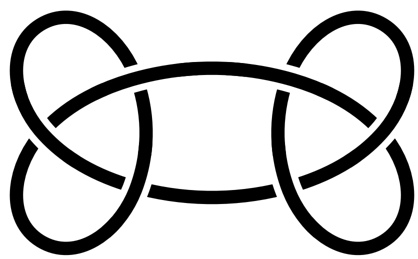
Slice Knots
and the
Concordance Group



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Contents

1. Introduction

1.1. Embeddings of D^2 and the Whitney Trick

The Whitney embedding theorem states that a smooth n-manifold M may be smoothly embedded into the euclidean space $\mathbb{R}^{2n}$, demonstrating that the "intrinsic" view of manifolds in terms of charts coincides with the "extrinsic" one of manifolds residing in finite-dimensional euclidean spaces. The construction of such of embedding is a subtle task: to begin, one finds an immersion $f : M \looparrowright \mathbb{R}^{2n}$ whose self-intersections are all transverse; the existence of such an immersion is a consequence of Whitney's immersion theorem, and that the double points may be be assumed transverse follows from a general position argument for manifolds of complementary dimension.

The method for converting such an immersion into an embedding proper relies on the *Whitney trick*. Suppose that M and N are submanifolds with complementary dimension of a simply-connected manifold P, and that M and N intersect transversely. The double points of the intersections of these manifolds are isolated points. Given double points $m \in M$ and $n \in N$ of opposite intersection sign, one may find paths α and β in M and N which join m and n. One then attempts to construct a "Whitney disk", an embedding $D^2 \hookrightarrow P$ whose boundary is $\alpha \cup -\beta$. If such a disk exists, one may then construct an isotopy which slides M along the embedded disk and results in the removal of the double points m and n. By pairing up double points in this manner, and introducing new ones by isotopies as necessary, a transverse immersion may be converted into an embedding. The trick is illustrated below for submanifolds of dimension 1 and 2 in $\mathbb{R}^3$, though in general the Whitney trick may not be applied in three dimensions because of the impossibility of finding an embedded Whitney disk.

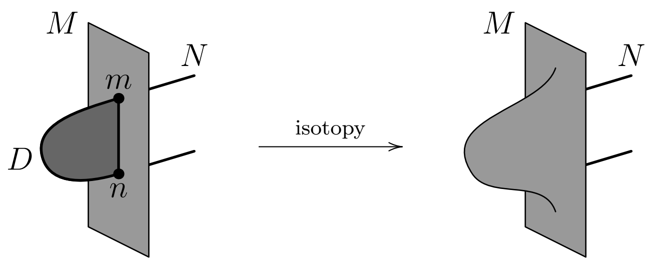

Figure 1.1: The Whitney trick in dimension 3 [9]

It may be proved that so long as the codimension of either M or N in P is at least 3, there exists a Whitney disk as required, and double points of opposite sign may be paired up and removed by isotopies [9]. If the dimension of the ambient manifold P is at least 5, then this will always be the case. This procedure lies at the heart of the proofs of not only the Whitney embedding theorem, but important classification results on manifolds in dimension at least 5. One such is the h-cobordism theorem, which plays a crucial role in the proof of results including the Poincaré conjecture in dimension $n \geq 5$.

Theorem 1.1 ([27]). *Suppose that M and N are manifolds of dimension $n \geq 5$, and that W is a compact cobordism of M and N. If $M \hookrightarrow W$ and $N \hookrightarrow W$ are homotopy equivalences, and M and N are simply connected, then M and N are diffeomorphic and $W \cong M \times [0,1]$.*

It is a result of Freedman that the h-cobordism theorem holds in dimension 4 if one works in the topological category, but the result is nonetheless false for smooth manifolds in this dimension. The investigations of Wall revealed the extent to which the methods used to prove the h-cobordism theorem remain valid in dimension 4, and provide an important starting point for much of the theory of 4-manifolds [9].

1.2. Ribbon Knots and Slice Knots

The failure of the Whitney trick in dimension 4 is due to the difficulty in finding embedded disks with given curves in a 4-manifold as their boundaries. A simpler problem might be to understand whether, given a knot $K \subset S^3$, there exists a disk Δ embedded in D^4 with K as its boundary. Certainly any knot bounds an immersed disk, but the Whitney trick may not in general be used to cancel the self-intersections of such a disk. The knots which bound such disks are called slice knots, and the detection of slice knots will be the focus of this thesis.

Definition 1. A knot $K \subset S^3$ is called a *slice knot* if there exists an embedded disk Δ in the interior of D^4 whose boundary is K. In this situation, Δ is called a *slice disk* for K.

Although only the unknot bounds a nonsingular disk $D^2 \to S^3$, all knots bound singular disks in S^3. A knot is termed a *ribbon knot* if its singularities are of a certain well-controlled type.

Definition 2. A knot K is a *ribbon knot* if it bounds a singular disk $f : D^2 \to S^3$ such that each component of the self-intersection of $f(D^2)$ is an arc, one of whose preimages lies in the interior of D^2.

The following diagram illustrates a typical ribbon knot. The importance of ribbon knots lies in their close relation to slice knots. Every ribbon knot is a slice knot: by sliding each singular arc $A \subset S^3$ into the interior of D^4 one obtains a slice disk for any ribbon knot K. A simple way to form a ribbon knot is by adding any knot K to its reverse rK. The resulting knot admits a ribbon disk, as illustrated in Figure 1.2.

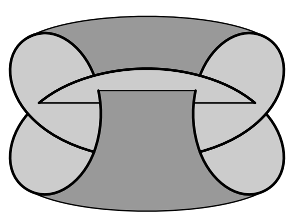

Figure 1.2: A ribbon disk

A longstanding conjecture is that slice knots and ribbon knots are really the same thing.

Conjecture 1. *Every slice knot is a ribbon knot.*

A variety of obstructions to a knot being ribbon have been developed, but the conjecture remains open in the general case. Nonetheless, the conjecture has been settled in the positive for several classes of knots: Lisca recently proved that a 2-bridge knot is ribbon if and only if it is slice using results of Donaldson on intersection forms on 4-manifolds [24], and Greene and Jabuka have announced a proof of the conjecture for 3-strand pretzel knots [15].

1.3. Slice Knots and 4-Manifolds

Slice knots first appeared in early work of Artin, but their investigation was initiated through the work of Kervaire and Milnor in the early 1960s [21]. An application of the Whitney trick demonstrates that any homotopy class in $\pi_n(M^{2n})$ is represented by a smooth embedding $S^n \to M^{2n}$, as long as $n \geq 3$. It was demonstrated by Kervaire and Milnor that the analogous result is not true in the case $n = 2$: there exist homology classes of certain 4-manifolds which are not represented by any smoothly embedded spheres. The main result in this direction is the following.

Theorem 1.2 ([21])**.** *Suppose that $\alpha \in H^2(M;\mathbb{Z})$ is dual to the Stiefel-Whitney class $w_2(M)$. If α is represented by a smoothly embedded sphere, then $[\alpha]^2 \equiv \sigma(M) \bmod 16$.*

The proof relies on the following deep result of Rohlin.

Proposition 1.3 ([21])**.** *If $w_2(M) = 0$, then $\sigma(M) \equiv 0 \bmod 16$.*

To illustrate Theorem 1.2, let $\mathbb{CP}^2$ be the complex projective plane, with coordinates $[X, Y, Z]$, and let α be a generator of $H_2(\mathbb{CP}^2;\mathbb{Z})$. Consider the class $\xi = 3\alpha$. Then $[\xi]^2 = 9$, and since $w_2(\mathbb{CP}^2) \neq 0$, the class ξ is dual to the Steifel-Whitney class $w_2(M)$. Since $\sigma(\mathbb{CP}^2) = 1$, it follows that $[\xi]^2 - \sigma(M) \equiv 8 \bmod 16$ and so this class is not represented by a smoothly embedded sphere.

However, the cuspidal cubic $C = \{[X, Y, Z] : Y^3 - X^2Z = 0\}$ is homeomorphic to $\mathbb{CP}^1 \cong S^2$. Since it is of degree 3 it represents the class ξ, and so by the above, does not admit a smooth representative. Consider a small sphere S^3 centered at the singular point $[0, 0, 1]$. One may check that $C \cap S^3$ is a right-handed trefoil in S^3. If the trefoil were a slice knot, then joining the component of C on the exterior of S^3 to a slice disk Δ would yield a homologous smooth representative of the class ξ, a contradiction: thus the trefoil does not admit a slice disk. In this manner, Milnor and Kervaire demonstrated that the trefoil is not a slice knot. In Section 4 we will encounter a simpler proof of this fact using the signature invariant.

1.4. The Concordance Group

A bit of care must be taken in defining these basic notions. We understand a knot K to be a smooth, oriented submanifold of S^3 which is diffeomorphic to S^1. Two knots K_1 and K_2 are called *smoothly concordant* if there is a smooth embedding $S^1 \times [0, 1] \to S^3 \times [0, 1]$ whose boundary is $K_1 \times \{0\} \amalg -K_2 \times \{1\}$. A knot is then termed *smoothly slice* if it is concordant to the unknot; it is easy to see that a knot K is smoothly slice if and only if there is a smoothly embedded disk $\Delta \subset \mathring{D}^4$ with $\partial\Delta = K$.

Although most classical work on the concordance group dealt with smooth concordance and we will work primarily in this setting, many of the results described here may be extended to the topological category. A knot is said to be *topologically slice* if it is the boundary of a locally flat disk $\Delta \subset D^4$; here this means that the embedding $\Delta \to D^4$ extends to a smooth embedding $\Delta \times D^2 \to D^4$. A smoothly slice knot is of course topologically slice, but the reverse implication does not hold in general [23].

