two vector fields in (b) $X_1, X_2$, respectively. Explicit computation shows that

$$[X_1, X_2] = X_2,$$

and so the non-zero structure constants are $f_{12}^2 = -f_{21}^2 = 1$. 
1. (AT) Let \( p : E \to B \) be a \( k \)-fold covering space, and suppose that the Euler characteristic \( \chi(E) \) is defined, nonzero, and relatively prime to \( k \). Show that any CW decomposition of \( B \) has infinitely many cells.

**Solution**: Suppose that \( B \) has a finite CW decomposition; in particular, \( \chi(B) \) is defined. Restricted to each of the cells of \( B \), the covering \( p \) is trivial, and the connected components of the total spaces of these restricted covers form a CW decomposition of \( E \). Counting cells, we find that \( \chi(E) = k \cdot \chi(B) \). Since \( \chi(E) \neq 0 \), it follows that \( k \) divides \( \chi(E) \), a contradiction.

2. (RA) Let \( W \) be Gumbel distributed, that is \( P(W \leq x) = e^{-e^{-x}} \). Let \( X_i \) be independent and identically distributed Exponential random variables with mean 1; that is, \( X_i \) are independent, with \( P(X_i \leq x) = \exp(-\max(x,0)) \).

Let \( M_n = \max_{i \leq n} X_i \).

Show that there are deterministic sequences \( a_n, b_n \) such that

\[
\frac{M_n - b_n}{a_n} \to W
\]

in law; that is, such that for any continuous bounded function \( F \),

\[
\mathbb{E} F \left( \frac{M_n - b_n}{a_n} \right) \to \mathbb{E} F(W).
\]

**Solution**: let \( b_n = \log n \) and \( a_n = 1 \). Then

\[
P(M_n - b_n \leq x) = P(X_i \leq x + \log n)^n
\]

since \( X_i \) are i.i.d. Now

\[
P(X \leq x + \log n)^n = (1 - P(X > x + \log n))^n
\]

\[
= \left( 1 - \int_{x+\log n}^{\infty} e^{-w} \, dw \right)^n
\]

\[
= \left( 1 - \frac{1}{n} \int_x^{\infty} e^{-w} \, dw \right)^n \to e^{-e^{-x}}
\]
As $e^{-e^{-x}}$ is continuous everywhere and

\[ P(M_n - b_n \leq x) \to P(W \leq x), \]

we see that $M_n - b_n \to W$ in law by the Portmanteau lemma.

3. (DG) Consider $\mathbb{R}^2$ as a Riemannian manifold equipped with the metric

\[ g = e^x dx^2 + dy^2. \]

(i) Compute the Christoffel symbols of the Levi-Civita connection for $g$.

(ii) Show that the geodesics are described by the curves $x(t) = 2 \log(At + B)$ and $y(t) = Ct + D$, for appropriate constants $A, B, C, D$.

(iii) Let $\gamma : \mathbb{R}_+ \to \mathbb{R}^2$, $\gamma(t) = (t, t)$. Compute the parallel transport of the vector $(1, 2)$ along the curve $\gamma$.

(iv) Are there two vector fields $X, Y$ parallel to the curve $\gamma$, such that $g(X(t), Y(t))$ is non-constant?

Solution:
Part (i). We can identify

\[ g^{-1} = \begin{pmatrix} e^{-x} & 0 \\ 0 & 1 \end{pmatrix}. \]

Denoting $x^1 = x$, $x^2 = y$, the only non-vanishing Christoffel symbol is

\[ \Gamma^1_{11} = \frac{1}{2} g^{-1}_{11} \partial_1 g_{11} = \frac{1}{2}. \]

Part (ii). Using part (i), the two ODE describing the geodesic $(x(t), y(t))$ are given by

\[ \frac{d^2x}{dt^2} + \frac{1}{2} \left( \frac{dx}{dt} \right)^2 = 0, \quad \frac{d^2y}{dt^2} = 0. \]

The second ODE is solved by $y(t) = Ct + D$. For the first ODE, we introduce $u(t) := \frac{dx}{dt}$ and obtain

\[ \frac{du}{dt} + \frac{1}{2} u^2 = 0. \]

By separation of variables, this is solved by $u(t) = \frac{2}{t + C_1}$. We integrate this to get $x$ and find

\[ x(t) = 2 \log(t + C_1) + C_2 = 2 \log(At + B), \]
where we redefined the constants in the second step.

