Qualifying Examination

HARVARD UNIVERSITY

Department of Mathematics

Tuesday, August 29, 2017 (Day 1)

PROBLEM 1 (DG)

a) Let M denote a compact, smooth manifold. Show that M can be imbedded in $\mathbb{R}^N$ if N is sufficiently large.

b) Let $\pi: E \to M$ denote a smooth, finite rank real vector bundle. Show that E is isomorphic to a subbundle of the product bundle $M \times \mathbb{R}^N$ if N is sufficiently large.

PROBLEM 2 (T)

Let X denote the quotient space of S^2 that is obtained by identifying two distinct points.

a) Compute the homology groups $H_*(X; \mathbb{Z})$.

b) What is the universal covering space of X?

PROBLEM 3 (AN)

Let $r = 2^{1/3}$. Let K be the cubic number field $\mathbb{Q}(r)$, and A its ring of integers. You may assume that $A = \mathbb{Z}[r]$.

a) Prove that if p is an prime such that $p = 5 \bmod (6)$, then the ideal pA of A factors as a product of two distinct prime ideals.

b) Find these two prime ideals with $p = 5$.

PROBLEM 4 (AG)

a) Find all common solutions (x, y) of the equations $f(x,y) = g(x,y) = 0$, where

$$f(x,y) = x^2y^3 - x^3y^2 \quad \textit{and} \quad g(x,y) = x^2 - 2x + y^2 - 2y + 1\,.$$

b) Let $\mathbb{C}[x,y]$ denote the ring of polynomials, and let $I = (f,g) \subset \mathbb{C}[x,y]$ denote the ideal generated by f and g. Find the radical $\sqrt{I}$ of the ideal I.

PROBLEM 5 (RA)

Suppose that (X,μ) is a measure space with $\mu(X)$ finite. Let p and q denote positive real numbers obeying $1 \le p < q \le \infty$.

a) Prove that $L^q(X)$ is a subset of $L^p(X)$.

b) Let $X = [0, 1]$ with μ denoting Lebesgue measure. Give an example of a function that is in L^p but not in L^q.

c) Give an example of a measure space X with $\mu(X)$ infinite such that the reverse inclusion holds: Every L^p function is an L^q function.

PROBLEM 6 (CA)

Prove that the infinite product $f(z) = \prod_{n=1}^{\infty}((1-\frac{z}{n})e^{z/n})$ defines an analytic function on the whole of $\mathbb{C}$ whose zeros are the positive integers.

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Wednesday, August 30, 2017 (Day 2)

PROBLEM 1 (DG)

Let $\mathcal{H}^2$ denote the upper half plane in $\mathbb{R}^2$; the set $\{(x,y) \in \mathbb{R}^2 : y > 0\}$. Supposing that α is a real number, equip $\mathcal{H}^2$ with the metric

$$g_\alpha = \frac{dx^2 + dy^2}{y^\alpha} \quad .$$

a) Show that g_α is not complete if $\alpha \neq 2$.

b) Let $z = x + iy$. Fix a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with real entries and determinant 1 (so an element in $SL(2,\mathbb{R})$). Show that the map $z \to \frac{az+b}{cz+d}$ maps $\mathcal{H}^2$ to itself as a diffeomorphism of $\mathcal{H}^2$, and that in so doing, it defines an isometry of the metric g_2.

PROBLEM 2 (T)

a) Show that for all i, the cohomology groups $H^i(S^1 \times S^2; \mathbb{Z})$ and $H^i(S^1 \vee S^2 \vee S^3; \mathbb{Z})$ are isomorphic.

b) Show that there does not exist a compact manifold that is homotopy equivalent to the wedge sum $S^1 \vee S^2 \vee S^3$.

PROBLEM 3 (AN)

a) Show that the polynomial $x^{11} - 1$ has discriminant -11^{11}.

b) Deduce that the polynomial $C(x) = (x^{11} - 1)/(x - 1)$ in $\mathbb{Q}[x]$ factors over $\mathbb{Q}(\sqrt{-11})$ as the product of two quintic polynomials, each with cyclic Galois group over $\mathbb{Q}(\sqrt{-11})$. (You may use without proof the irreducibility of cyclotomic polynomials over $\mathbb{Q}$.)

PROBLEM 4 (AG)

a) Define the Hilbert polynomial of a projective variety X in $\mathbb{P}^n$.

b) Let $X \subset \mathbb{P}^4$ be a variety given as the intersection of a quadric and a cubic hypersurface with no common component. Show that the Hilbert polynomial of $X \subset \mathbb{P}^4$ is the polynomial $d \to 3d^2+2$.

PROBLEM 5 (RA)

Let S^1 denote the circle, $\mathbb{R}/(2\pi\mathbb{Z})$; and let $x \in S^1$ denote an irrational multiple of 2π.

a) Suppose that $f: S^1 \to \mathbb{C}$ is a finite linear combination of functions from the set $\{e^{inx}: n \in \mathbb{Z}\}$. Prove the identity

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(kx) = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt \ .$$

b) Prove that this identity also holds for any continuous function f on S^1 whose Fourier coefficients are absolutely summable. (This means that f can be written as $f(t) = \sum_{n\in\mathbb{Z}} a_n e^{int}$ with $\sum_{n\in\mathbb{Z}} |a_n|$ being a convergent sequence.)