It is evident that the relation of concordance (either smooth or topological) defines an equivalence relation on the set of knots. Moreover, there is an operation of connected sum of knots which gives the knots modulo the relation of concordance a group structure. Given two knots $K_1, K_2 \subset S^3$, define the *connected sum* $K_1 \# K_2$ by removing balls $(D_1^3, D_1^1) \subset (S^3, K_1)$ and $(D_2^3, D_2^1) \subset (S^3, K^2)$, and identifying the resulting boundary pairs by a diffeomorphism $h : (D_2^3, D_2^1) \to (D_1^3, D_1^1)$ which is orientation-reversing on (D_2^3, D_2^1). Since $S^3 \# S^3 \cong S^3$, the resulting space is a pair (S^3, K), and the knot $K = K_1 \# K_2$ is the called the connected sum of K_1 and K_2. This procedure is illustrated below for the connected sum of two trefoils.

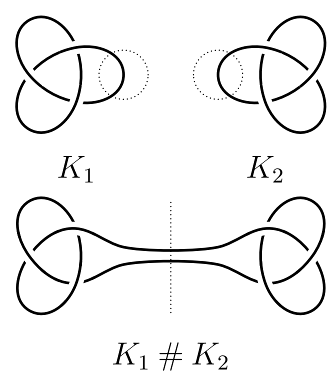

Figure 1.3: Connected sum in S^3

To see that the knots in S^3 modulo concordance form a group with respect to this operation, it is necessary to exhibit an identity element and inverses. Observe that the operation of connected sum is clearly associative of knots and well-defined on classes, and that $K \# U = K$, where U is the unknot. Thus $[U]$, the concordance class of slice knots, serves as he identity element. Since the connected sum of a knot K with its reverse rK is a ribbon knot and thus a slice knot, it follows $K \# rK$ is concordant to the identity element. The disk of Figure 1.2 is a ribbon disk for the connected sum of a trefoil and its reverse illustrated in Figure 1.3. In general, a ribbon disk for $K \# rK$ may be obtained as the union of the segments linking points in K to their reflections in $K \# rK$. Thus the set of concordance classes of knots with operation of connected sum is in fact a group, called the *concordance group* and denoted $\mathcal{C}_1^3$.

1.5. Historical Background

The systematic investigation of the concordance group was initiated by Fox and Milnor [10]. Applications of commutative knot invariants in work by Levine and Kervaire led to a complete understanding of the structure of the analogous concordance group $\mathcal{C}_n^{n+2}$ of knotted spheres S^n in S^{n+2}, for dimensions $n \geq 2$ [23],[20]. J. Levine's work demonstrated the existence in the classical case $n = 1$ of a surjective homomorphism $\phi : \mathcal{C}_1^3 \to \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$ [23]. The knots in the kernel of this homomorphism are the *algebraically slice knots*, whose failure to be slice can not be determined from certain algebraic data provided by the Seifert form. It was demonstrated by Casson and Gordon that there exist knots which are algebraically slice but not smoothly slice [4],[5], and subsequent work by Gilmer, Letsche and others elaborated on the techniques of Casson and Gordon to give a number of finer obstructions to algebraically slice knots being slice and computational tools for the Casson-Gordon invariants. The major breakthrough after the work of Casson-Gordon was the construction of new sliceness obstructions by Cochran, Orr, and Teichner. These obstructions yield an infinite, nontrivial filtration

$$\cdots \subset \mathcal{F}_{2.0} \subset \mathcal{F}_{1.5} \subset \mathcal{F}_{1.0} \subset \mathcal{F}_{0.5} \subset \mathcal{F}_0 \subset \mathcal{C}_1^3$$

in both the smooth and topological categories. Previously known sliceness obstructions fit conveniently into this filtration: the algebraically slice knots are exactly those in $\mathcal{F}_{0.5}$, while every knot in $\mathcal{F}_{1.5}$ has vanishing Casson-Gordon invariants [7]. More recent work has demonstrated that each quotient $\mathcal{F}_n/\mathcal{F}_{n.5}$ is of infinite rank [6].

The initial work on the concordance group is contained in the papers of Fox and Milnor [10], Kervaire [20], and Levine [23]. Some of the sliceness obstructions elucidated by this early work arise as consequences of more sophisticated invariants, as is the case of Theorem 3.3 here. The important work of Casson and Gordon on slice knots was introduced in [4] and further developed in [5]. The former paper is more algebraic in nature, and the main computation of this thesis is closer in spirit to [5]. There followed a variety of elaborations on the Casson-Gordon theme, providing computational techniques for wider classes of knots, an excellent reference for which is the survey of Livingston [26]. References for the recent work of Cochran, Teichner, and Orr include their original paper, as well as others published by various of the authors [6],[7],[30]. This thesis begins with a discussion of some elementary knot theory, branched covers of S^3, and signature invariants, drawing from the exposition of these topics by Rolfsen [29] and Kauffman [19]. We then outline the construction of the Casson-Gordon invariant as developed in [4] and [5] and discussed in [19], culminating in a computation in the spirit of [5] simplified by the employment of constructions first developed for more recent work on the Cochran-Orr-Teichner filtration [6]. The class of knots to which this computation applies has previously been the subject of considerable attention [25],[26].

1.6. Organization

Section 2 introduces Seifert surfaces and the Seifert pairing, objects associated to a knot which give rise to the most basic sliceness obstructions. Among these is the condition of algebraic sliceness, which is a complete invariant for detecting concordance classes in higher-dimension knots, but whose vanishing is no longer a sufficient invariant in the classical dimension $n = 1$. The rest of the discussion is devoted to constructing examples of knots which are algebraically slice but not slice. Section 3 introduces a variety of 3- and 4-manifolds associated to a knot K as covering spaces, and discusses the applications of these spaces to the computation of algebraic concordance classes. Section 4 introduces a variety of signature invariants of knots and 4-manifolds, whose computation is of crucial importance in the description of non-slice knots whose failure to be slice is not detected by the abelian invariants of the preceding sections. In particular, we present an elementary proof of the Atiyah-Singer

G-signature theorem for 4-manifolds with cyclic actions due to Gordon [14]. Section 5 introduces the methods of Casson and Gordon based on the signatures of metabelian covering spaces associated to a knot, and adapts these methods to give a proof that certain knots are not slice, making use of computations on a cobordism employed in the work of Cochran, Harvey, and Leidy [6]. At last, Section 6 provides a short overview of more recent sliceness obstructions and the filtration of the concordance group constructed by Cochran, Orr, and Teichner.

2. Algebraically Slice Knots

2.1. Seifert Surfaces and the Seifert Pairing

Given a knot K, a connected, compact, bicollared surface $F \subset S^3$ with $\partial F = K$ is called a *Seifert surface* for F. Observe that the requirement of bicollaredness implies that F is orientable, and admits a unique orientation which restricts to that of K. The following fact is standard and can be proved by a direct construction. The interested reader is referred to Rolfsen's text for details [29].

Proposition 2.1. *Every knot $K \subset S^3$ admits a Seifert surface.*

The Seifert surface of a knot is not unique; given any Seifert surface a new one may be obtained by, for example, removing two disks D^2 and adding a handle between the resulting boundary components. Proposition 2.2 describes the extent to which the Seifert surface of K fails to be unique, thereby making possible knot invariants defined in terms of a Seifert surface which do not depend on the choice of this surface.

Observe that the Seifert surface of K is a compact 2-dimensional manifold with boundary diffeomorphic to S^1. Attaching a disk along this boundary yields a 2-dimensional manifold without boundary; thus F is homeomorphic to a surface of some genus g with a disk removed. The *genus* of K is defined to be the minimum of this g over all choices of Seifert surface F for K.

The simplest nontrivial knot is the trefoil, whose properties will play a crucial role in a later construction. As indicated in the introduction, the trefoil is not a slice knot, a fact which will eventually be proved using the signature invariant.

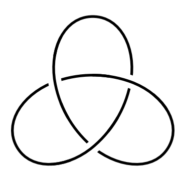

Figure 2.1: The right-handed trefoil.

Observe that the trefoil is either right- or left-handed, depending on the choice of orientation: the left-handed trefoil appears as the mirror image of Figure 2.1. An invariant which we will encounter in Section 4, the *signature*, provides the means to distinguish these two orientations.

Like any knot, the trefoil admits a Seifert surface, illustrated in Figure 2.2. Its boundary is easily verified to be equivalent to the trefoil pictured in Figure 2.1. Gluing a disc D^2 along the boundary of this Seifert surface, we obtain a torus, and so the genus of the Seifert surface for the trefoil given in the preceding illustration is 1. Since the trefoil is nontrivial, it admits no genus 0 Seifert surface and so is in fact of genus 1.

It follows from the preceding observations that the group $H_1(F;\mathbb{Z})$ of a Seifert surface F is free on $2g$ generators $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$. Define the *Seifert pairing* $\theta : H_1(F;\mathbb{Z}) \to$

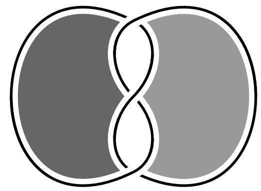

Figure 2.2: A Seifert surface for the trefoil.

$H_1(F;\mathbb{Z}) \to \mathbb{Z}$ by $\theta(\alpha,\beta) = \mathrm{lk}(\alpha,\beta_+)$, where β_+ is obtained from β by pushing this curve in the positive direction along the normal field of F into a bicollar neighborhood.

