

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Tuesday September 1, 2015 (Day 1)

1. (A) The integer 8871870642308873326043363 is the 13th power of an integer n . Find n .
2. (AG) Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree 4.
 - (a) Describe the canonical bundle of C in terms of line bundles on \mathbb{P}^2 . What are the effective canonical divisors on C ?
 - (b) What is the genus of C ? Explain how you obtain this formula.
 - (c) Prove that C is not hyperelliptic.
3. (DG) Let M be a C^∞ manifold, TM its tangent bundle, and $T^{\mathbb{C}}M = \mathbb{C} \otimes_{\mathbb{R}} TM$ the complexified tangent bundle. An almost complex structure on M is a C^∞ bundle map $J : TM \rightarrow TM$ such that $J^2 = -1$.
 - (a) Show that an almost complex structure naturally determines, and is determined by, each of the following two structures:
 - i) the structure of a complex C^∞ vector bundle – i.e., with fibres that are complex vector spaces – on TM compatible with its real structure.
 - ii) a C^∞ direct sum decomposition $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ with $T^{0,1}M = \text{complex conjugate of } T^{1,0}M$.
 - (b) Show that every almost complex manifold is orientable.
 - (c) If S is a C^∞ , orientable, 2-dimensional, Riemannian manifold, construct a natural almost complex structure on S in terms of its Riemannian structure, but one that depends only on the underlying conformal structure of S .
 - (d) Does the almost complex structure constructed in (c) determine the conformal structure of S ? You need NOT give a detailed answer to this question; a heuristic one- or two-sentence answer suffices.
4. (RA) In this problem V denotes a Banach space over \mathbb{R} or \mathbb{C} .
 - (a) Show that any finite dimensional subspace $U_0 \subset V$ is closed in V .
 - (b) Now let $U_1 \subset V$ a closed subspace, and $U_2 \subset V$ a finite dimensional subspace. Show that $U_1 + U_2$ is closed in V .
5. (AT) Consider the following three topological spaces:

$$A = \mathbb{H}\mathbb{P}^3, \quad B = S^4 \times S^8, \quad C = S^4 \vee S^8 \vee S^{12}.$$

($\mathbb{H}\mathbb{P}^3$ denotes quaternionic projective 3-space.)

- (a) Calculate the cohomology groups (with integer coefficients) of all three.
 - (b) Show that A and B are not homotopy equivalent.
 - (c) Show that C is not homotopy equivalent to any compact manifold.
- 6.** (CA) Let $f(z)$ be a function which is analytic in the unit disc $D = \{|z| < 1\}$, and assume that $|f(z)| \leq 1$ in D . Also assume that $f(z)$ has at least two fixed points z_1 and z_2 . Prove that $f(z) = z$ for all $z \in D$.

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Wednesday September 2, 2015 (Day 2)

1. (AT) Let $\mathbb{C}\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$ be n dimensional complex projective space.

(a) Show that every map $f : \mathbb{C}\mathbb{P}^{2n} \rightarrow \mathbb{C}\mathbb{P}^{2n}$ has a fixed point. (Hint: Use the ring structure on cohomology.)

(b) For every $n \geq 0$, give an example of a map $f : \mathbb{C}\mathbb{P}^{2n+1} \rightarrow \mathbb{C}\mathbb{P}^{2n+1}$ without any fixed points and describe its induced map on cohomology.

2. (A) Let A be a commutative ring with unit. Define what it means for A to be *Noetherian*. Prove that the ring of continuous functions $f : [0, 1] \rightarrow \mathbb{R}$ (with pointwise addition and multiplication) is *not* Noetherian.

3. (CA) Let $S \subset \mathbb{C}$ be the open half-disc $\{x + iy : y > 0, x^2 + y^2 < 1\}$.

(a) Construct a surjective conformal mapping $f : S \rightarrow D$, where D is the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$.

(b) Construct a harmonic function $h : S \rightarrow \mathbb{R}$ such that:

- $h(x + iy) \rightarrow 0$ as $y \rightarrow 0$ from above, for all real x with $|x| < 1$, and
- $h(re^{i\theta}) \rightarrow 1$ as $r \rightarrow 1$ from below, for all real θ with $0 < |\theta| < \pi$.

4. (AG) Let Q be the complex quadric surface in \mathbb{P}^3 defined by the homogeneous equation $x_0x_3 - x_1x_2 = 0$.

(a) Show that Q is non-singular.

(b) Show that through each point of Q there are exactly two lines which lie on Q .

(c) Show that Q is rational, but not isomorphic to \mathbb{P}^2 .