Part (iii). The equation for parallel transport $\nabla_{\gamma'}(a^1, a^2) = 0$, with $\gamma(t) = (t, t)$, becomes

$$\frac{da^1}{dt} + \frac{1}{2} a^1 = 0, \quad \frac{da^2}{dt} = 0.$$ 

These are solved by $a^1(t) = Ae^{-t/2}$ and $a^2(t) = B$, respectively. To satisfy the initial condition $(a^1(0), a^2(0)) = (1, 2)$, we take $A = 1$ and $B = 2$. The solution is thus

$$(a^1(t), a^2(t)) = (e^{-t/2}, 2).$$

Part (iv). No. Since $\nabla$ is the Levi-Civita connection, the scalar product of two vectors is preserved by parallel transport.

4. (A) Let $G$ be a group of order 78.

(a) Show that $G$ contains a normal subgroup of index 6.

(b) Show by example that $G$ may contain a subgroup of index 13 that is not normal.

Solution: (a) Sylow theory guarantees the existence of a 13-Sylow subgroup $H \leq G$, which has index 6. This Sylow subgroup is unique and hence normal; indeed, the number of such divides 6 and is congruent to 1 mod 13 by Sylow’s theorems. (b) Take $G$ to be the semidirect product $C_{13} \rtimes S_3$ of the cyclic group of order 13 and the symmetric group on 3 letters, where $S_3$ acts via the composite

$$S_3 \xrightarrow{\text{sgn}} C_2 \xrightarrow{\text{inv}} \text{Aut}(C_{13})$$

of the sign homomorphism and the inversion homomorphism (we use that $C_{13}$ is Abelian). We claim that the subgroup $S_3 \leq G$ is not normal. To see why this is so, let $\sigma \in S_3$ be an odd permutation and $\rho \in C_{13}$ a generator, and compute that

$$(\rho, e)(e, \sigma)(\rho^{-1}, e) = (\rho^2, \sigma) \notin S_3.$$ 

5. (AG) Let $K$ be an algebraically closed field of characteristic 0, and consider the curve $C \subset \mathbb{A}^3$ over $K$ given as the image of the map

$$\phi : \mathbb{A}^1 \to \mathbb{A}^3$$

$$t \mapsto (t^3, t^4, t^5).$$
Show that no neighborhood of the point \( \phi(0) = (0,0,0) \in C \) can be embedded in \( \mathbb{A}^2 \).

**Solution:** Suppose \( f(x,y,z) \) is any polynomial on \( \mathbb{A}^3 \) vanishing on \( C \). The constant term of \( f \) must be zero, since \( f \) vanishes at \( (0,0,0) \in C \), and the linear terms of \( f \) must also be zero, since the pullback to \( \mathbb{A}^1 \) of any monomial in \( x, y \) and \( z \) of degree 2 or more must vanish to order at least 6. In other words, the ideal \( I(C) \) is contained in the square \( (x,y,z)^2 \) of the maximal ideal of the origin. In particular, the Zariski tangent space to \( C \) at \( (0,0,0) \) is three dimensional, and hence no neighborhood of this point is embeddable in \( \mathbb{A}^2 \).

6. (CA) Let \( f(z) \) be an entire function such that

a) \( f(z) \) vanishes at all points \( z = n, n \in \mathbb{Z} \);

b) \( |f(z)| \leq e^{\pi|\text{Im} \, z|} \) for all \( z \in \mathbb{C} \).

Prove that \( f(z) = c \sin \pi z \), with \( c \in \mathbb{C} \), \( |c| \leq 1 \).

**Solution:** Define \( h(z) = (\sin \pi z)^{-1} f(z) \). The hypotheses imply that \( h(z) \) is entire. Then, for \( \text{Im} \, z > 0 \),

\[
|h(z)| = |\sin \pi z|^{-1} |f(z)| \leq |\sin \pi z|^{-1} e^{\pi|\text{Im} \, z|} \leq 2(1 - e^{-2\pi \text{Im} \, z})^{-1}.
\]

Since the hypotheses are invariant under the substitution \( z \mapsto -z \), we get the analogous bound for \( \text{Im} \, z < 0 \). Thus \( h(z) \) is uniformly bounded on \( |\text{Im} \, z| \geq \delta \), \( \delta > 0 \). On the vertical lines \( \text{Re} \, z = (n + 1/2)\pi, \, n \in \mathbb{Z} \), \( |\sin \pi z|^{-1} e^{\pi|\text{Im} \, z|} = 2(1 + e^{-2\pi|\text{Im} \, z|})^{-1} \), which is bounded by 2. Applying the maximum principle to \( h(z) \) on the rectangles with sides \( \text{Im} \, z = \pm 1 \), \( \text{Re} \, z = (n \pm 1/2)\pi \), we find that \( h(z) \) is a bounded entire function, hence \( f(z) = c \sin \pi z \) with \( c \in \mathbb{C} \). Evaluating the inequality \( |f(z)| = c |\sin \pi z| \leq e^{\pi|\text{Im} \, z|} \) at \( z = 1/2 \) leads to \( |c| \leq 1 \).