A quick sketch reveals that the right-handed trefoil has the Seifert matrix (with respect to the Seifert surface of Figure 2.2)

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

As remarked earlier, the Seifert surface is not itself a knot invariant. However, any two Seifert surfaces for a knot may satisfy an equivalence relation termed *S-equivalence*. Given a Seifert surface F for K, one can remove two disks D_1 and D_2 and glue in a cylinder $S^1 \times I$ along their boundaries to obtain a new Seifert surface. This operation is termed *1-surgery*. To invert this, one may pick a curve on F that bounds a disk in $S^3 \setminus F$, cut out $\alpha \times I$, and cap the ends with disks. This operation is called *0-surgery*. If F and F' are two Seifert surfaces, and F' may be obtained from F through some combination of these operations, then F and F' are called S-equivalent.

Proposition 2.2 ([19])**.** *Let F and F' be two Seifert surfaces for K. Then F and F' are S-equivalent.*

There is a simple relation between the Seifert matrices associated to these surfaces. Suppose that F' is obtained from F be a 1-surgery. Choosing new basis vectors α and β in $H_1(F';\mathbb{Z})$ which are a meridian and longitude of the connecting bridge, we obtain

$$A' = \left(\begin{array}{c|cc} A & 0 & 0 \\ \hline 0 & 0 & 1 \\ C & 0 & 0 \end{array}\right).$$

The operation of 0-surgery corresponds to removing a block of this form with respect to some choice of basis. That any two Seifert surfaces are related in this manner makes it possible to demonstrate that a knot invariant defined by some property of a Seifert surface F, and whose value is invariant under the operations of 0- and 1-surgery, is a well-defined invariant of the knot.

2.2. A Few Facts about Slice Knots

The information captured by the Seifert surface is useful in characterizing the slice knots. One of the most elementary applications of this fact is Theorem 2.3, which gives a means of converting a Seifert surface of a genus 1 knot into a slice disk through surgery along an embedded curve.

Theorem 2.3. *[19] Suppose that $\alpha \in S^3$ is a curve in a surface which represents a nontrivial class in $H_1(F;\mathbb{Z})$. Suppose that $\theta(\alpha,\alpha) = 0$ and that $\alpha \subset S^3$ is a slice knot. Then there is a surface $F' \subset D^4$ such that $\partial F' = \partial F$ and $g(F') = g(F) - 1$.*

Proof. Since $\theta(\alpha,\alpha) = 0$, there is an embedding $\alpha \times [0,1] \to F$. Cut out an annulus in F between $\alpha \times \{0\}$ and $\alpha \times \{1\}$. Let Δ and Δ' be slice disks for $\alpha \times \{0\}$ and $\alpha \times \{1\}$. Glue these along their boundaries to the boundary of the removed annulus and let F' be the resulting surface. Then α is homologically trivial in F' and so $g(F') = g(F) - 1$, and F' is the desired surface. □

Corollary 2.4. *Suppose that K is a knot of genus 1. If there exists a homologically nontrivial curve $\alpha \in F$ with self-linking 0 for some Seifert surface of K such that α is slice, then K is slice.*

Proof. The surface F' of the proposition is of genus 0 and thus furnishes a slice disk for K. □

Important to our investigations will be the knot 9_{46} illustrated in Figure 2.3, which was central to the constructions of Cochran, Harvey, and Leidy [6]. It is clear that 9_{46} is a slice knot as a consequence of Corollary 2.4: a path α around one of its bands is the unknot, and performing surgery along a slice disk for this α in the manner described yields a slice disk for 9_{46}.

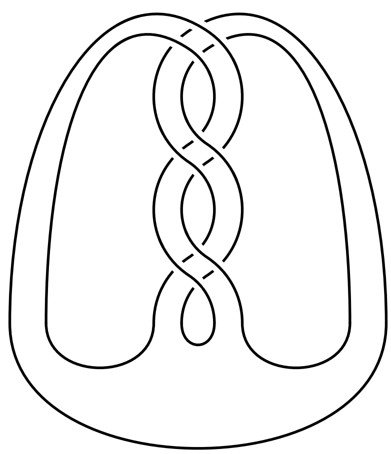

Figure 2.3: The knot 9_{46}.

The Seifert form for 9_{46} with respect to the obvious Seifert surface is

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}.$$

One may more generally verify the following.

Proposition 2.5 ([19])**.** *Suppose that $\alpha_1, \ldots, \alpha_g$ span a half-rank submodule of $H_1(F;\mathbb{Z})$ on which θ vanishes, and that each α_i is slice. Then K is slice.*

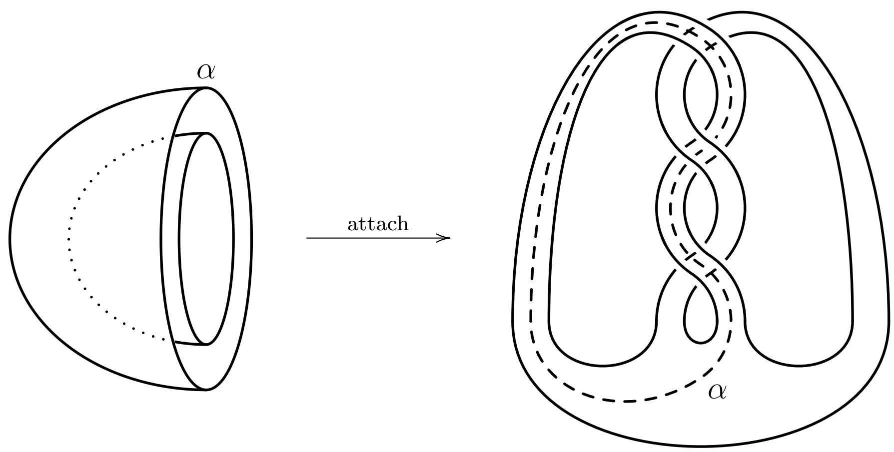

Figure 2.4: Surgery along a slice homology element in a Seifert surface for 9_{46}

We will see later that every slice knot admits a basis for $H_1(F;\mathbb{Z})$ on which the Seifert pairing vanishes, but it is not in general possible to arrange for representative curves of this basis to be themselves slice.

2.3. Algebraically Slice Knots

The result of Proposition 2.5 centered on knots possessing Seifert pairing which vanishes on a half-rank submodule of $H_1(F;\mathbb{Z})$. We will see soon that in fact every slice knot has this property, and it is natural to ask whether this algebraic property in fact completely characterizes the slice knots. The following definition encapsulates this intuition.

Definition 3. A knot K is called *algebraically slice* if for some Seifert surface F of K, the Seifert pairing $\theta : H_1(F;\mathbb{Z}) \times H_1(F;\mathbb{Z}) \to \mathbb{Z}$ vanishes on a submodule with half the rank of $H_1(F;\mathbb{Z})$.

The important result on algebraically slice knots is that all slice knots are algebraically slice, and so the easily computable obstruction of algebraic sliceness provides a means to show that many knots are not slice. This fact will follow from a more general lemma on 3-manifolds with boundary.

Lemma 2.6 ([19])**.** *Let M be a connected, oriented, 3-manifold with boundary, and let the inclusion $i : \partial M \to M$ induce $i_* : H_1(\partial M;\mathbb{Q}) \to H_1(M;\mathbb{Q})$. Then*

$$\dim \ker(i_*) = \frac{1}{2} \dim H_1(\partial M;\mathbb{Q}).$$

Proof. It follows from Poincaré-Lefschetz duality that

$$1 = \dim H_3(M,\partial M;\mathbb{Q}) = \dim H_0(M;\mathbb{Q}) \qquad a = \dim H_2(M,\partial M;\mathbb{Q}) = \dim H_1(M;\mathbb{Q})$$
$$b = \dim H_1(M,\partial M;\mathbb{Q}) = \dim H_2(M;\mathbb{Q}) \qquad c = \dim H_2(\partial M;\mathbb{Q}) = \dim H_0(\partial M;\mathbb{Q})$$

Then a dimension count on the homology long exact sequence sequence for the pair $(M, \partial M)$ yields

$$
\begin{aligned}
0 &= 1 - c + b - a + \dim H_1(\partial M; \mathbb{Q}) - a + b - c + 1 \\
\dim H_1(\partial M; \mathbb{Q}) &= 2(a - b + c - 1).
\end{aligned}
$$

The kernel itself has dimension

$$\dim \ker(i_*) = \dim H_1(\partial M; \mathbb{Q}) - a + b - c + 1,$$

whence $\dim \ker(i_*) = \frac{1}{2} \dim H_1(\partial M; \mathbb{Q})$, as claimed. □

The most important property of algebraic sliceness follows as a consequence.

Theorem 2.7. *Every slice knot is algebraically slice.*

Proof. From the proof of S-equivalence of Seifert surfaces for a knot K, one constructs a 3-manifold $M \subset D^4$ with $\partial M = D \cup F$, where D is a slice disk for $K \subset S^3 = \partial D^4$ and F is a Seifert surface [29].

Let $V = \ker(H_1(\partial M; \mathbb{Q}) \to H_1(M; \mathbb{Q})$. It follows from the preceding that $\dim V = \frac{1}{2} \dim H_1(\partial M; \mathbb{Q})$. But $H_1(\partial M; \mathbb{Q}) \cong H_1(D; \mathbb{Q}) \oplus H_1(F; \mathbb{Q}) \cong H_1(F; \mathbb{Q})$, so $\dim V = \frac{1}{2} \dim H_1(F; \mathbb{Q}) = g$. Let $W \in H_1(F; \mathbb{Z})$ be the associated integral submodule.