PROBLEM 6 (CA)

Supposing that $a > \sqrt{2}$, let $I(a) = \int_0^{2\pi} \frac{1}{a+\sin(\theta)+\cos(\theta)} d\theta$. Use contour integration to prove that $I(a) = \frac{2\pi}{\sqrt{a^2-2}}$.

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Thursday, August 31, 2017 (Day 3)

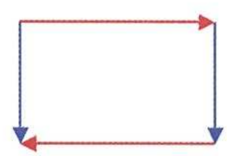
PROBLEM 1 (DG)

Suppose that S is an embedded surface in $\mathbb{R}^3$.

a) Define the first and second fundamental forms of S.
b) Define the principle curvatures of S at a given point.
c) Prove that if S is compact, then the product of the principle curvatures of S can not be negative at every point of S.

PROBLEM 2 (T)

Let K denote the Klein bottle. It is obtained from a rectangle by making the edge identifications as indicated in the following picture:



Take the closed rectangle and identify the left blue side with the right head-to-head and tail-to-tail; and identify the upper red side with the lower red side head-to-head and tail-to tail.

a) For which topological spaces X does there exist a finite-sheeted covering map $X \to K$? (Hint: You may use the fact that if there exists an n-sheeted covering map from X to a compact manifold Y, then $\chi(X) = n\chi(Y)$ with χ denoting the Euler characteristic.)
b) How many connected 2-sheeted covering spaces does K have (up to automorphism)? How many of them are orientable?

PROBLEM 3 (AN)

Let k denote a finite field of q elements with $q > 2$, and let G denote the group of permutations of k that have the form $g_{a,b}$: $x \to ax+b$ with $a,b \in k$ and $a \neq 0$.

a) Prove that two *nonidentity* permutations $g_{a,b}$ and $g_{a',b'}$ are conjugate in G if and only if $a = a'$. In particular, explain why this proves that G has q conjugacy classes.

b) Let (V, ρ) denote the associated permutation representation over $\mathbb{C}$, and V_0 the complement of the 1-dimensional trivial representation $V^G \subset V$. Prove that $\langle \chi_V, \chi_V \rangle = 2$, and deduce that V_0 is irreducible.

c) How many distinct homomorphisms are there from k* to $\mathbb{C}^*$? Show that any such homomorphism (call it φ) yields a 1-dimensional representation $g_{a,b} \to \varphi(a)$ of G. Explain why these respresentations together with V_0 give the complete set of isomorphism classes of irreducible representations of G.

PROBLEM 4 (AG)

Let C denote a smooth, projective curve, let K_C denote the canonical bundle of C; and let $\mathcal{L}$ denote a holomorphic line bundle on C; . The Riemann-Roch theorem says:

$$h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}^{-1} \otimes K_C) = \deg(\mathcal{L}) + 1 - g(C)$$

where $\deg(\mathcal{L})$ denotes the degree of $\mathcal{L}$ and g(C) denotes the genus of C. Use the Riemann-Roch theorem to prove that every curve C has a non-constant map to $\mathbb{P}^1$ of degree $g(C)+1$ or less.

PROBLEM 5 (RA)

Let g denote a smooth function on $\mathbb{R}^3$ with compact support. Let f denote the function given by the formula $f(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} g(y)\, dy$.

a) Prove that the integral that defines f converges for each $x \in \mathbb{R}^3$ if g is a square integrable function on $\mathbb{R}^3$ with compact support.

b) Prove that f is differentiable and that the gradient of f is given by the formula $\nabla f|_x = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} (\nabla g)|_y\, dy$.

c) Prove that f obeys $-\Delta f = g$ with Δ denoting here $\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$. (You are being asked to prove that the function $x \to \frac{1}{4\pi} \frac{1}{|x-y|}$ is the Green's function for the Laplacian with pole at y.)

PROBLEM 6 (CA):

Suppose that f is a C^1 map from an open disk in $\mathbb{R}^2$ to $\mathbb{R}^2$. Prove that the following are equivalent assertions (be careful not to give a circular proof):

a) f obeys the Cauchy-Riemann equations.

b) When viewed as a $\mathbb{C}$-valued function on a disk in $\mathbb{C}$ (with $\mathbb{C}$ identified with $\mathbb{R}^2$ in the usual way) the function f has a complex derivative in the following sense: Fix any point z in the disk where f is defined, and then a non-zero h such that z + h is in this disk. Then

$$\lim_{t \to 0} \frac{f(z+th) - f(z)}{th}$$

exists and it is independent of h. (The limit is taken along the line segment that is parametrized by the interval $[0, 1] \subset \mathbb{R}$ via the map $t \to z + th$.)