QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday September 1, 2009 (Day 1)

1. (RA) Let H be a Hilbert space and $\{u_i\}$ an orthonormal basis for H. Assume that $\{x_i\}$ is a sequence of vectors such that

$$\sum ||x_n - u_n||^2 < 1.$$

Prove that the linear span of $\{x_i\}$ is dense in H.

Solution. To show that the linear span $L = \text{span}\{x_i\}$ is dense, it suffices to show $\bar{L}^{\perp} = 0$. Suppose not, then there exists $v \neq 0$ with $v \perp x_i$ for all i. We may assume $\|v\| = 1$. Then

$$v = \sum_{i=0}^{\infty}(v, u_i)u_i$$

so $\|v\|^2 = \sum |(v, u_i)|^2$. On the other hand, by the Cauchy-Schwarz inequality

$$|(v, u + i)|^2 = |(v, x_i - u_i)|^2 \leq \|v\|\|x_i - u_i\| = \|x_i - u_i\|.$$

Thus

$$1 = \|v\|^2 = \sum |(v, u_i)|^2 \leq \sum \|x_i - u_i\|^2 < 1$$

a contradiction.

2. (T) Let $\mathbb{CP}^n$ be complex projective n-space.

(a) Describe the cohomology ring $H^*(\mathbb{CP}^n, \mathbb{Z})$ and, using the Kunneth formula, the cohomology ring $H^*(\mathbb{CP}^n \times \mathbb{CP}^n, \mathbb{Z})$.

(b) Let $\Delta \subset \mathbb{CP}^n \times \mathbb{CP}^n$ be the diagonal, and $\delta = i_*[\Delta] \in H_{2n}(\mathbb{CP}^n \times \mathbb{CP}^n, \mathbb{Z})$ the image of the fundamental class of Δ under the inclusion $i : \Delta \to \mathbb{CP}^n \times \mathbb{CP}^n$. In terms of your description of $H^*(\mathbb{CP}^n \times \mathbb{CP}^n, \mathbb{Z})$ above, find the Poincaré dual $\delta^* \in H^{2n}(\mathbb{CP}^n \times \mathbb{CP}^n, \mathbb{Z})$ of δ.

Solution.

(a) The cohomology ring $H^*(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}[\alpha]/\alpha^{n+1}$, where deg α=2. Since $H^*(\mathbb{CP}^n, \mathbb{Z})$ are $\mathbb{Z}$-free, the Künneth formula implies

$$H^*(\mathbb{CP}^n \times \mathbb{CP}^n, \mathbb{Z}) \cong H^*(\mathbb{CP}^n, \mathbb{Z}) \otimes H^*(\mathbb{CP}^n, \mathbb{Z})$$

(here $\otimes$ is the graded tensor product)

$$\cong \mathbb{Z}[\alpha]/\alpha^{n+1} \otimes \mathbb{Z}[\beta]/\beta^{n+1} = \mathbb{Z}[\alpha, \beta]/(\alpha^{n+1}, \beta^{n+1}),$$

with deg α = deg $\beta = 2$.

(b) Put $X = \mathbb{CP}^n \times \mathbb{CP}^n$.
Poincaré duality allows us to define a pushforward map $I_* : H^k(\Delta) \to H^{k+2n}(X)$ by the diagram

$$\begin{array}{ccc} H^k(\Delta) & \xrightarrow{i_*} & H^{k+2n}(X) \\ \downarrow{\scriptstyle PD} & & \uparrow{\scriptstyle PD} \\ H_{2n-k}(\Delta) & \xrightarrow{i_*} & H_{2n-k}(X) \end{array}$$

were the lower horizontal map is the usual i_* by functoriality. This construction satisfies the projection formula

$$i_*(a \cup i^*b) = i_*a \cup b$$

for cohomology classes a, b. Observe that by construction $i_*1 = \delta^*$. Now put $i_*1 = \delta^* = \sum c_k \alpha^k \cup \beta^{n-k}$. Then

$$c_{n-k}\alpha^n\beta^n = i_*1 \cup \alpha^k \cup \beta^{n-k} = i_*\gamma^n$$

where we denote by γ the generator of of the cohomology algebra $H^*(\Delta) = H^*(\mathbb{CP}^n)$ as in the previous part of the question (note that $i^*\alpha = i^*\beta = \gamma$, since the composition of the diagonal with each projection is the identity). But γ^n is the Poincaré dual of the class of a point in $H_0(\Delta)$, so that $i_*\gamma^n$ is the dual of the class of a point in $H^*(X)$, which is $\alpha^n\beta^n$. Thus $c_{n-k} = 1$, and $\delta^* = \sum \alpha^k\beta^{n-k}$.

3. (AG) Let $X \subset \mathbb{P}^n$ be an irreducible projective variety, $\mathbb{G}(1,n)$ the Grassmannian of lines in $\mathbb{P}^n$, and $F \subset \mathbb{G}(1,n)$ the variety of lines contained in X.

(a) If X has dimension k, show that

$$\dim F \leq 2k - 2,$$

with equality holding if and only if $X \subset \mathbb{P}^n$ is a k-plane.

(b) Find an example of a projective variety $X \subset \mathbb{P}^n$ with $\dim X = \dim F = 3$.

Solution.