To see that the Seifert pairing θ vanishes on W, observe that there are surfaces $A, B \subset M$ with $\partial A = \alpha$ and $\partial B = \beta$. The operation $x \mapsto x_+$ shifting along the positive normal to F extends to $F \to D^4 \setminus F$, and $\partial A_+ = \alpha_+$. Then $\theta(\alpha, \beta) = \operatorname{lk}(\alpha_+, \beta) = A_+ \cdot B = 0$, proving the claim. □

The condition of algebraic sliceness is equivalent to sliceness for knots of odd dimension $n \geq 3$, and all even-dimensional knots are slice [20]. It was not known until the work of Casson and Gordon in the late 1970s whether this was also true in the classical case $n = 1$. The first counterexample constructed by Casson and Gordon demonstrated that certain doubles of the unknot were algebraically slice but not slice [4].

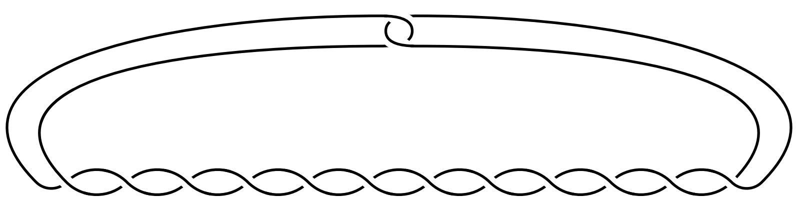

Figure 2.5: A knot which is algebraically slice but not slice

2.4. Infection along Curves

One method to produce knots which are candidates to be algebraically slice but not slice is to tie non-slice knots into the bands of Seifert surfaces for known slice knots. This procedure is termed *infection* and has provided numerous examples in the study of the concordance group [26].

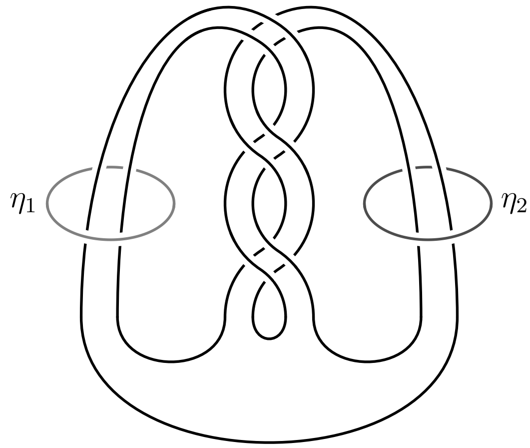

Figure 2.6: Paths along which 9_{46} is infected to yield $J(K)$

The intuition for the construction of the infection of a knot K by a knot L along a curve η is as follows. Suppose that η is a curve in S^3 disjoint from K and with $\mathrm{lk}(K, \eta) = 0$, bounding a disk that meets K transversely. Cut K along this disk bounded by η. Tie the cut strands of L into the knot K and reglue them with no twisting, in the sense that a representative for a homology class laying on a band of the Seifert surface retains the same self-linking number after surgery. The resulting knot is called the infection of K by L along η. A more precise and often useful description of the infection construction is by means of a surgery maneuver.

Definition 4. Let K be a knot and η be an unknotted curve in S^3 disjoint from K. Let L be the infecting knot. Take U to be a tubular neighborhood of η and T a tubular neighborhood of L. Form the space $S^3 \setminus U \amalg S^3 \setminus T$ and identify the two components along their boundaries such that a longitude of η is identified with a meridian of K, and a meridian of η with the reverse of a longitude of K. The resulting space is homeomorphic to S^3, and the image of K is the infected of K by L along η.

That this quotient space is another sphere S^3 is clear since S^3 is unknotted, and so $S^3 \setminus U$ is itself a solid torus, which is then glued along the boundary of the removed knot L. We will occasionally speak of performing infection along two curves η_1 and η_2 by knots L_1 and L_2. This is simply the result of performing these two constructions in succession; the order in which these are performed does not alter the result.

A good way to get algebraically slice knots is by infection of the knot 9_{46} by a knot K along the loops η_1 and η_2 illustrated in Figure 2.3. The resulting knot is then denoted $J(K)$. In the particular case that K is trivial, we obtain $J(U) = 9_{46}$. The usefulness of the infection construction in forming knots which are slice but not algebraically slice lies in the following observations.

Lemma 2.8. *Let $J(K)$ denote the infection of 9_{46} along each of the loops η_1, η_2 by the knot K. Then*

1. *$J(K)$ has the same Seifert form as 9_{46}.*

2. *$J(K)$ is slice if K is slice.*

3. *$J(K)$ is algebraically slice for any K.*

Proof. The first point follows from the observation that $J(K)$ has a Seifert surface given by tying K into one the two bands of the Seifert surface for 9_{46}. That the linking numbers of distinct homology generators in this surface coincide with those of 9_{46} is clear, and that fact that the self-linking numbers of the homology generators for $J(K)$ coincide with those of 9_{46} is a consequence of the fact that no twisting is introduced into this band during the infection procedure. The third point is an immediate consequence of this observation, as the condition of algebraic sliceness is one of the Seifert form. At last, the second point follows from Theorem 2.3 and the fact that the homology generators of $H_1(F;\mathbb{Z})$ have zero self-linking. □

It should be noted that all of the results of Lemma 2.8 hold even if only one of the bands of 9_{46} is infected by K. However, in this case, the result is slice, no matter the knot K: this follows from Lemma 2.4 applied to a homology generator for the non-infected band. Since we seek to construct algebraically slice knots which are not slice, consideration of $J(K)$ will be more fruitful.

3. Cyclic Coverings and Abelian Knot Invariants

3.1. Cyclic Coverings

To a knot K are associated several branched and unbranched covering spaces of S^3 and related spaces, which are introduced in this section. When K is a slice knot, these covering spaces bound covers of D^4, branched over a slice disk. We will see that the signatures of these 4-manifolds associated to a slice knots are particularly well-behaved, a fact which will be exploited in Section 5 to prove that certain knots are not slice.

Associated to the complement $S^3 \setminus K$ of a knot K in S^3 are cyclic covering spaces U_K^k of order k, as well as an infinite cyclic cover U_K^∞. These are the covering spaces associated to the map $\pi_1(S^3 \setminus K) \to H_1(S^3 \setminus K) \to C_k$. The cyclic covers of the knot complement have an easy geometric description in terms of a choice of Seifert surface for K, described by Rolfsen [29].

Fix a Seifert surface $F \subset S^3$ for K, and let $N : \mathring{F} \times (-1,1) \to S^3$ be the interior of a bicollar neighborhood M of F, such that $\mathring{F} = N(\mathring{F} \times \{0\})$. Then define the following sets, which will be glued together to obtain the desired covering space.

$$\begin{aligned}
N &= N(\mathring{F} \times (-1,1)) \\
N^+ &= N(\mathring{F} \times (0,1)) \\
N^- &= N(\mathring{F} \times (-1,0)) \\
Y &= S^3 \setminus M \\
X &= S^3 \setminus K.
\end{aligned}$$

Then let $\widetilde{N} = \coprod_{i=-\infty}^{\infty} N_i$ and $\widetilde{Y} = \coprod_{i=-\infty}^{\infty} Y_i$ be countable disjoint unions of each of these sets. Define the space U_K^∞ by by identifying $N_i^+ \subset Y_i$ with $N_i^+ \subset N_i$, and $N_i^- \subset Y_i$ with $N_{i+1}^- \subset N_{i+1}$. The obvious map $\pi : U_K^\infty \to S^3 \setminus K$ is readily verified to be a regular covering map. This space admits a covering action $\tau : U_K^\infty \to U_K^\infty$ which sends Y_i and N_i to the corresponding points in Y_{i+1} and N_{i+1} respectively, and is the infinite cyclic covering of $S^3 \setminus K$. A similar construction then yields k-fold cyclic coverings U_K^k of $S^3 \setminus K$. These spaces are equivalently obtained either as the quotient of U_K^∞ by the action of τ^k, or by identifying k copies of Y_i and N_i along their intersections in a manner analogous to that above.

A map $f : X \to Y$ of manifolds of equal dimension is called a *branched covering* if there exist sets $B \subset X$ and $C \subset Y$ (called the *branch sets*) such that $f(B) = C$ and $f : X \setminus B \to Y \setminus C$ is an honest covering space, and $Y \setminus C$ is the full set of points which are evenly covered, and such that components of preimages of open sets in Y are a basis for the topology of X [29].

One may obtain from the construction of unbranched covers of $S^3 \setminus K$ corresponding cyclic branched covers of S^3, branched over K. Let T be a tubular neighborhood of K in S^3. Then ∂T is homeomorphic to a torus, and we see directly from the preceding construction that $\partial \widetilde{T} = \partial \pi^{-1}(T)$ is also homeomorphic to a torus. The meridian of T has a preimage which which is a single meridian wrapping around k times from the covering map, while the preimage of a longitude is k parallel longitudes on $\widetilde{T}$

Define L_K^k by gluing a solid torus $S^1 \times D^2$ so that a meridian is attached to the preimage of a meridian in $\partial\widetilde{T}$, and a longitude is glued along a longitude upstairs. This extends to a branched covering map $L_K^k \to S^3$ by sending $D^2 \times S^1$ to T with the identity on the S^1 component and $z \mapsto z^k/|z|^{k-1}$ on the D^2 component. This is branched over $\{0\} \times S^1$ and provides the desired branched covering. The space L_K^k is called the k-fold branched cover of S^3 branched over K. L_K^k is in fact the boundary of a cover V_K^k, the k-fold cover of D^4 branched over Δ, a slice disk for K. The space V_K^k will be constructed explicitly in Section 5 by similar means.

A third, related, set of cyclic covering spaces may be associated to a knot. These are the unbranched coverings of the result of 0-framed surgery along the knot.

Definition 5. Let K be a knot. We define M_K, the result of *nulhomologous* or 0-*framed* surgery on K to be the 3-manifold obtained by removing tubular neighborhood $T \cong S^1 \times D^2$ of K, and attaching $S^1 \times D^2$ along $\partial S^3 \setminus K$, such that a longitude of K is glued along a meridian, and the meridian of K along the opposite of a longitude.

