Qualifying exam, Fall 2006, Day 1

(1) Let G_1 and G_2 be finite groups, and let V_i be a finite dimensional complex representation of G_i, for $i = 1, 2$. Give $V_1 \otimes_{\mathbb{C}} V_2$ the structure of a representation of the direct product $G_1 \times G_2$ by the rule

$$(g_1, g_2)(v_1 \otimes v_2) := (g_1 v_1) \otimes (g_2 v_2).$$

(a) Show that if V_1 and V_2 are irreducible representations of G_1 and G_2, respectively, then $V_1 \otimes V_2$ is an irreducible representation of $G_1 \times G_2$.

(b) Show that every irreducible representation of $G_1 \times G_2$ arises in this way.

(2) Let R be the polynomial ring on 9 generators $\mathbb{C}[a_{11}, a_{21}, ..., a_{23}, a_{33}]$, and let A be a matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

with values in R. Let I be the ideal in R generated by the entries of A^3.

(a) Show that the subvariety X of $\mathbb{A}^9$ defined by I is irreducible.

(b) Let J be the ideal of polynomials in R that vanish identically on X. Does J equal I?

(3) Prove that for $n = 1, 2, 3, \ldots$

$$\frac{1}{2\pi} \int_0^{2\pi} \cos(n\theta - 2\sin\theta) d\theta = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}.$$

Hint: consider the function $z \mapsto e^{z - 1/z}$.

(4) Prove that π_1 of a topological group is abelian.

(5) Let $f : Y \to X$ be a smooth embedding of a manifold Y into a manifold X. Let X be equipped with a Riemannian metric $\overline{g}$ with the associated Levi-Cevita connection $\overline{\nabla}$ on TX. Let $g = f^*\overline{g}$ be the induced metric on Y, with Levi-Cevita connection ∇. For $\eta, \xi \in TY$ define

$$\Psi(\eta, \xi) = \overline{\nabla}_{(f_*\eta)}(f_*\xi) - f_*(\nabla_\eta \xi) \in TX|_Y.$$

Show that Ψ is a well-defined tensor field in $Sym^2(T^*Y) \otimes \mathcal{N}_{Y/X}$, where $\mathcal{N}_{Y/X}$ is the normal to Y in X, i.e., $\mathcal{N}_{Y/X} := TY^\perp \subset TX|_Y$.

(6) Let B be the unit ball in $\mathbb{R}^n$. Prove that the embedding $C^{k+1}(B) \to C^k(B)$ is a compact operator.

Qualifying exam, Fall 2006, Day 2

*All problems are worth 10 points. Problems marked with * will give extra bonus*

(1) Let R be a Noetherian commutative domain, and let M be a torsion-free R-module. (I.e., for $0 \neq r \in R$ and $0 \neq m \in M$ implies $r \cdot m \neq 0$.)
(a) Show that if R is a Dedekind domain and M is finitely generated, then M is a projective R-module.
(b) Give examples showing that M may not be projective if either R is not Dedekind or M is not finitely generated.

(2) Let X be the blow-up of $\mathbb{A}^2$ at 0, and let $Y \subset X$ be the exceptional divisor (i.e., the preimage of 0). Consider the line bundles $\mathcal{L}_n := \mathcal{O}_X(n \cdot Y)$ for $n \in \mathbb{Z}$. Calculate $\Gamma(X, \mathcal{L}_n)$.

(3) Does there exist a nonconstant holomorphic function f on $\mathbb{C}$ such that $f(z)$ is real whenever $|z| = 1$?

(4) Let X be the union of the unit sphere in $\mathbb{R}^3$ and the straight line segment connecting the south and north poles.
(a) Calculate $\pi_1(X)$.
(b*) Calculate $\pi_2(X)$, and describe $\pi_2(X)$ as a $\mathbb{Z}[\pi_1(X)]$-module.

(5) Show that a curve in $\mathbb{R}^3$ lies in a plane if and only if its torsion τ vanishes identically. Identify those curves with vanishing torsion *and* constant curvature k.

(6) Let B be the unit ball in $\mathbb{R}^n$. Recall that if $f : B \to \mathbb{C}$ is a measurable function we define, for $0 < p < \infty$, the $L^p(B)$ norm of f by

$$\|f\|_p = \left(\int_B |f|^p dx \right)^{1/p},$$

and the L^∞ norm of f by

$$\|f\|_\infty = \inf \{ a \geq 0 : \{ x \in B : |f(x)| > a \} \text{ has Lebesgue measure } 0 \}.$$

The spaces $L^p(B)$ and $L^\infty(B)$ are the spaces of measurable functions on B with finite L^p and L^∞ norms, respectively. Show that if $f \in L^\infty$ then

$$\|f\|_\infty = \lim_{q \to \infty} \|f\|_q.$$

Qualifying exam, Fall 2006, Day 3

*All problems are worth 10 points. Problems marked with * will give extra bonus*

(1) Let G be a finite p-group, N a normal subgroup, Z the center of G. Prove that $Z \cap N$ is non-trivial.

(2) Let $\mathrm{Gr}(k, n)$ be the Grassmannian of k-planes in $\mathbb{C}^n$, and let W be a fixed d-plane in $\mathbb{C}^n$ with $k + d \geq n$. Let S_i be the subset of $\mathrm{Gr}(k, n)$, consisting of k-planes V, for which $\dim(V + W) \leq n - i$.
(a) Show that S_i is a closed subvariety of $\mathrm{Gr}(k, n)$.
(b) Find the dimension of S_i.
(c*) Show that the singular locus of S_i is contained in S_{i+1}.

(3) Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + x + 1} dx.$$

(4) Formulate the Poincaré duality theorem for orientable compact manifolds with boundary.

(5) Let G be a Lie group. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g} \subset \mathrm{Lie(G)}$. Show that there exists a unique Lie subgroup $H \subset G$ with $\mathfrak{h} = \mathrm{Lie(H)}$.

(6) Let $f \in L_1(\mathbb{R})$ and $f_\epsilon := \epsilon^{-1} f(x/\epsilon)$. Prove that $\lim_{\epsilon \to +0} f_\epsilon$ exists in the space $\mathcal{D}'(\mathbb{R})$ and find it. Calculate the following limits in $\mathcal{D}'(\mathbb{R})$:

$$\lim_{\epsilon \to +0} \frac{1}{\sqrt{\epsilon}} e^{-\frac{x^2}{\epsilon}}, \quad \lim_{\epsilon \to +0} \frac{\epsilon}{x^2 + \epsilon^2}.$$