(a) Consider the incidence correspondence $\Sigma = \{(l, x) : x \in l\} \subset \mathbb{G}(1,n) \times \mathbb{P}^n$.
Since $pr_1^{-1}(F) \to F$ has fiber isomorphic to $\mathbb{P}^1$, we have

$$\dim\ \Sigma \cap pr_1^{-1}(F) = \dim\ F + 1.$$

We now estimate the dimension of the fibers of $pr_2 : pr_1^{-1}(F) \to X$, noting that pr_2 maps $pr_1^{-1}(F)$ into X by definition.
Let L_x be the fiber over x. Then $\Sigma \cap pr_1^{-1}(L_x)$ projects via pr_2 into X

with finite fibers over points in $X \setminus \{x\}$ (in fact, generically one-to-one), because a line through $x \neq y \in X$ is uniquely determined.
Thus

$$\dim\, pr_1^{-1}(L_x) \cap \Sigma = \dim\, L_x + 1 \leq \dim\, X = k\dim\, F + 1.$$

It follows that $\dim\, \Sigma \cap pr_1^{-1}(F) \leq \dim\, X + \dim\, X - 1$, so $\dim\, F \leq 2k-2$. Equality can occur only if for generic $x \in X$, the projection $pr_2 : \Sigma \cap pr_1^{-1}(L_x) \to X$ is dominant (hence surjective). In particular this implies that for all $x \neq y \in X$, the line xy is in X. The only X with such property is are linear subspaces of $\mathbb{P}^n$. (To see the last point, take a maximal set of independent points in X, then X must be contained in the plane spanned by them, but also contains the plane spanned by them).

(b) Let X be the cone over the smooth quadric $Q \subset \mathbb{P}^3$ with apex P. Clearly $\dim\, X = 2 + 1 = 3$.
We know Q is ruled by two $\mathbb{P}^1$ family of lines and any line in Q lies in one of those families.
It follows that a line in $X = C(Q)$ must lie in the plane spanned by P and a line in Q since the only rational curves on Q are the ruling lines. But the dimension of the variety of lines in $\mathbb{P}^2$ is 2, and we have two $\mathbb{P}^1$ family of planes which intersect each other only at P, hence $\dim\, F(X) = 2 + 1 = 3 = \dim\, X$.

4. (CA) Let $\Omega \subset \mathbb{C}$ be the open set

$$\Omega = \{z : |z| < 2 \text{ and } |z-1| > 1\}.$$

Give a conformal isomorphism between Ω and the unit disc $\Delta = \{z : |z| < 1\}$.

Solution. The Möbius map $z \mapsto \frac{2z}{2-z}$ sends the disk $\{|z| < 2\}$ to $\{\Re z < 1\}$, the disk $\{|z-1| < 1\}$ to $\{\Re z < 0\}$, hence it sends Ω biholomorphically to the strip $\{0 < \Re z < 1\}$.
Now the map $z \mapsto e^{2\pi i z}$ sends this strip biholomorphically to the upper half-plane, since we can write down an inverse by taking a branch of $\frac{1}{2\pi i}\log$ in the complement of the negative imaginary axis.
Finally $z \mapsto \frac{-z+i}{z+i}$ maps the upper half-plane biholomorphically to the unit disk Δ.
Thus the map

$$z \mapsto \frac{-e^{\frac{2\pi i z}{z-2}} + i}{e^{\frac{2\pi i z}{z-2}} + i}$$

defines a conformal isomorphism between Ω and Δ.

5. (A) Suppose ϕ is an endomorphism of a 10-dimensional vector space over $\mathbb{Q}$ with the following properties.

1. The characteristic polynomial is $(x-2)^4(x^2-3)^3$.
2. The minimal polynomial is $(x-2)^2(x^2-3)^2$.
3. The endomorphism $\phi - 2I$, where I is the identity map, is of rank 8.

Find the Jordan canonical form for ϕ.

Solution.

(a) Denote the underlying vector space by V. We will determine V as a $\mathbb{Q}[x]-$module, where x acts via ϕ. As such, V splits into a direct sum of cyclic modules, of form $\mathbb{Q}[x]/P(x)^k$ for irreducible P.
Since ϕ has characteristic polynomial $(x-2)^4(x^2-3)^2$, dim $V = 10$.
Since ϕ has minimal polynomial $(x-2)^2(x^2-3)^2$, V must be a direct sum of factros $\mathbb{Q}[x]/(x-2)^k$ with $k \leq 2$ and $\mathbb{Q}[x]/(x^2-3)^l$ with $l \leq 2$, and at least one factor for which $k = 2$, $l = 2$.
This gives only two possibilities:

$$V \cong \mathbb{Q}[x]/(x^2-3)^2 \oplus \mathbb{Q}[x]/(x^2-3) \oplus \mathbb{Q}[x]/(x-2)^2 \oplus \mathbb{Q}[x]/(x-2)^2$$

or

$$V \cong \mathbb{Q}[x]/(x^2-3)^2 \oplus \mathbb{Q}[x]/(x^2-3) \oplus \mathbb{Q}[x]/(x-2)^2 \oplus \mathbb{Q}[x]/(x-2) \oplus \mathbb{Q}[x]/(x-2).$$

Noting the $\phi - 2I$ has rank $6+1+1 = 8$ in the first case and $6+1+0+0 = 7$ in the second case, we conclude that the first case occurs. Thus

$$V \otimes \mathbb{C} \cong \mathbb{C}[x]/(x-\sqrt{3})^2 \oplus \mathbb{C}[x]/(x+\sqrt{3})^2 \oplus \mathbb{C}[x]/(x-\sqrt{3}) \oplus \mathbb{C}[x]/(x+\sqrt{3}) \oplus (\mathbb{C}[x]/(x-2)^2)^2$$

so that the Jordan normal form of ϕ is

$$\begin{pmatrix}
\sqrt{3} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\sqrt{3} & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{pmatrix}.$$

6. (DG) Let $\gamma : (0,1) \to \mathbb{R}^3$ be a smooth arc, with $\gamma' \neq 0$ everywhere.