The resulting M_K is a 3-manifold without boundary. There is another view of this construction. Let $S^3 = \partial D^4$, and attach a handle $D^2 \times D^2$ along the boundary by a map $f : D^2 \times S^1 \to S^3 = \partial D^4$ with image a tubular neighborhood of K. Consider the boundary of the resulting 4-manifold. It is S^3, but without $f(D^2 \times S^1)$, but with a copy of $S^1 \times D^2$ resulting from the gluing. Observe too that if K is a slice knot, then $M_K \cong D^4 \setminus \bar{\Delta}$ for $\bar{\Delta}$ a regular neighborhood of a slice disk Δ.

Note that $\pi_1(M_K) \cong \mathbb{Z}$ is generated by the image of a meridian of K in $H_1(S^3 \setminus K; \mathbb{Z})$, and that M_K has the homology of $S^1 \times S^2$. One may construct the k-fold cyclic, unbranched covering spaces associated to $\pi_1(M_K) \to C_k$. By construction, the inclusion-induced map $H_1(S^3 \setminus K) \to H_1(M_K)$ is an isomorphism, and then the k-fold unbranched cover of $S^3 \setminus K$ extends to a cover $M_K^k \to M_K$ by attaching preimages of the attached torus along the preimages of the boundary. The following theorem relates the homology groups of these spaces to those of the branched covers of (S^3, K).

Theorem 3.1 ([11])**.** *Let $\Lambda = \mathbb{C}[t, t^{-1}]$. There are isomorphisms*

$$\begin{aligned} H_1(L_K^k) &= H_1(M_K^\infty; \mathbb{Z})/(t^k - 1), \\ H_1(M_K^k) &= H_1(L_K^k) \oplus \mathbb{Z} = H_1(M_K^\infty, \mathbb{Z})/(t^k - 1) \oplus \mathbb{Z}. \end{aligned}$$

Here $H_1(M_K^\infty)$ is regarded with the Λ-module structure induced by the covering translation. There is a canonical identification of $H_1(M_K^\infty)$ with $H_1(M_K; \Lambda)$, where the latter is homology with local coefficients.

Proof. The first isomorphism is clear, since the first homology of M_K^k is that of L_K^k with a free component generated by the image of a meridian of K. To understand the second, observe that there is a short exact sequence of chain complexes

$$0 \longrightarrow C_* M_K^\infty \xrightarrow{t^k - 1} C_*(M_K^\infty) \xrightarrow{\pi_\sharp} C_*(M_K) \longrightarrow 0$$

This gives rise to a long exact sequence in homology

$$\cdots \longrightarrow H_i(M_K^\infty) \xrightarrow{t^k - 1} H_i(M_K^\infty) \longrightarrow H_i(M_K^k) \longrightarrow \cdots$$

At $i = 1$, we have $H_0(M_K^\infty) \cong \mathbb{Z}$ and so $H_1(M_K^k) \cong H_1(M_K^\infty)/(t^k - 1) \oplus \mathbb{Z}$. Then $H_1(M_K^k) \cong H_1(M_K^\infty)/(t^k - 1) \oplus \mathbb{Z}$. □

3.2. The Alexander Invariant

Let U_K^∞ denote the infinite cyclic cover of $S^3 \setminus K$, as previously constructed. Set $\Lambda = \mathbb{Z}[t, t^{-1}]$. Consider now the homology $H_*(U_K^\infty; \mathbb{Z})$. Let $\tau : U_K^\infty \to U_K^\infty$ be a generator of the group of covering translations. With this, one may define a Λ-action on $H_*(U_K^\infty; \mathbb{Z})$: given $p(t) \in \Lambda$ and $\alpha \in H_i(U_K^\infty)$, write $p(t) = \sum c_i t^i$ and set

$$p(t)\,\alpha = \sum_i c_i \tau_*^i \alpha.$$

This gives $H_1(U_K^\infty; \mathbb{Z})$ a Λ-module structure, and this is termed the *Alexander module.* This module is finitely presentable, and a presentation matrix is called the Alexander matrix: it may be verified that since Λ is a PID this matrix generates an ideal of relations, and we define the *Alexander polynonmial* $\Delta_K(t)$ to be a generator of the ideal of relations. The choice of a generator is defined only up to multiplication by units, i.e. factors of $\pm t^n$. We will follow the convention that the polynomial is fixed under $t \mapsto t^{-1}$ and has positive constant term.

In fact, the Alexander polynomial can be expressed in terms of the Seifert pairing.

Theorem 3.2 ([29])**.** *Let K be a knot with Seifert pairing matrix A with respect to some Seifert surface F. Then, up to multiplication by a unit,*

$$\Delta_K(t) = \det(A^T - tA).$$

Proof. Let F be a Seifert surface for K with generators $\alpha_1, \ldots, \alpha_{2g}$ of $H_1(F)$. Then $A_{ij} = \mathrm{lk}(\alpha_i, \alpha_j^+)$. Let $\beta_1, \ldots, \beta_{2g}$ be a basis for $H_1(S^3 \setminus F)$ dual to the α_i (so $\mathrm{lk}(\alpha_i, \beta_j) = \delta_{ij}$).

Now, $H_1(U_K^\infty)$ is generated by $\alpha_1, \ldots, \alpha_{2g}$ subject to $\alpha_i^- = t\alpha_i^+$. This implies

$$\sum_j \mathrm{lk}(a_i^-, a_j) = t \sum_j \mathrm{lk}(a_i^+, a_j)\alpha_j.$$

Because $\mathrm{lk}(a_i^-, a_j) = \mathrm{lk}(a_i, a_j^+) = a_{ij}$ this shows that $A^T - tA$ is a relation matrix for $H_1(U_K^\infty)$. This establishes the theorem. □

Theorem 2.7 now implies one of the first known sliceness obstructions, due to Fox and Milnor.

Theorem 3.3 (Fox-Milnor [10])**.** *The Alexander polynomial of a slice knot, normalized as in the definition, may be factored as $\Delta_K(t) = p(t)p(t^{-1})$ for some polynomial p.*

Proof. Extending a basis for the submodule constructed in Theorem 2.7 on which the pairing vanishes to a basis for $H_1(F; \mathbb{Z})$, we may assume that the Seifert matrix is of the form

$$A = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}.$$

Then

$$\Delta_K(t) = \det(A^T - tA) = \det \begin{pmatrix} 0 & C^T - tB \\ B^T - tC & D^T - tD \end{pmatrix} = \det(C^T - tB)\det(B^T - tC)$$
$$= (-t)^g \det(C^T - tB)\det(C^T - t^{-1}B).$$

This is the factorization required. □

Evaluating the Alexander at $t = -1$ in light of the factorization of Theorem 3.3 implies that the knot determinant $|\Delta(-1)| = |H_1(L_K^2)|$ of a slice knot is a perfect square. One may compute that the figure-eight knot has Alexander polynomial $\Delta_K(t) = -t^{-1} + 3 - t$, so $|\Delta_K(-1)| = 5$; thus the figure-eight knot is not slice. However, it is isotopic to its reverse, and so therefore represents a concordance class of order 2 in the concordance group.

In fact, the homology groups of the branched cyclic covers may be either finite or infinite, and may be computed explicitly in terms of the values of the Alexander polynomial on S^1. We state here a theorem relating this information: Section 5.2 will give some indication as to the origin of this result.

Proposition 3.4 ([13])**.**

$$\left|H_1(L_K^k)\right| = \left|\prod_{j=1}^{k} \Delta_K(e^{2\pi i j/k})\right|.$$

The vanishing of this product corresponds to $H_1(L_K^k)$ having a free summand.

3.3. The Torsion Linking and Blanchfield Pairings

Two pairings, the torsion linking pairing and Blanchfield pairings, on the homology of branched covers of a knot play a role in the detection of slice knots. The main result of Casson and Gordon is stated in terms of the vanishing of the torsion linking pairing on a certain subgroup of $H_1(L_K^k)$ for K a slice knot.

Definition 6. Suppose that k is an integer for which $H_1(L_K^k)$ is finite. There exists a pairing, the *torsion linking pairing*

$$\lambda_{L,k} : H_1(L_K^k) \times H_1(L_K^k) \to \mathbb{Q}/\mathbb{Z}.$$

Given α, β, find a 2-chain c such that $\partial c = n\beta$ and define

$$(\alpha, \beta) \mapsto \frac{1}{n}\alpha \cdot \beta,$$

In the case that k is a prime power, this defines a non-singular pairing on $H_1(L_K^k) \cong TH_1(M_K^k)$, where $TH_1(M_K^k)$ denotes the torsion part of this homology group. A submodule $P \subset H_1(L_K^k)$ is called a *metabolizer* of λ_k if $P = P^\perp$ with respect to this pairing. The following characterization is important in the work of Casson and Gordon. Recall that for a slice knot K, the manifold V_K^k is the k-fold cover of D^4 branched over a slice disk Δ for K.

Theorem 3.5 ([11])**.** *Suppose that K is slice and k is a prime power. Let $G = \ker i_* : H_1(L_K^k) \to H_1(V_K^k)$. Then $\lambda_{L,k}$ vanishes on $G \times G$.*

Proof. It follows from Lemma 3.7 that $\widetilde{H}_*(W_n;\mathbb{Q}) = 0$, so that $H_1(W_n;\mathbb{Z})$ is torsion. For any $b \in G$ there is a $c \in C_2(L_K^{(k)})$ such that $c = \partial b$, since c is 0 in $H_1(V_K^{(k)})$. Thus we can take $n = 1$ in the definition, and λ_L vanishes on G. $\square$

To define the Blanchfield pairing, let M_K^∞ denote the infinite cyclic cover of M_K, and consider the homology $H_1(M_K^\infty;\mathbb{Z}) \cong H_1(M_K;\Lambda)$. There is an involution on Λ which sends t to t^{-1}. Let $S = \{f \in \Lambda : f(1) = 1\}$. Then $H_1(M_K^\infty)$ is entirely S-torsion, since it is annihilated by the Alexander polynomial $\Delta_K(t) \in S$.

