QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday September 4, 2018 (Day 1)

1. (AT)

(a) Let X and Y be compact, oriented manifolds of the same dimension n. Define the *degree* of a continuous map $f : X \to Y$.

(b) Let $f : \mathbb{CP}^3 \to \mathbb{CP}^3$ be any continuous map. Show that the degree of f is of the form m^3 for some integer m.

(c) Show that conversely for any $m \in \mathbb{Z}$ there is a continuous map $f : \mathbb{CP}^3 \to \mathbb{CP}^3$ of degree m^3.

2. (A) Let G be a group.

(a) Prove that, if V and W are irreducible G-representations defined over a field $\mathbb{F}$, then a G-homomorphism $f : V \to W$ is either zero or an isomorphism.

(b) Let $G = D_8$ be the dihedral group with 8 elements. What are the dimensions of its irreducible representations over $\mathbb{C}$?

3. (CA) Let f_n be a sequence of analytic functions on the unit disk $\Delta \subset \mathbb{C}$ such that $f_n \to f$ uniformly on compact sets and such that f is not identically zero. Prove that $f(0) = 0$ if and only if there is a sequence $z_n \to 0$ such that $f_n(z_n) = 0$ for n large enough.

4. (AG) Let K be an algebraically closed field of characteristic 0, and let $\mathbb{P}^n$ be the projective space of homogeneous polynomials of degree n in two variables over K. Let $X \subset \mathbb{P}^n$ be the locus of n^{th} powers of linear forms, and let $Y \subset \mathbb{P}^n$ be the locus of polynomials with a multiple root (that is, a repeated factor).

(a) Show that X and $Y \subset \mathbb{P}^n$ are closed subvarieties.

(b) What is the degree of X?

(c) What is the degree of Y?

5. (DG) Given a smooth function $f : \mathbb{R}^{n-1} \to \mathbb{R}$, define $F : \mathbb{R}^n \to \mathbb{R}$ by

$$F(x_1, \dots, x_n) := f(x_1, \dots, x_{n-1}) - x_n$$

and consider the preimage $X_f = F^{-1}(0) \subset \mathbb{R}^n$.

(a) Prove that X_f is a smooth manifold which is diffeomorphic to $\mathbb{R}^{n-1}$.

(b) Consider the two examples X_f and $X_g \subset \mathbb{R}^3$ with $f(x_1, x_2) = x_1^2 + x_2^2$ and $g(x_1, x_2) = x_1^2 - x_2^2$. Compute their normal vectors at every point $(x_1, x_2, x_3) \in X_f$ and $(x_1, x_2, x_3) \in X_g$.

6. (RA) Let $K \subset \mathbb{R}^n$ be a compact set. Show that for any measurable function $f : K \to \mathbb{C}$, it holds that

$$\lim_{p \to \infty} \|f\|_{L^p(K)} = \|f\|_{L^\infty(K)}.$$

(Recall that $\|f\|_{L^p(K)} = \left(\int_K |f|^p \mathrm{d}x\right)^{1/p}$ and that $\|f\|_{L^\infty(K)}$ is the essential supremum of f, i.e., the smallest upper bound if the behavior of f on null sets is ignored.)

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Wednesday September 5, 2018 (Day 2)

1. (AG) Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree d.

(a) Let K_C be the canonical bundle of C. For what integer n is it the case that $K_C \cong \mathcal{O}_C(n)$?

(b) Prove that if $d \geq 4$ then C is not hyperelliptic.

(c) Prove that if $d \geq 5$ then C is not trigonal (that is, expressible as a 3-sheeted cover of $\mathbb{P}^1$).

2. (CA) (The 1/4 theorem). Let $\mathcal{S}$ denote the class of functions that are analytic on the disk and one-to-one with $f(0) = 0$ and $f'(0) = 1$.

(a) Prove that if $f \in \mathcal{S}$ and w is not in the range of f then

$$g(z) = \frac{wf(z)}{(w - f(z))}$$

is also in $\mathcal{S}$.

(b) Show that for any $f \in \mathcal{S}$, the image of f contains the ball of radius 1/4 around the origin. You may use the elementary result (Bieberbach) that if $f(z) = z + \sum_{k\geq 2} a_k z^k$ in $\mathcal{S}$ then $|a_2| \leq 2$.

3. (A) Find a polynomial $f \in \mathbb{Q}[x]$ whose Galois group (over $\mathbb{Q}$) is D_8, the dihedral group of order 8.

4. (RA)

(a) Let $a_k \geq 0$ be a monotone increasing sequence with $a_k \to \infty$, and consider the ellipse,

$$E(a_k) = \{v \in \ell^2(\mathbb{Z}) : \sum a_k v_k^2 \leq 1\}.$$

Show that $E(a_n)$ is a compact subset of $\ell^2(\mathbb{Z})$.