(a) Define the *curvature* and *torsion* of the arc.

(b) Characterize all such arcs for which the curvature and torsion are constant.

Solution. We assume γ is parameterized by arc-length, so that $|\gamma'(t)| = 1$.

(a) The curvature of γ is
$$\kappa(t) = |\gamma''(t)|.$$
Put $v = \gamma'$. Because $(\gamma', \gamma') = 1$, differentiating with respect to t gives $(\gamma', \gamma'') = 0$, hence $v \perp \gamma''$. Define the normal n to be the unit vector in the direction of γ'', so
$$n(t) = \frac{\gamma''(t)}{|\gamma''(t)|}.$$
(If $\gamma'' = 0$, γ lies on a line and has curvature 0 and we do not define n in this case).
We have
$$\gamma''(t) = \kappa(t)\gamma'(t),$$
that is
$$n = \kappa v.$$
The binormal is defined to be $b = v \wedge n$. Note that v, b, n is a positively oriented orthonormal frame at each point of γ.
Finally the torsion τ is defined by
$$n' = -\kappa v + \tau b.$$

(b) We have
$$B' = v' \wedge n + v \wedge n' = v \wedge (-\kappa v + \tau b) = -\tau n.$$
So
$$\begin{pmatrix} v' \\ n' \\ b' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} v \\ n \\ b \end{pmatrix}.$$
Suppose that κ, τ are constant along γ. We will show that γ is a helix (or a circle or a line for degenerate values of κ, τ) by setting up a differential equation that γ satisfies. Assume κ, $\tau \neq 0$.
We have $v = \gamma'$, $v' = \gamma'' = \kappa n$, so $n = \frac{\gamma''}{\kappa}$, $b = \frac{\gamma' \wedge \gamma''}{\kappa}$. Also $n' = \frac{\gamma'''}{\kappa} = -\kappa\gamma''' + \tau\frac{\gamma' \wedge \gamma''}{\kappa}$. This gives
$$\begin{pmatrix} \gamma_1''' \\ \gamma_2''' \\ \gamma_3''' \end{pmatrix} = -\kappa^2 \begin{pmatrix} \gamma_1'' \\ \gamma_2'' \\ \gamma_3'' \end{pmatrix} + \tau \begin{pmatrix} \gamma_2'\gamma_3'' - \gamma_3'\gamma_2'' \\ \gamma_3'\gamma_1'' - \gamma_1'\gamma_3'' \\ \gamma_1'\gamma_2'' - \gamma_2'\gamma_1'' \end{pmatrix}$$
This is a third order system of ODE, so it suffices to find helices $\gamma(z) = \frac{1}{r}(a\cos z, a\sin z, bz)$ (here $a^2 + b^2 = r^2$) with arbitrary given constant curvature κ, torsion τ and initial vectors $\gamma(0)$, $\gamma'(0)$, $\gamma''(0)$ such that $\gamma'(0) \perp \gamma''(0)$.

Looking at the helix $\gamma(z) = (a\cos z, a\sin z, bz)$, $a^2 + b^2 = 1$, $a > 0$ we have

$$v = \gamma'(z) = (-a\sin z, a\cos z, b)$$

$$\gamma'(0) = (0, a, b)$$

$$\gamma''(z) = (-a\cos z, a\sin z, b)$$

$$\gamma''(0) = (-a, 0, 0).$$

This gives $\kappa = a$.

$$n = (\cos z, \sin z, 0)$$

$$n' = (-\sin z, \cos z, 0)$$

$$\underline{b} = v \wedge n = (-b\sin z, b\cos z, -a)$$

so $\tau = b$.
This shows that we can arrange the helix to have arbitrary constant curvature and torsion κ, τ.
It is clear that by shifting z and scaling we can arrange the helix so that $\gamma'(0)$, $\gamma''(0)$ are an arbitrary pair of orthogonal vectors. By a translation, we can arrange for $\gamma(0)$ to be any given point. This shows that we can arrange our initial conditions to be arbitrary, and hence all space curves with non-zero constant curvature and torsion are helices. One can also easily show that curves with torsion 0 are circles, and curves with curvature 0 are lines.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Wednesday September 2, 2009 (Day 2)

1. (CA) Let $\Delta = \{z : |z| < 1\} \subset \mathbb{C}$ be the unit disc, and $\Delta^* = \Delta \setminus \{0\}$ the punctured disc. A holomorphic function f on Δ^* is said to have an *essential singularity* at 0 if $z^n f(z)$ does not extend to a holomorphic function on Δ for any n.

Show that if f has an essential singularity at 0, then f assumes values arbitrarily close to every complex number in any neighborhood of 0—that is, for any $w \in \mathbb{C}$ and $\forall \epsilon$ and $\delta > 0$, there exists $z \in \Delta^*$ with

$$|z| < \delta \quad \text{and} \quad |f(z) - w| < \epsilon.$$

Solution. Suppose the contrary, so there exists $\delta > 0$, $\epsilon > 0$, $w \in \mathbb{C}$ such that $|f(z) - w| > \epsilon$ for all $|z| < \delta$.
Then the function $g(z) = \frac{1}{f(z)-w}$ is defined and holomorphic in $|z| < \delta$.
Furthermore $|g(z)| < \frac{1}{\epsilon}$ for such z. Hence the singularity of g at 0 is removable, so $g(z)$ is holomorphic at 0. But that means $g(z) = z^n h(z)$ for holomorphic h in $|z| < \delta$ and $h(0) \neq 0$, hence $\frac{1}{h}$ is holomorphic at 0.
Now

$$f(z) = w + \frac{1}{g(z)} = w + \frac{1}{z^n h(z)}$$

so $z^n f(z)$ extends to a holomorphic function on Δ, a contradiction.