Then define the Blanchfield pairing by

$$\begin{aligned}\lambda_{Bl} : H_1(M_K;\Lambda) \times H_1(M_K;\Lambda) &\to S^{-1}\Lambda/\Lambda \\ (a,b) &\mapsto \frac{1}{q(t)}\sum_i (a \cdot t^i c)t^{-i}.\end{aligned}$$

Here c is a 2-chain chosen such that $\partial c = q(t)b$ for some $q(t) \in S$.

Given a submodule $P \subset H_1(M_K;\Lambda)$ define the orthogonal complement

$$P^\perp = \{v \in H_1(M_K;\Lambda) : \lambda_{Bl}(v,w) = 0 \text{ for all } w \in P\}.$$

A submodule $P \subset H_1(M;\Lambda)$ is said to be a metabolizer, and the Blanchfield pairing metabolic, if $P = P^\perp$. Characterizations such as the following have played a large role in recent approaches to the problem of detecting slice knots.

Theorem 3.6 ([22])**.** *The Blanchfield pairing λ_{Bl} is metabolic if and only if K is algebraically slice. In particular, if D is a slice disk for K, then*

$$P = \ker\{H_1(M_K;\Lambda) \to FH_1(D^4 \setminus \bar{\Delta};\Lambda)\},$$

where FH_1 is the free part of the homology, is a metabolizer.

Generalizations of this characterization lie at the heart of the methods of Cochran, Orr, and Teichner.

3.4. Bounds on the Homology of Covering Spaces

Here we prove some technical results on the homology of these various covering spaces that will be required in the implementation of the computations in the coming sections.

Lemma 3.7 ([5])**.** *Let Δ be a 2-disc in D^4 and let $V_K^{(k)}$ be the q^k-fold branched cover of (B^4, D) for some prime q. Then $\widetilde{H}_*(V_K^{(k)};\mathbb{Q}) = 0$.*

Proof. Let X_K^∞ be the infinite cyclic cover of $D^4 \setminus \Delta$ and $X_K^{(k)}$ the q^n-fold cyclic cover of $D^4 \setminus \Delta$, with the covering translation τ. The following is a short exact sequence of chain complexes.

$$0 \longrightarrow C_*(X_K^\infty) \xrightarrow{\tau_\sharp^{q^k}-1} C_*(X_K^\infty) \longrightarrow C_*X_K^{(k)} \longrightarrow 0$$

This gives rise to a long exact sequence in homology:

$$\cdots \longrightarrow H_i(X_K^\infty;\mathbb{Z}_q) \xrightarrow{\tau_*^{q^k}-1} H_i(X_K^\infty;\mathbb{Z}_q) \longrightarrow H_i(X_K^k;\mathbb{Z}_q) \longrightarrow H_{i-1}(X_K^\infty;\mathbb{Z}_q) \longrightarrow \cdots$$

Since $X_K^{(k)}$ and $V_K^{(k)}$ have identical homology in dimension 1, so $H_1(X_K^{(k)};\mathbb{Z}_q) \cong H_1(V_K^{(k)};\mathbb{Z}_q)$, the following sequence is exact in reduced homology:

$$\cdots \longrightarrow \widetilde{H}_i(X_K^\infty;\mathbb{Z}_q) \xrightarrow{\tau_*^{q^k}-1} \widetilde{H}_i(X_K^\infty;\mathbb{Z}_q) \longrightarrow \widetilde{H}_i(V_K^{(k)};\mathbb{Z}_q) \longrightarrow \widetilde{H}_{i-1}(X_K^\infty;\mathbb{Z}_q) \longrightarrow \cdots$$

Note that $\tau - 1$ is a homology isomorphism on $\widetilde{H}_*(X_K^\infty;\mathbb{Z})$. Thus with coefficients in $\mathbb{Z}_q$, the map $\tau^{q^n} - 1 = (\tau-1)^{q^n}$ is as well. So $\widetilde{H}_*(V_n;\mathbb{Z}_q) = 0$. Thus $\widetilde{H}_*(V_n;\mathbb{Q}) = 0$ as well; if this space had positive dimension, the integral homology would have a free summand, and $\widetilde{H}_*(V_n;\mathbb{Z}_q) = 0$ would not vanish. □

By much the same arguments we obtain $H_*(M_K^{(k)};\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$, where $M_K^{(k)}$ is the q^k-fold cyclic cover of S^3 branched over K for some prime q. As a result, if k is a prime power then the result of Theorem 3.1 may be made more explicit.

Theorem 3.8 ([11])**.** *Again let $M_K^{(k)}$ denote the q^k-fold cyclic cover of S^3 branched over K for some prime q. Then*

$$\begin{aligned} H_1(M_K^{(k)}) &= TH_1(M_K^{(k)}) \oplus \mathbb{Z} \\ TH_1(M_K^{(k)}) &= H_1(L_K^{(k)}) = H_1(M_K;\Lambda)/(t^k-1) \end{aligned}$$

Moreover, if Δ is a slice disk for K, and $V_K^{(k)}$ is the q^k-fold cover of D^4 branched over Δ,, then $H_1(V_K^{(k)}) \cong TH_1(V_K^{(k)}) \oplus \mathbb{Z} = H_1(V_K^{(k)}k;\Lambda)/(t^k-1) \oplus \mathbb{Z}$, and the kernel of the inclusion-induced map $TH_1(M_K^k) \to (\alpha,\beta) \mapsto \frac{1}{n}\alpha\cdot\beta$ is a metabolizer for the linking pairing.

Proof. The first two parts are immediate. Theorem 3.5 established the third in the case of slice knots, which is the most relevant case. The general proof is given by Friedl [11]. □

Lemma 3.9. *Let V be a rational homology 4-ball. Suppose that the image of the inclusion $H_1(\partial V) \to H_1(V)$ has order ℓ. Then $H_1(\partial V)$ has order ℓ^2.*

Proof. By Poincaré duality and universal coefficients, we have $H_2(\partial V;\mathbb{Z}) \cong H^1(\partial V;\mathbb{Z}) \cong \mathrm{Hom}(H_1(\partial V,\mathbb{Z}),\mathbb{Z}) = 0$. The sequence

$$0 \longrightarrow H_2(V;\mathbb{Z}) \longrightarrow H_2(V,\partial V;\mathbb{Z}) \longrightarrow H_1(\partial V) \longrightarrow H_1(V;\mathbb{Z}) \longrightarrow H_1(V,\partial V;\mathbb{Z}) \longrightarrow 0$$

is exact. Poincaré-Lefschetz duality implies that $a = |H_2(V;\mathbb{Z})| = |H_1(V,\partial V;\mathbb{Z})|$ and $b = |H_2(V,\partial V;\mathbb{Z})| = |H_1(V;\mathbb{Z})|$. Exactness on the left side gives that the image of $H_2(V,\partial V;\mathbb{Z})$ in $H_1(\partial V)$, and thus the kernel of $H_1(\partial V;\mathbb{Z}) \to H_1(V;\mathbb{Z})$ is order b/a. So too is the image of this map, by exactness on the right, whence $b/a = \ell$. Thus $|H_1(\partial V)| = (b/a)^2 = \ell^2$, as claimed. □

The following topological lemmas will also be employed. They provide a means of bounding the size of the second homology groups (and thus the signatures) of certain 4-manifolds which bound covers of M_K in the case that K is a slice knot. These bounds will be of central importance in Section 5.

Lemma 3.10 ([4]). *Suppose that $\widetilde{X}$ is a p-fold cover of X. If $H_*(X;\mathbb{Z}_p)$ is finite, then $H_*(\widetilde{X};\mathbb{Z}_p)$ is finite.*

Proof. The spectral sequence of the covering spaces has $E^2_{i,j} = H_i(C_p; H_j(\widetilde{X};\mathbb{Z}_p))$ and $\bigoplus_{i+j=k} E^\infty_{i,j} \cong H_k(X;\mathbb{Z}_p)$. Thus $E^\infty_{0,k}$ is finite for all k, as it is a summand of $\bigoplus_{i+j=k} E^\infty_{i,j}$ and $H_K(X;\mathbb{Z}_p)$ is assumed finite. For an induction, suppose that $H_j(\widetilde{X};\mathbb{Z}_p)$ is finite for $j < k$. It follows that $E^2_{i,j}$ is finite for $j < k$, and so $d^r : E^r_{r,k+1-r} \to E^r_{0,k}$ has finite rank if $r \geq 2$. Thus $E^2_{0,k} = H_0(C_p; H_k(\widetilde{X};\mathbb{Z}_p))$ is finite. The covering transformation τ of induces $\tau_* : H_k(\widetilde{X};\mathbb{Z}_p) \to H_k(\widetilde{X};\mathbb{Z}_p)$. Since $E^2_{0,k} = \operatorname{coker}(\tau_* - 1)$ and $(\tau_* - 1)^p = \tau_*^p - 1 = 0$, we obtain $\dim H_k(\widetilde{X};\mathbb{Z}_p) \leq p \dim E^2_{0,k}$. Induction on j completes the proof. □

Lemma 3.11 ([4]). *Suppose that X is an infinite cyclic covering of a finite complex. If $H_*(X;\mathbb{Z}_p)$ is prime for any prime p, then $H_*(X;\mathbb{Q})$ is finite-dimensional.*