5. (DG) Let Ω be the 2-form on $\mathbb{R}^3 - \{0\}$ defined by

$$\Omega = \frac{1}{x^2 + y^2 + z^2} (x dy \wedge dz + y dz \wedge dx + z dx \wedge dy).$$

(a) Prove that Ω is closed.

(b) Let $f : \mathbb{R}^3 - \{0\} \rightarrow S^2$ be the map which sends (x, y, z) to $(\frac{1}{x^2 + y^2 + z^2})^{1/2}(x, y, z)$. Show that Ω is the pull-back via f of a 2-form on S^2 .

(c) Prove that Ω is not exact.

6. (RA) Consider the linear ODE $f'' + P f' + Q f = 0$ on the interval $(a, b) \subset \mathbb{R}$, with P, Q denoting C^∞ real valued functions on (a, b) . Recall the definition of the Wronskian $W(f_1, f_2) = f_1 f_2' - f_1' f_2$ associated to any two solutions f_1, f_2 of this differential equation.
- (a) Show that $W(f_1, f_2)$ either vanishes identically or is everywhere nonzero, depending on whether the two solutions f_1, f_2 are linearly dependent or not.
 - (b) Now suppose that f_1, f_2 are linearly independent, real valued solutions. Show that they have at most first order zeroes, and that the zeroes occur in an alternating fashion: between any two zeroes of one of the solutions there must be a zero of the other solution.

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Thursday September 3, 2015 (Day 3)

1. (DG) Consider the graph S of the function $F(x, y) = \cosh(x) \cos(y)$ in \mathbb{R}^3 and let

$$\Phi : \mathbb{R}^2 \rightarrow S \subset \mathbb{R}^3$$

be its parametrization: $\Phi(x, y) = (x, y, \cosh(x) \cos(y))$.

- (a) Write down the metric on \mathbb{R}^2 that is defined by the rule that the inner product of two vectors v and w at the point (x, y) is equal to the inner product of $\Phi_*(v)$ and $\Phi_*(w)$ at the point $\Phi(x, y)$ in \mathbb{R}^3 .
- (b) Define the Gaussian curvature of a general surface embedded in \mathbb{R}^3 .
- (c) Compute the Gaussian curvature of the surface S at the point $(0, 0, 1)$.
2. (RA) Let $f(x) \in C(\mathbb{R}/\mathbb{Z})$ be a continuous \mathbb{C} -valued function on \mathbb{R}/\mathbb{Z} and let $\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}$ be its Fourier series.

- (a) Show that f is C^∞ if and only if $|a_n| = O(|n|^{-k})$ for all $k \in \mathbb{N}$.
- (b) Prove that a sequence of functions $\{f_n\}_{n \geq 1}$ in $C^\infty(\mathbb{R}/\mathbb{Z})$ converges in the C^∞ topology (uniform convergence of functions and their derivatives of all orders) if and only if the sequences of k -th derivatives $\{f_n^{(k)}\}_{n \geq 1}$, for all $k \geq 0$, converge in the L^2 -norm on \mathbb{R}/\mathbb{Z} .

3. (AG) Let C be a smooth projective curve over \mathbb{C} and $\omega_C^{\otimes 2}$ the square of its canonical sheaf.

- (a) What is the dimension of the space of sections $\Gamma(C, \omega_C^{\otimes 2})$?
- (b) Suppose $g(C) \geq 2$ and $s \in \Gamma(C, \omega_C^{\otimes 2})$ is a section with simple zeros. Compute the genus of $\Sigma = \{x^2 = s\}$ in the total space of the line bundle ω_C , i.e. the curve defined by the 2-valued 1-form \sqrt{s} .

4. (AT) Show (using the theory of covering spaces) that every subgroup of a free group is free.

5. (CA)

- (a) Define Euler's Gamma function $\Gamma(z)$ in the half plane $\operatorname{Re}(z) > 0$ and show that it is holomorphic in this half plane.
- (b) Show that $\Gamma(z)$ has a meromorphic continuation to the entire complex plane.

- (c) Where are the poles of $\Gamma(z)$?
- (d) Show that these poles are all simple and determine the residue at each pole.
6. (A) Let G be a finite group, and $\rho : G \rightarrow GL_n(\mathbb{C})$ a linear representation. Then for each integer $i \geq 0$ there is a representation $\wedge^i \rho$ of G on the exterior power $\wedge^i(\mathbb{C}^n)$. Let W_i be the subspace $(\wedge^i(\mathbb{C}^n))^G$ of $\wedge^i(\mathbb{C}^n)$ fixed under this action of G .

Prove that $\dim W_i$ is the T^i coefficient of the polynomial

$$\frac{1}{|G|} \sum_{g \in G} \det(\mathbf{1}_n + T\rho(g))$$

where $\mathbf{1}_n$ is the $n \times n$ identity matrix.