2. (AG) Let $S \subset \mathbb{P}^3$ be a smooth algebraic surface of degree d, and $S^* \subset \mathbb{P}^{3^*}$ the *dual surface*, that is, the locus of tangent planes to S.

(a) Show that no plane $H \subset \mathbb{P}^3$ is tangent to S everywhere along a curve, and deduce that S^* is indeed a surface.

(b) Assuming that a general tangent plane to S is tangent at only one point (this is true in characteristic 0), find the degree of S^*.

Solution. We assume $d > 1$ throughout.
The projective tangent plane to $F(X, Y, Z, T) = 0$ at $P \in \{F = 0\}$ has equation $\frac{\partial F}{\partial X}X + ... + \frac{\partial F}{\partial T}T = 0$ (the partial derivatives are evaluated at P).
In terms of coordinates, the Gauss map $S \mapsto S^* \subset \mathbb{P}^{3*}$ is given by

$$[X : Y : Z : T] \mapsto [\frac{\partial F}{\partial X} : \frac{\partial F}{\partial Y} : \frac{\partial F}{\partial Z} : \frac{\partial F}{\partial T}]$$

which is a morphism because S is smooth.

(a) Suppose $H \subset \mathbb{P}^3$ is a plane tangent to S everywhere along a curve γ. We can arrange so that H is $T = 0$. Then H is tangent to $F = 0$ along γ means $\frac{\partial F}{\partial X} = \frac{\partial F}{\partial Y} = \frac{\partial F}{\partial Z} = 0$ along γ.
But dim $\gamma = 1$, dim $(\frac{\partial F}{\partial X} = 0) \geq 2$, so $\frac{\partial F}{\partial X} = 0$ must intersect γ at some point P. But for this P we have $\frac{\partial F}{\partial X} = ... = \frac{\partial F}{\partial T} = 0$, so P is a singular point of S, a contradiction.

(b) Put $\phi : S \to S^* \subset \mathbb{P}^{3*}$ for the Gauss map. Since a general tangent plane to S is tangent at only one point of S, ϕ is generically one-to-one.
We can find deg S^* by intersecting S^* with a generic line $l \in \mathbb{P}^{3*}$.
Arrange coordinates so that our line has equations $Z^* = T^* = 0$ for dual coordinates $[X^* : Y^* : Z^* : T^*]$ of $\mathbb{P}^{3*}$.
The image of $[X : Y : Z : T] \in S$ is in the intersection iff $\frac{\partial F}{\partial Z} = \frac{\partial F}{\partial T} = F = 0$ at that point. Since $\frac{\partial F}{\partial Z} = 0$, $\frac{\partial F}{\partial T} = 0$ are generically hypersurfaces of degree $d-1$, and F has degree d, a generic choice of the line (reflected in the generic choice of coordinates) makes the hypersurfaces intersect at $(d-1)^2 d$ points. We can arrange so that $l \cap S^*$ lies in the open dense subset where ϕ^{-1} is a singleton, because a generic point Q will be the only point in S with image $\phi(Q)$.
Hence this shows deg $S^* = (d-1)^2 d$.

(Alternatively, one has $\phi^*\mathcal{O}(1) \cong \mathcal{O}(d-1)$. Computing the Hilbert function we have $\chi(\mathcal{O}_{S^*}(n)) = \chi(\phi_*\mathcal{O}_S(n))$ up to terms of degree < 2 in n, because ϕ is generically one-to-one.
But $\phi_*(\mathcal{O}_S \otimes \phi^*\mathcal{O}(1)) \cong \phi_*\mathcal{O}_S \otimes \mathcal{O}(1)$, so

$$\chi(\phi_*\mathcal{O}_S(n)) = \chi(\mathcal{O}_S((d+1)n)).$$

Since $\chi(\mathcal{O}_S(n)) = \frac{1}{2!}dn^2 + ...$, we have $\chi(\mathcal{O}_{S^*}(n)) = \frac{1}{2!}d(d-1)^2 n^2 + ...$, giving the degree of S^* to be $d(d-1)^2$.)

3. (T) Let $X = S^1 \vee S^1$ be a figure 8, $p \in X$ the point of attachment, and let α and $\beta : [0,1] \to X$ be loops with base point p (that is, such that $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = p$) tracing out the two halves of X. Let Y be the CW complex formed by attaching two 2-discs to X, with attaching maps homotopic to

$$\alpha^2\beta \quad \text{and} \quad \alpha\beta^2.$$

(a) Find the homology groups $H_i(Y, \mathbb{Z})$.

(b) Find the homology groups $H_i(Y, \mathbb{Z}/3)$.

Solution. Y has cell structure with one 0-cell p, two 1-cells α, β and two 2-cells A, B. The cellular chain complex of Y is

$$0 \longrightarrow \mathbb{Z}A \oplus \mathbb{Z}B \xrightarrow{d_2} \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \xrightarrow{0} \mathbb{Z}p \longrightarrow 0$$

where $d_1 = 0$ because we must get a $\mathbb{Z}$ in H_0, as Y is connected.