Proof. As a consequence of the universal coefficients theorem,

$$H_n(X;\mathbb{Q}) \cong H_n(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \oplus \operatorname{Tor}_{\mathbb{Z}}(H_{n-1}(X);\mathbb{Q})) \cong H_n(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Since X covers a finite complex, its homology groups with coefficients in $\mathbb{Z}[t,t^{-1}]$ are finitely generated modules. Since $\mathbb{Q}[t,t^{-1}]$ is a PID, it follows that $H_n(X;\mathbb{Z}) \otimes \mathbb{Q}$ is a direct sum of a finite number of cyclic $\mathbb{Q}[t,t^{-1}]$-modules. If $H_*(X;\mathbb{Q})$ has infinite dimension, then at least one of these summands must be free. The projection onto this summand defines a surjective map $H_n(X;\mathbb{Z}) \otimes \mathbb{Q} \to \mathbb{Q}[t,t^{-1}]$. On elements $\alpha \otimes 1$ this restricts to a homomorphism $f : H_n(X;\mathbb{Z}) \to \mathbb{Q}[t,t^{-1}]$, and since $H_n(X;\mathbb{Z})$ is finitely generated as a $\mathbb{Z}[t,t^{-1}]$-module, there exists an integer $k \in \mathbb{Z}$ so that the image of $H_n(X;\mathbb{Z})$ under kf lies in $\mathbb{Z}[t,t^{-1}]$. This image is an ideal in $\mathbb{Z}[t,t^{-1}]$, and so $(kf)(H_n(X;\mathbb{Z}))$ is additively a free abelian group of infinite rank. Then $H_n(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is infinite. But this is isomorphic to a subgroup of $H_n(X;\mathbb{Z}_p)$, which was assumed finite. So it must be that $H_*(X;\mathbb{Q})$ is finite-dimensional. □

Theorem 3.12 ([4]). *Suppose that X is an infinite cyclic covering of a finite complex Y, and that $\widetilde{X}$ is a regular p^r-fold covering of X. If $H_*(Y;\mathbb{Z}_p) \cong H_*(S^1;\mathbb{Z}_p)$, then $H_*(\widetilde{X};\mathbb{Q})$ is finite-dimensional.*

Proof. It is first necessary to verify that $H_*(X;\mathbb{Z}_p)$ is finite. Let τ generate the group of covering translations. The long exact sequence in homology for the covering has

$$H_{n+1}(Y;\mathbb{Z}_p) \longrightarrow H_n(X;\mathbb{Z}_p) \xrightarrow{\tau_*-1} H_n(X;\mathbb{Z}_p) \xrightarrow{\tau_*-1} H_n(Y;\mathbb{Z}_p)$$

If $n \geq 1$ the groups on the ends vanish be assumption on Y, and so $\tau_* - 1$ acts as an automorphism on $H_n(X;\mathbb{Z}_p)$. But $H_n(X;\mathbb{Z}_p)$ is a finite module over $\mathbb{Z}_p[t,t^{-1}]$ and so $H_n(X;\mathbb{Z}_p)$ is finite. Now let $\widetilde{X}$ be a p-fold cover of X. Lemma 3.10 implies that $H_*(X;\mathbb{Z}_p)$ is finite. Let G be the image of $H_1(\widetilde{X};\mathbb{Z}_p)$ in $H_1(X;\mathbb{Z}_p)$. The latter group is finite, and there is therefore an integer k such that $(h_*)^r(G) = G$. Then $h^r : X \to X$ has a lift to a homeomorphism of $\widetilde{X}$, and $\widetilde{X}$ is an infinite cyclic covering of the finite complex $\widetilde{X}/\widetilde{h}$. Then by Lemma 3.11 $\widetilde{X}$ $H_*(\widetilde{X};\mathbb{Q})$ is finite-dimensional. By factoring a p^r-fold cover into p-fold covers and repeatedly applying the preceding method, we obtain the result of the theorem. □

Lemma 3.13 ([19])**.** *Let X be a finite connected complex. Suppose that $\pi_1(X)$ is finitely generated, that its abelianization $H_1(X;\mathbb{Z})$ is finite, and that $H_1(X;\mathbb{Z}_p)$ is cyclic for some prime p. Let $\widetilde{X} \to X$ be a p^a-fold (unbranched) cyclic covering. Then $H_1(\widetilde{X};\mathbb{Q}) = 0$.*

Proof. We adopt the method of Kauffman. Let τ generate the covering group of $\widetilde{X}$. The exact sequence of chain complexes

$$0 \longrightarrow C_*(\widetilde{X}) \xrightarrow{\tau_\sharp - 1} C_*(\widetilde{X}) \longrightarrow C_*(X) \longrightarrow 0$$

Gives rise to a long exact sequence in homology, terminating with

$$H_1(\widetilde{X}) \xrightarrow{\tau_* - 1} H_1(\widetilde{X}) \longrightarrow H_1(X) \longrightarrow H_0(\widetilde{X}) \longrightarrow 0$$

The map $H_1(X) \to H_0(\widetilde{X})$ is an isomorphism, for both are isomorphic to the field F. Thus the map $\tau_* - 1$ is surjective, whence $H_1\widetilde{X}$ is finitely generated by a finite set of generators over $\mathbb{Z}[F]$, since X was assumed finite. It follows that $\dim H_1(\widetilde{X}, F)$ is itself finite. $\square$

4. Signature Invariants

A hermitian form $h : V \times V \to \mathbb{C}$ on a complex vector space V has a well-defined signature $\sigma(h)$, defined by

$$\sigma(h) = \dim V^+ - \dim V^-,$$

where V^+ is a maximal subspace on which h is positive-definite, and V^- is a maximal subspace on which h is negative-definite. Equivalently, the signature is given by the number of positive eigenvalues minus the number of eigenvalues of a matrix associated to h.

A variety of signature invariants associated to the covering spaces of Section 3 will play a role in our study of the concordance group. We begin with a consideration of signatures of knots defined in terms of signatures of certain hermitian forms arising from the Seifert pairing. These signatures in turn are related to more sophisticated invariants on $4n$-manifolds, including the G-signature. After introducing these knot invariants, we give a proof of a version of the G-signature theorem for 4-manifolds and indicate its application to the detection of slice knots.

4.1. Tristram-Levine Knot Signatures

The *signature* of a knot K is defined by $\sigma(K) = \sigma(A + A^T)$, where A is a Seifert matrix for K. $A + A^T$ is symmetric, and that this is well-defined follows from the properties of S-equivalence of Seifert surfaces described in Section 2.1. It is clear that the signature is additive, for a Seifert surface for $K_1 \# K_2$ is given by the boundary connected sum of Seifert surfaces for the respective knots, and thus the Seifert matrices are related by $A = A_1 \oplus A_2$. We will see too that the signature vanishes on slice knots, and thus gives a well-defined homomorphism $\sigma : \mathcal{C}_1^3 \to \mathbb{Z}$. More general signatures were studied by Tristram and Levine to show that the concordance group was not finitely generated. Given $\omega \in S^1$, set

$$\sigma_\omega(K) = \sigma(A(1-\omega) + A^T(1-\bar{\omega})).$$

The matrix $A(1-\omega) + A^T(1-\bar{\omega})$ is hermitian, so this signature is in fact defined. It is easily checked that this function is constant neighborhood of $1 \in S^1$, and so $\sigma_1(K)$ is defined to make the signature function continuous here [23]. Observe that the signature is locally constant except where the matrix $A(1-\omega) + A^T(1-\bar{\omega})$ is singular, for the eigenvalues vary continuously in ω. Since generally $(1-\bar{\omega})/(1-\omega) = -\bar{\omega}$, one obtains

$$\det(A(1-\omega) + A^T(1-\bar{\omega})) = \det((1-\omega)(A - \bar{\omega}A^T)) = (1-\omega^2)\Delta_K(\omega).$$

Neither $\Delta_K(1)$ nor $\Delta_K(-1)$ ever vanishes, for these are the knot determinant and 1. Thus the matrix may be singular only for those ω at which $\Delta_K(\omega) = 0$.

As an illustration, we compute the signatures of the trefoil. The Alexander polynomial is given by $\Delta_K(\omega) = \omega^2 - \omega + 1$. This vanishes at $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. To find the signatures of the trefoil it is necessary only to compute the signature at these two roots and a point in each of the intervals which they bound. Explicit computations of the matrices $A(1-\omega) + A^T(1-\bar{\omega})$ and at a point in each interval between them, with the results indicated in Figure 4.1.

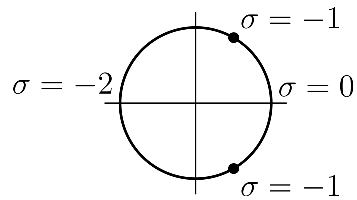

Figure 4.1: Signatures of the trefoil on S^1

There is a related invariant which arises as the average of the Tristram-Levine signatures over S^1. This is the first of the von Neumann ρ-invariants utilized in [7] and is defined by

$$\rho_0(K) = \int_{S^1} \sigma_\omega(K)\, d\omega.$$

Slice knots have vanishing Tristram-Levine signatures, a result which follows from an easy fact in linear algebra.

Lemma 4.1. *Suppose that M is hermitian. Then $\sigma(M) = \sigma(PM\bar{P}^T)$ for any nonsingular P. Moreover, if $\det M \neq 0$ and*

$$M = \begin{pmatrix} 0 & B \\ \bar{B}^T & D \end{pmatrix},$$

then $\sigma(C) = 0$.