(b) Let $\mathbb{T}$ denote the one-dimensional torus; that is, $\mathbb{R}/2\pi\mathbb{Z}$, or $[0, 2\pi]$ with the ends identified. Recall that the space $H^1(\mathbb{T})$ is the closure of $C^\infty(\mathbb{T})$ in the norm

$$\|f\|_{H^1(\mathbb{T})} = \sqrt{\|f\|_{L^2(\mathbb{T})} + \|\frac{d}{dx}f\|_{L^2(\mathbb{T})}}.$$

Use part (a) to conclude that the inclusion $i : H^1(\mathbb{T}) \hookrightarrow L^2(\mathbb{T})$ is a compact operator.

5. (AT) Consider the following topological spaces:

$$A = S^1 \times S^1 \qquad\qquad B = S^1 \vee S^1 \vee S^2.$$

(a) Compute the fundamental group of each space.

(b) Compute the integral cohomology ring of each space.

(c) Show that B is not homotopy equivalent to any compact orientable manifold.

6. (DG) Consider the set

$$G := \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & x & y \\ 0 & 0 & 1 \end{pmatrix} : x \in \mathbb{R}_+,\, y \in \mathbb{R} \right\},$$

and equip it with a smooth structure via the global chart that sends $(x, y) \in \mathbb{R}_+ \times \mathbb{R}$ to the corresponding element of G.

(a) Show that G is a Lie subgroup of the Lie group $GL(\mathbb{R}, 3)$.

(b) Prove that the set

$$\left\{ x\frac{\partial}{\partial x}, x\frac{\partial}{\partial y} \right\}$$

forms a basis of left-invariant vector fields on G.

(c) Find the structure constants of the Lie algebra $\mathfrak{g}$ of G with respect to the basis in (b).

QUALIFYING EXAMINATION

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Thursday September 6, 2018 (Day 3)

1. (AT) Let $p : E \to B$ be a k-fold covering space, and suppose that the Euler characteristic $\chi(E)$ is defined, nonzero, and relatively prime to k. Show that any CW decomposition of B has infinitely many cells.

2. (RA) Let W be Gumbel distributed, that is $P(W \leq x) = e^{-e^{-x}}$. Let X_i be independent and identically distributed Exponential random variables with mean 1; that is, X_i are independent, with $P(X_i \leq x) = \exp(-\max x, 0)$.

Let

$$M_n = \max_{i \leq n} X_i.$$

Show that there are deterministic sequences a_n, b_n such that

$$\frac{M_n - b_n}{a_n} \to W$$

in law; that is, such that for any continuous bounded function F,

$$\mathbb{E}F\left(\frac{M_n - b_n}{a_n}\right) \to \mathbb{E}F(W).$$

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3. (DG) Consider $\mathbb{R}^2$ as a Riemannian manifold equipped with the metric

$$g = e^x \mathrm{d}x^2 + \mathrm{d}y^2.$$

(i) Compute the Christoffel symbols of the Levi-Civita connection for g.

(ii) Show that the geodesics are described by the curves $x(t) = 2\log(At + B)$ and $y(t) = Ct + D$, for appropriate constants A, B, C, D.

(iii) Let $\gamma : \mathbb{R}_+ \to \mathbb{R}^2$, $\gamma(t) = (t, t)$. Compute the parallel transport of the vector $(1, 2)$ along the curve γ.

(iv) Are there two vector fields X, Y parallel to the curve γ, such that $g(X(t), Y(t))$ is non-constant?

4. (A) Let G be a group of order 78.

(a) Show that G contains a normal subgroup of index 6.

(b) Show by example that G may contain a subgroup of index 13 that is not normal.

5. (AG) Let K be an algebraically closed field of characteristic 0, and consider the curve $C \subset \mathbb{A}^3$ over K given as the image of the map

$$\begin{aligned} \phi : & \mathbb{A}^1 \to \mathbb{A}^3 \\ & t \mapsto (t^3, t^4, t^5). \end{aligned}$$

Show that no neighborhood of the point $\phi(0) = (0, 0, 0) \in C$ can be embedded in $\mathbb{A}^2$.

6. (CA) Let $f(z)$ be an entire function such that

a) $f(z)$ vanishes at all points $z = n$, $n \in \mathbb{Z}$;

b) $|f(z)| \leq e^{\pi |\operatorname{Im} z|}$ for all $z \in \mathbb{C}$.

Prove that $f(z) = c \sin \pi z$, with $c \in \mathbb{C}$, $|c| \leq 1$.