To compute d_2, we have $A = n_{A\alpha} + n_{A\beta}\beta$, where $n_{A\alpha}$, $n_{A\beta}\beta$ are degrees of the maps $\partial A \to X/\beta$, $\partial A \to X/\alpha$, hence $d_2(A) = 2\alpha+\beta$. Similarly $d_2(B) = \alpha+2\beta$. Thus the cellular chain complex if Y is

$$0 \longrightarrow \mathbb{Z}A \oplus \mathbb{Z}B \xrightarrow{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}} \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \xrightarrow{0} \mathbb{Z}p \longrightarrow 0$$

The cellular chain complex for $\mathbb{Z}/3$ coefficient is obtained by reducing the above mod 3. This gives:

(a) We already know $H_0(Y, \mathbb{Z}) = \mathbb{Z}$.
$H_1(Y, \mathbb{Z}) = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta/(2\alpha + \beta = 0, \alpha + 2\beta = 0) = \mathbb{Z}/3$.
$H_2(Y, \mathbb{Z}) = 0$ since $2x + y = x + 2y = 0$ imples $x = y = 0$ in $\mathbb{Z}$.
All other homology groups vanish.

(b) $H_0(Y/\mathbb{Z}/3) = \mathbb{Z}/3$.
$H_1(Y, \mathbb{Z}/3)$ is the cokernel of $(\mathbb{Z}/3)^2 \to (\mathbb{Z}/3)^2$ given by $(x, y) \mapsto (-x + y, x - y)$, so $H_1(Y, \mathbb{Z}/3) = \mathbb{Z}/3$.
$H_2(Y, \mathbb{Z}/3) = \mathbb{Z}/3$, since $2x + y = x + 2y = 0$ imples $x = y$ in $\mathbb{Z}/3$.
All other homology groups vanish.

4. (DG) Let $f = f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be smooth, and let $S \subset \mathbb{R}^3$ be the graph of f, with the Riemannian metric ds^2 induced by the standard metric on $\mathbb{R}^3$. Denote the volume form on S by ω.

(a) Show that

$$\omega = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}.$$

(b) Find the curvature of the metric ds^2 on S

Solution.

(a) We have a parameterization of S given by $\phi : (x, y) \mapsto (x, y, f(x, y))$.
We have (the lower index denotes the variable with respect to which we differentiate)

$$\phi_x = (1, 0, f_x)$$

$$\phi_x = (0, 1, f_y)$$

The first fundamental form is $Edx^2 + 2Fdxdy + Gdy^2$ with

$$E = (\phi_x, \phi_x) = 1 + f_x^2$$

$$F = (\phi_x, \phi_y) = f_x f_y$$

$$G = (\phi_y, \phi_y) = 1 + f_y^2$$

hence the volume form

$$\omega = \sqrt{EF - G^2} = \sqrt{(1+f_x^2)(1+f_y)^2 - (f_x f_y)^2} = \sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}.$$

(b) The normal vector is

$$\underline{N} = \frac{\phi_x \wedge \phi_y}{|\phi_x \wedge \phi_y|} = \frac{1}{1 + f_x^2 + f_y^2}(-f_x, -f_y, 1).$$

$$\phi_{xx} = (0, 0, f_{xx})$$
$$\phi_{xy} = (0, 0, f_{xy})$$
$$\phi_{yy} = (0, 0, f_{yy})$$

The second fundamental form $Ldx^2 + 2Mdxdy + Ndy^2$ is given by

$$L = -(\underline{N}, \phi_{xx}) = -f_{xx}$$
$$M = -(\underline{N}, \phi_{xy}) = -f_{xy}$$
$$N = -(\underline{N}, \phi_{yy}) = -f_{yy}$$

The Gaussian curvature is given by

$$K(x,y) = \frac{LN - M^2}{EG - F^2} = \frac{\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - (\frac{\partial^2 f}{\partial x \partial y})^2}{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}$$

5. (RA) Suppose that $\mathcal{O} \subset \mathbb{R}^2$ is an open set with finite Lebesgue measure. Prove that the boundary of the closure of $\mathcal{O}$ has Lebesgue measure 0.

Solution. We will construct a counter-example. First we will construct a "fat" Cantor set X in the interval $I = [0,1]$. Let X_1 be I with the interval of length $\frac{1}{4}$ in the middle removed. X_1 consists of two intervals of length $\frac{1}{2}(1-\frac{1}{4})$. Inductively suppose we get X_n consisting of 2^n intervals of length $\frac{1}{2^n}(1 - \frac{1}{4} - ...\frac{2^{n-1}}{4^n})$. Then X_{n+1} is obtained by removing the middle intervals of length $\frac{1}{4^{n+1}}$ in each interval of X_n. Thus X_{n+1} consists of 2^{n+1} intervals of length $\frac{1}{2^{n+1}}(1 - \frac{1}{4} - ... - \frac{1}{4^{n+1}})$. The X_n form a decreasing sequence of non-empty compact subset of I, hence $X = \cap X_n$ is a non-empty compact subset of I. Note that $\mu(X) = 1 - \frac{1}{4} - \frac{2}{4^2} - ... = \frac{1}{2}$. X is also nowhere dense, since X_n can not contain an interval of size $> \frac{1}{2^n}$.

