QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday 20 September 2005 (Day 1)

1. Let X be the CW complex constructed as follows. Start with $Y = S^1$, realized as the unit circle in $\mathbb{C}$; attach one copy of the closed disc $D = \{z \in \mathbb{C} : |z| \leq 1\}$ to Y via the map $\partial D \to S^1$ given by $e^{i\theta} \mapsto e^{4i\theta}$; and then attach another copy of the closed disc D to Y via the map $\partial D \to S^1$ given by $e^{i\theta} \mapsto e^{6i\theta}$.

(a) Calculate the homology groups $H_*(X, \mathbb{Z})$.

(b) Calculate the homology groups $H_*(X, \mathbb{Z}/2\mathbb{Z})$.

(c) Calculate the homology groups $H_*(X, \mathbb{Z}/3\mathbb{Z})$.

2. Show that if a curve in $\mathbb{R}^3$ lies on a sphere and has constant curvature then it is a circle.

3. Let $X \cong \mathbb{P}^5$ be the space of conic curves in $\mathbb{P}^2_{\mathbb{C}}$; that is, the space of nonzero homogeneous quadratic polynomials $F \in \mathbb{C}[A, B, C]$ up to scalars. Let $Y \subset X$ be the set of quadratic polynomials that factor as the product of two linear polynomials; and let $Z \subset X$ be the set of quadratic polynomials that are squares of linear polynomials.

(a) Show that Y is a closed subvariety of $X \cong \mathbb{P}^5$, and find its dimension and degree.

(b) Show that Z is a closed subvariety of $X \cong \mathbb{P}^5$, and find its dimension and degree.

4. We say that a linear functional F on $\mathcal{C}([0, 1])$ is *positive* if $F(f) \geq 0$ for all non-negative functions f. Show that a positive F is continuous with the norm $||F|| = F(1)$, where 1 means the constant function 1 on $[0, 1]$.

5. Let D_8 denote the dihedral group with 8 elements.

(a) Calculate the character table of D_8.

(b) Let V denote the four dimensional representation of D_8 corresponding to the natural action of the dihedral group on the vertices of a square. Decompose $\mathrm{Sym}^2 V$ as a sum of irreducible representations.

6. Let f be a holomorphic function on $\mathbb{C}$ with no zeros. Does there always exist a holomorphic function g on $\mathbb{C}$ such that $\exp(g) = f$?

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Wednesday 21 September 2005 (Day 2)

1. Let n be a positive integer. Using Cauchy's Integral Formula, calculate the integral

$$\oint_C \left(z - \frac{1}{z}\right)^n \frac{dz}{z}$$

where C is the unit circle in $\mathbb{C}$. Use this to determine the value of the integral

$$\int_0^{2\pi} \sin^n t \, dt$$

2. Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a smooth plane curve of degree $d > 1$. Let $\mathbb{P}^{2^*}$ be the dual projective plane, and $C^* \subset \mathbb{P}^{2^*}$ the set of tangent lines to C.

(a) Show that C^* is a closed subvariety of $\mathbb{P}^{2^*}$.

(b) Find the degree of C^*.

(c) Show that not every tangent line to C is bitangent, i.e., that a general tangent line to C is tangent at only one point. (Note: this is false if $\mathbb{C}$ is replaced by a field of characteristic $p > 0$!)

3. Find all surfaces of revolution $S \subset \mathbb{R}^3$ such that the mean curvature of S vanishes identically.

4. Find all solutions to the equation $y''(t) + y(t) = \delta(t+1)$ in the space $\mathcal{D}'(\mathbb{R})$ of distributions on $\mathbb{R}$. Here $\delta(t)$ is the Dirac delta-function.

5. Calculate the Galois group of the splitting field of $x^5 - 2$ over $\mathbb{Q}$, and draw the lattice of subfields.

6. A covering space $f : X \to Y$ with X and Y connected is called *normal* if for any pair of points $p, q \in X$ with $f(p) = f(q)$ there exists a deck transformation (that is, an automorphism $g : X \to X$ such that $g \circ f = f$) carrying p to q.

(a) Show that a covering space $f : X \to Y$ is normal if and only if for any $p \in X$ the image of the map $f_* : \pi_1(X, p) \to \pi_1(Y, f(p))$ is a normal subgroup of $\pi_1(Y, f(p))$.

(b) Let $Y \cong S^1 \vee S^1$ be a figure 8, that is, the one point join of two circles. Draw a normal 3-sheeted covering space of Y, and a non-normal three-sheeted covering space of Y.

QUALIFYING EXAMINATION

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Thursday 22 September 2005 (Day 3)

1. Let $f : \mathbb{CP}^m \to \mathbb{CP}^n$ be any continuous map between complex projective spaces of dimensions m and n.

(a) If $m > n$, show that the induced map $f_* : H_k(\mathbb{CP}^m, \mathbb{Z}) \to H_k(\mathbb{CP}^n, \mathbb{Z})$ is zero for all $k > 0$.

(b) If $m = n$, the induced map $f_* : H_{2m}(\mathbb{CP}^m, \mathbb{Z}) \cong \mathbb{Z} \to H_{2m}(\mathbb{CP}^m, \mathbb{Z}) \cong \mathbb{Z}$ is multiplication by some integer d, called the *degree* of the map f. What integers d occur as degrees of continuous maps $f : \mathbb{CP}^m \to \mathbb{CP}^m$? Justify your answer.

2. Let a_1, $a_2, \ldots$, a_n be complex numbers. Prove there exists a real $x \in [0, 1]$ such that

$$\left| 1 - \sum_{k=1}^{n} a_k e^{2\pi i k x} \right| \geq 1.$$

3. Suppose that ∇ is a connection on a Riemannian manifold M. Define the torsion tensor τ via

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

where X, Y are vector fields on M. ∇ is called symmetric if the torsion tensor vanishes. Show that ∇ is symmetric if and only if the Christoffel symbols with respect to any coordinate frame are symmetric, i.e. $\Gamma^k_{ij} = \Gamma^k_{ji}$. Remember that if $\{E_i\}$ is a coordinate frame, and ∇ is a connection, the Christoffel symbols are defined via

$$\nabla_{E_i} E_j = \sum_k \Gamma^k_{ij} E_k.$$

4. Recall that a commutative ring is called *Artinian* if every strictly descending chain of ideals is finite. Let A be a commutative Artinian ring.

(a) Show that any quotient of A is Artinian.

(b) Show that any prime ideal in A is maximal.

(c) Show that A has only finitely many prime ideals.

5. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d > 1$, and let $\Lambda \cong \mathbb{P}^k \subset X$ a k-dimensional linear subspace of $\mathbb{P}^n$ contained in X. Show that $k \leq (n-1)/2$.

6. Let $l^\infty(\mathbb{R})$ denote the space of bounded real sequences $\{x_n\}$, $n = 1, 2, \dots$. Show that there exists a continuous linear functional $L \in l^\infty(\mathbb{R})^*$ with the following properties:

a) $\inf x_n \leq L(\{x_n\}) \leq \sup x_n$,
b) If $\lim_{n\to\infty} x_n = a$ then $L(\{x_n\}) = a$,
c) $L(\{x_n\}) = L(\{x_{n+1}\})$.

Hint: Consider subspace $V \subset l^\infty(\mathbb{R})$ generated by sequences $\{x_{n+1} - x_n\}$. Show that $\{1, 1, \dots\} \notin \bar{V}$ and apply Hahn-Banach.