We now construct $\mathcal{O}$ as the complement of $X \times [0,1]$ inside the square $[-2,2] \times [2,2]$, which is open as $X \times [0,1]$ is compact. It has finite measure. Because X is nowhere dense, $X \times [0,1]$ can not contain any rectangle, so $\mathcal{O}$ is dense in $[-2,2] \times [-2,2]$. It follows that the boundary of $\mathcal{O}$ will contain $X \times [0,1]$ which has measure $\frac{1}{2}$. This gives the desired counter-example.

6. (A) Let R be the ring of integers in the field $\mathbb{Q}(\sqrt{-5})$, and S the ring of integers in the field $\mathbb{Q}(\sqrt{-19})$.

(a) Show that R is not a principal ideal domain

(b) Show that S is a principal ideal domain

Solution.

(a) We have $\alpha = a + b\sqrt{-5} \in R$ iff $Tr_{\mathbb{Q}(\sqrt{-5})/\mathbb{Q}}(\alpha) \in \mathbb{Z}$, $N_{\mathbb{Q}(\sqrt{-5})/\mathbb{Q}}(\alpha) \in \mathbb{Z}$ iff $2a \in \mathbb{Z}$, $a^2 + 5b^2 \in \mathbb{Z}$. This implies $2a, 2b \in \mathbb{Z}$ and $4a^2 + 20b^2 \in 4\mathbb{Z}$, so $2a, 2b \in 2\mathbb{Z}$, so $a, b \in \mathbb{Z}$.
Hence $R = \mathbb{Z}[\sqrt{-5}]$.

If R is a PID, it must be UFD. But in R we have $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$.
We claim these are two essentially different factorizations. Indeed any unit $\epsilon \in R$ has norm 1, and $N(2) = 4$, $N(3) = 9$, $N(\pm\sqrt{-5}) = 6$ (we write N as a shorthand for the norm). There are no elements of norm 2 or 3 in R (because $a^2 + 5b^2 = 2$ or 3 has no solution mod 5), hence all factors in the above factorization are non-associated irreducible elements of R.
Thus R is not a UFD, hence not a PID.

(b) By the Minkowski bound, every class in the ideal class group of $\mathcal{O}_K$ of a number field K contains an integral ideal of norm $\leq M_K = \sqrt{|D|}(\frac{4}{\pi})^{r_2}\frac{n!}{n^n}$, where $n = [K : \mathbb{Q}]$, $2r_2$ is the number of complex embeddings of K and D the discriminant of K.
For $K = \mathbb{Q}(\sqrt{-19})$ we have $n = 2$, $r_2 = 1$. To compute D, an argument as in (a) shows that $S = \mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$. Thus a $\mathbb{Z}-$basis for S is given by 1, $\frac{1+\sqrt{-19}}{2}$. Hence $D = -19$. The Minkowski bound is thus $M_K = \frac{2}{\pi}\sqrt{19} < 4$.
To show that S is a PID it suffices to show that all ideal classes are trivial. By the Minkowski bound, it suffices to check that all prime ideals in S of norm < 4 are principal. A prime ideal $\mathfrak{p}$ of norm 2, 3 must lie above 2. 3 respectively. Note that $S \cong \mathbb{Z}[x]/(x^2 - x + 5)$, and has discriminant -19 which is coprime to 2, 3, so the splitting behavior of 2, 3 in S are determined by the factorization of $x^2 - x + 5$ in $\mathbb{F}_2$, $\mathbb{F}_3$. But one easily checks that $x^2 - x + 5$ has no solution in $\mathbb{F}_2$, $\mathbb{F}_3$, hence stay irreducible there. It follows that 2, 3 are inert in S, hence there are no prime ideals in S of norm 2, 3, hence we are done.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Thursday September 3, 2009 (Day 3)

1. (A) Let $c \in \mathbb{Z}$ be an integer not divisible by 3.

(a) Show that the polynomial $f(x) = x^3 - x + c \in \mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$.

(b) Show that the Galois group of f is the symmetric group $\mathfrak{S}_3$.

Solution.

(a) If f is reducible in $\mathbb{Q}[x]$, it is reducible in $\mathbb{Z}[x]$ by Gauss' lemma, hence reducible ober $\mathbb{F}_3[x]$. But f mod 3 has no zeroes in $\mathbb{F}_3$, hence can not factorize there. Thus f is irreducible in $\mathbb{Q}[x]$.

(b) Because f has degree 3, its splitting field K has degree at most 6 over $\mathbb{Q}$. Since f is irreducible, $3|[K : \mathbb{Q}]$, hence $[K : \mathbb{Q}] = 3$ or 6, depending on whether $\mathrm{Gal}(K/\mathbb{Q})$ is A_3 or S_3 (it is a subgroup of S_3 via the transitive action on the roots of f in $\bar{\mathbb{Q}}$.

If $[K : \mathbb{Q}] = 3$, the discriminant $\Delta(f)$ must be a square in $\mathbb{Q}$. But $\Delta(f) = 27c^2 - 4 = -1$ mod 3, hence $\Delta(f) \in \mathbb{Z}$ is not s square in $\mathbb{Z}$, hence not a square in $\mathbb{Q}$. Hence the Galois group is S_3.

2. (CA) Let τ_1 and $\tau_2 \in \mathbb{C}$ be a pair of complex numbers, independent over $\mathbb{R}$, and $\Lambda = \mathbb{Z}\langle\tau_1, \tau_2\rangle \subset \mathbb{C}$ the lattice of integral linear combinations of τ_1 and τ_2. An entire meromorphic function f is said to be *doubly periodic* with respect to Λ if

$$f(z + \tau_1) = f(z + \tau_2) = f(z) \quad \forall z \in \mathbb{C}.$$

(a) Show that an entire holomorphic function doubly periodic with respect to Λ is constant.

(b) Suppose now that f is an entire meromorphic function doubly periodic with respect to Λ, and that f is either holomorphic or has one simple pole in the closed parallelogram

$$\{a\tau_1 + b\tau_2 : a, b \in [0, 1] \subset \mathbb{R}\}.$$

Show that f is constant.

Solution.

(a) The lattice Λ has fundamental domain $D = \{x\tau_1 + y\tau_2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ which is compact. If f is doubly periodic with respect to Λ, put $M = \max_D |f|$.

For any $z \in \mathbb{C}$, there is a $z_0 \in D$ with $f(z) = f(z_0)$, hence $f(z) \leq M$ for all z. Hence f is a bounded entire function, hence constant by Liouville's theorem.

(b) Suppose f is not constant. Translating a fundamental domain if necessary and using (a), we can assume that f has a simple pole inside the fundamental domain D. By the residue theorem

$$\int_{\partial D} f dz = \sum_{z \in Int(D)} 2\pi i \mathrm{Res}_z(f)$$

with the right-hand side non-zero, because f can have non-zero residue only at the unique pole, and the residue there is non-zero since the pole is simple. But $\Lambda-$periodicity implies that in the integral on the left the integral along opposite edges of D cancel each other, so the left-hand side is 0, a contradiction.

3. (DG) Let M and N be smooth manifolds, and let $\pi : M \times N \to N$ be the projection; let α be a differential k-form on $M \times N$. Show that α has the form $\pi^*\omega$ for some k-form ω on N if and only if the contraction $\iota_X(\alpha) = 0$ and the derivative $\mathcal{L}_X(\alpha) = 0$ for any vector field X on $M \times N$ whose value at every point is in the kernel of the differential $d\pi$.

Solution. If $\alpha = \pi^*\omega$

$$\iota_X(\alpha)(Y_1, ..., Y_{k-1}) = \pi^*\omega(X, , Y_1, ..., Y_{k-1}) = \omega(d\pi(X), ..., d\pi(Y_{k-1}) = 0$$

for arbitrary vector fields $Y_1, ..., Y_{k-1}$ on $M \times N$ and X whose value at every point is in ker $d\pi$.

Also

$$\mathcal{L}_X(\alpha)(Y_1, ..., Y_k) = X(\pi^*\omega(Y_1, ...Y_k)) = \pi_* X(\omega(Y_1, ..., Y_k)) = 0$$

because $\pi_* X = 0$ for X as above.

We now show the converse.

The problem is local on both M and N (by taking partitions of unity on M, N and take their product as a partition of unity on $M \times N$), so we work in a neighborhood of $M \times N$ which is of the form $U \times V$ for coordinate neighborhoods U, V of M, N. Call the corresponding local coordinates x_1,...,x_m amd y_1,...,y_n. We will also use standard multi-index notation.
Let

$$\alpha = \sum_{|I|+|J|=k, I \subset \{1,...,m\}, J \subset \{1,...,n\}} f_I dx_I \wedge dy_J$$

be a $k-$form satisfying the necessary conditions. A general vector field X killed by π_* is of form $\sum a_i \frac{\partial}{\partial x_i}$.

We have

$$\iota_{\frac{\partial}{\partial x_i}}(f dx_I \wedge dy_J) = z = \begin{cases} 0 & if \;\; i \notin I \\ \pm f dx_{I\setminus\{i\}} \wedge dy_J & if \;\; i \in I \end{cases}$$

Hence the condition $\iota_{\frac{\partial}{\partial x_i}}\alpha = 0$ implies $f_{IJ} = 0$ for all $I \ni i$. Since i is arbitrary, $f_{IJ} = 0$ for all $I \neq \emptyset$. This means

$$\alpha = \sum_{|J|=k, J\subset\{frm[o]--,...,n\}} f_J(x_1, ..., x_m, y_1, ...y_n) dy_J.$$

But

$$0 = \mathcal{L}_{\frac{\partial}{\partial x_i}}\alpha(\frac{\partial}{\partial y_{i_1}}, ..., \frac{\partial}{\partial y_{i_k}}) = \frac{\partial}{\partial x_i}(f_{i_1 \ldots i_k})$$

for $i_1 < ... < i_k$, hence the functions f_J are functions on the y_j only, so they are of form $\pi^* g_J$ for smooth functions g_J on N. Since the dy_J are pulled back from N, we have $\alpha = \pi^*\omega$ for some k-form ω on N.

4. (RA) Show that the Banach space ℓ^p can be embedded as a summand in $L^p(0,1)$—in other words, that $L^p(0,1)$ is isomorphic as a Banach space to the direct sum of ℓ^p and another Banach space.

Solution. Choose disjoint intervals $I_n \subset (0,1)$ and let f_n be a positive multiple of the characteristic function of I_n, normalized by $\|f_n\|_p = 1$. Define an embedding $\iota : l^p \to L^p(0,1)$ by $(a_n) \mapsto \sum a_n f_n$. Observe that

$$\int |\sum_r^s a_n f_n|^p = \sum_r^s \int_{I_n} |a_n|^p |f_n|^p = \sum_r^s |a_n|^p$$

This shows that if $(a_n) \in l^p$ then $\sum a_n f_n$ converges in $L^p(0,1)$ and has the same norm. This shows that our map is defined and is an isometric embedding. Therefore it remains to write down a continuous projection P splitting it. Define $P : L^p(0,1) \to l^p$ by $f \mapsto (\int f|f_n|^{p-1})_n$. The map is defined since $f \in L^1(0,1)$. Now we have by Hölder's inequality (note that $(p-1)q = p$)

$$\sum_n |\int_{I_n} f|f_n|^{p-1}|^p \leq \sum_n (\int_{I_n} |f|^p)(\int |f_n|^{(p-1)q})^{\frac{p}{q}} = \sum_n \int_{I_n} |f|^p \leq \|f\|_p^p$$

This shows that P is continuous (indeed of norm ≤ 1), and clearly $P\iota = id$. This gives the desired decomposition of $L^p(0,1)$ with one summand l^p.

5. (T) Find the fundamental groups of the following spaces:

(a) $SL_2(\mathbb{R})$

(b) $SL_2(\mathbb{C})$

(c) $SO_3(\mathbb{C})$

Solution.

(a) By the polar decomposition, any $A \in GL_2(\mathbb{R})$ can be written uniquely as $A = PU$ with $U \in O_2(\mathbb{R})$ and P a positive definite symmetric matrix. If $A \in SL_2(\mathbb{R})$ thn $U \in SO_2(\mathbb{R})$.
This gives a homeomorphism

$$SL_2(\mathbb{R}) \cong P_+ \times SO_2(\mathbb{R}).$$

The space P_+ of positive definite symmetric matrices is contractible since it is an open cone in a real vector space while $SO_2(\mathbb{R}) \cong S^1$. Thus $\pi_1(SL_2(\mathbb{R})) \cong \pi_1(S^1) \cong \mathbb{Z}$.

(b) Similar to (a), the complex polar decomposition gives a unique decomposition $A = PU$ for $A \in SL_2(\mathbb{C})$, where $U \in SU_2(\mathbb{C})$ and P a positive definite Hermitian matrix. Again, the space of positive definite Hermitian matrix is contractible, hence

$$\pi_1(SL_2(\mathbb{C})) \cong \pi_1(SU_2(\mathbb{C})) \cong \pi_1(S^3) = 0$$

noting that $SU_2(\mathbb{C}) \cong S^3$ via $(a, b) \mapsto \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix}$ where a, b are complex numbers such that $|a|^2 + |b|^2 = 1$.

(c) $SL_2(\mathbb{C})$ acts on $\mathfrak{sl}_2 \cong \mathbb{C}^3$ (the subspace of $M_2(\mathbb{C})$ consisting of trace 0 matrices). This action preserves the non-degenerate symmetric bilinear form given by $K(A, B) = Tr(ad(A).ad(B))$ where $ad(A)$ is the operator $X \mapsto [A, X] = AX - XA$ on sl_2. This gives a morphism $SL_2(\mathbb{C}) \to SO_3(\mathbb{C})$ whose kernel is $\pm I$. Hence we get $PSL_2(\mathbb{C}) \hookrightarrow SO_3(\mathbb{C})$. Since both sides are connected Lie groups of the same complex dimension, the map is an isomorphism. From (b) we know that $SL_2(\mathbb{C})$ is simply connected, hence is the universal cover of $SO_3(\mathbb{C})$, so $\pi_1(SO_3(\mathbb{C})) \cong \mathbb{Z}/2$.

6. (AG) Let $X \subset \mathbb{A}^n$ be an affine algebraic variety of pure dimension r over a field K of characteristic 0.

(a) Show that the locus $X_{\text{sing}} \subset X$ of singular points of X is a closed subvariety.

(b) Show that X_{sing} is a proper subvariety of X.

Solution.

(a) Let $I(X) = (f_1, ..., f_m)$. Then $x \in X$ is singular iff the Jacobian matrix $J = (\frac{\partial f_i}{x_j})$ has rank $<$ codim $(X) = n - r$ at x. This happens iff every $(n-r) \times (n-r)$ minors of $J(x)$ vanish. Since these are regular functions, X_{sing} is a closed subvariety of X.

(b) It suffices to treat the case X irreducible. In characteristic 0, X is birational to a hypersurface $F = 0$ in some affine space $\mathbb{A}^n$. To see this, observe that the function field $K(X)$ is a simple extension of a purely transcendental field $k(t_1, ...t_r)$, by the primitive element theorem.

Hence $K(X) = k(t_1, ...t_r, u)$ with u algebraic over $k(t_1, ...t_r)$. Note $t_1, ...t_r$ is a transcendental basis of $K(X)$. If G is the minimal polynomial of u over $k(t_1, ..t_r)$, after clearing denominators we see that $K(X)$ is the function field of a hypersurface $F = 0$ in $\mathbb{A}^{r+1}$. In particular they have some isomorphic dense open subsets.

Thus we are reduced to the case X is a hypersurface $F = 0$ in $\mathbb{A}^{r+1}$. In this case X_{sing} is the locus $\frac{\partial F}{\partial X_i} = 0$ and $F = 0$. If $X_{sing} = X$, using the UFD property of $k[X_1, ..., X_{r+1}]$ and the fact F is irreducible, we deduce that $F | \frac{\partial F}{\partial X_i}$. This forces $\frac{\partial F}{\partial X_i} = 0$ for degree reasons. But this can not happen in characteristic 0, as can be seen by looking at a maximal monomial appearing in F with respect to the lexicographic order. This shows that X must contain non-singular points.