Proof. The first part is Sylvester's law of inertia, and the second follows from a simple dimension count: any space on which M is positive definite has dimension at most n, since otherwise it would intersect the space on which M vanishes. Similarly any space on which M is negative-definite is of dimension at most n. Since the $2n$-dimensional vector space decomposes into these two components because the pairing in nonsingular, it follows that there are n positive and n negative eigenvalues. □

Corollary 4.2. *Suppose that K is algebraically slice, and $\omega \in S^1$ is such that $\Delta_K(\omega) \neq 0$. Then $\sigma_\omega(K) = 0$.*

Proof. If $\Delta_K(\omega) \neq 0$, then $A(1-\omega) + A^T(1-\bar{\omega})$ is a nonsingular hermitian form. It follows from the Lemma 4.1 that the signature of this form vanishes, as required. □

The Alexander polynomial is non-vanishing when ω is a prime-power root of unity (roughly by Theorem 3.7), and so evaluation of σ_ω at any such root of unity defines a homomorphism $\sigma_\omega : \mathcal{C}_1^3 \to \mathbb{Z}$. It was by establishing the independence of these signatures that Levine first demonstrated that the concordance group $\mathcal{C}_1^3$ is infinitely generated [23].

4.2. Signatures of 4-Manifolds

Given a $4n$-dimensional manifold M, one may define a hermitian pairing on the middle-dimensional cohomology group $H^{2n}(M;\mathbb{C})$ by

$$h(\alpha, \beta) = (\alpha \smile \beta)([M]).$$

The symmetry of this form is a consequence of the identity $\alpha \smile \beta = (-1)^{pq}\beta \smile \alpha$ for $\alpha \in H^p(M;\mathbb{C})$ and $\beta \in H^q(M;\mathbb{C})$. The *signature* of M is then the signature of this pairing on cohomology. Equivalently, the signature of a manifold may be dually interpreted as the signature of the intersection pairing on the middle-dimensional homology groups. There is similarly a signature operator defined on 4-manifolds with boundary, given as the signature of the form $H^{2n}(M,\partial M;\mathbb{C}) \times H^{2n}(M,\partial M;\mathbb{C}) \to \mathbb{C}$ defined by $(a,b) \mapsto (a \cup b)([M,\partial M])$. The following are standard results on the behavior of this signature, and detailed proofs may be found in Kauffman's text [19].

Theorem 4.3. *Suppose that M is a $4n$-manifold. Then*

1. $\sigma(M_1 \times M_2) = \sigma(M_1) + \sigma(M_2)$.
2. (Novikov additivity) *If $M = M_1 \cup M_2$ is a decomposition of a M as a union of two $4n$-manifolds with boundary, such that their common boundary $M_1 \cap M_2$ is a $4n-1$ manifold, then $\sigma(M) = \sigma(M_1) + \sigma(M_2)$.*
3. *If $M = \partial N$ for some $4n+1$-manifold N, then $\sigma(M) = 0$. Thus signature is a cobordism invariant.*

Proof. The first claim is an easy consequence of the Künneth theorem; the latter two follow from Mayer-Vietoris arguments. □

The Atiyah-Singer index theorem is the crucial result which relates the signature of a manifold to the geometric data about M captured by its characteristic classes. Broadly, the Atiyah-Singer index theorem relates that the analytic data about an elliptic operator on a manifold is equal to topological information. Given an elliptic operator $D : C^\infty(E) \to C^\infty(F)$, the analytic index is defined by

$$\operatorname{index}(D) = \dim\ker(D) - \dim\operatorname{coker}(D) = \dim\ker(D) - \dim\ker(D^*).$$

That these dimensions are finite is a consequence of the elliptic condition. The topological index is defined by certain geometric data associated to characteristic classes of vector bundles over M. The Atiyah-Singer index theorem asserts that in the case of elliptic operators, these two indices in fact coincide [1].

We will interpret the signature of a $4n$-manifold as the index of a certain elliptic operator in order to indicate the relevance of the general index theorem to the situation at hand. When the Atiyah-Singer index theorem is applied to the signature operator, the result is the older Hirzebruch signature theorem. The signature of a $4n$-manifold may be interpreted as the index of certain elliptic operator, and thus computed by characteristic class methods. Suppose that M is has real dimension $4n$ and is endowed with a Riemannian metric. Let $\mathcal{A}^{p,q}$ denote the set of (p,q)-forms, and $* : \mathcal{A}^{p,q} \to \mathcal{A}^{2n-q,2n-p}$ the Hodge star operator. There is an inner product on the space $\mathcal{A}^{p,q}$ of (p,q)-forms defined by

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\bar{\beta},$$

with respect to which the exterior differential d has an adjoint d^*. The Laplacian is then defined by $\Delta = dd^* + d^*d$, and is an elliptic operator which preserves the degree of differential forms. The *harmonic forms* $\mathcal{H}^i(M)$ are the solutions of $\Delta u = 0$, and the Hodge theorem asserts that $\mathcal{H}^i(M) = H^i_{DR}(M;\mathbb{C})$. There is a related self-adjoint first-order operator $D = d + d^*$ which satisfies $D^2 = d^*d^* + dd^* + d^*d + d^2 = d^*d + dd^* = \Delta$, and so the harmonic forms are exactly those satisfying $Du = 0$, for if $\Delta u = 0$, then $(\Delta u, u) = (Du, Du) = 0$, and $Du = 0$. It is readily verified that $*(*\alpha) = (-1)^i\alpha$ where i is the degree of α, and that $d^*\alpha = - * d * \alpha$ [16]. Now introduce $\tau : \Omega^p \to \Omega^{4n-p}$ by setting

$$\tau(\alpha) = i^{p(p-1)+2n} * \alpha.$$

One may then count powers of i to determine that the map τ is an involution on $\Omega = \oplus\Omega^p$ which anticommutes with τ, and thus induces an eigenspace decomposition of this space as $\Omega = \Omega^+ \oplus \Omega^-$. Let $D^+ : \Omega^- \to \Omega^+$ and $D^- : \Omega^+ \to \Omega^-$ be the restrictions to D to these spaces. The harmonic forms $\mathcal{H}^- \subset \Omega^-$ are exactly the kernel of D^+, and $\mathcal{H}^+ \subset \Omega^+$ the kernel of D^-. Write $h^+ = \dim H^+$ and $h^- = \dim H^-$, so that $\operatorname{index} D^- = h^+ - h^-$. Now, τ fixes $H^k \oplus H^{4n-k}$ and so these space makes no contribution to the index. It follows that $\operatorname{index} D^- = h^{2n}_+ - h^{2n}_-$. Given $\alpha \in \mathcal{H}^n$ we have $\tau\alpha = *\alpha$, and so for $\alpha \in \mathcal{H}^n_+$ real and nonzero we obtain $\langle\alpha,\alpha\rangle > 0$, and similarly for $\alpha \in H^n_-$ we have $\langle\alpha,\alpha\rangle < 0$. This inner product coincides with the cup product form, and thus $\operatorname{index} D^- = \sigma(M)$. Then the force of the G-signature theorem is available, and relates this signature to information about the fixed point set and characteristic classes of M.

The G-signature theorem of Atiyah and Singer then provides a means of computing the G-signature in terms of data about the fixed point sets of the G action and the Pontrjagin classes of M [2]. In the case that the group G is trivial, this reduces to the Hirzebruch signature theorem, while the general form involves information about the fixed point set of the G-action. We will develop here a version of the G-signature theorem applicable to 4-manifolds admitting cyclic actions; the discussion here largely follows that of Kauffman [19], though this method originates with Gordon [13] and plays a central role in the work of Casson and Gordon [4]. The result of this 4-dimensional case was derived from the more general G-signature theorem by Atiyah and Singer in their original work [2].

4.3. The G-Signature Theorem for 4-Manifolds

The Riemann-Hurwitz theorem of complex analysis relates the Euler characteristics of two Riemann surfaces to information about their behavior around branch points of associated branched covers. In this section, we develop a version of the Atiyah-Singer G-signature theorem for 4-manifolds use it to relate the signatures of a branched cover $\widetilde{N}$ of a 4-manifold N to the signature of N and the self-intersection number of the branch set.

Let M be a 4-manifold and $G = C_m$ a cyclic group acting on M. Let $I : H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \to \mathbb{R}$ denote the intersection pairing, and observe that I is compatible with the group action in the sense that $I(gx, gy) = I(x, y)$.

The first step in defining the G-signature is to give a decomposition of $H_2(M;\mathbb{R})$ into two G-invariant subspaces, such that I is positive-definite on one and negative-definite on the other. Define a pairing

$$\langle x, y\rangle = \frac{1}{m}\sum_{g\in G}\langle gx, gy\rangle_0$$

where $\langle \cdot, \cdot \rangle_0$ is some inner product on $H_2(M;\mathbb{R})$. Then $\langle \cdot, \cdot \rangle$ is a G-equivariant positive-definite inner product. This induces a linear map $B : H_2(M;\mathbb{R}) \to H_2(M;\mathbb{R})$ defined by $\langle Ax, y \rangle = I(x, y)$.

We first verify that the map B commutes with the G-action. This follows from the G-equivariance of I and $\langle \cdot, \cdot \rangle$. Forming the inner product with an arbitrary element gy,

$$\langle Bgx, gy \rangle = I(gx, gy) = I(x, y) = \langle Bx, y \rangle = \langle gBx, gy \rangle .$$

Moreover, the map $B : H_2(M;\mathbb{R}) \to H_2(M;\mathbb{R})$ is self-adjoint:

$$\langle B^* x, y \rangle = \langle x, By \rangle = \langle By, x \rangle = I(y, x) = I(x, y) = \langle Bx, y \rangle .$$

Suppose that v is in the λ-eigenspace of B, so that $Bv = \lambda v$. Then $Bgv = gBv = g\lambda v = \lambda gv$, and so gv is in this eigenspace as well. Set H^+ to be the direct sum of all positive eigenspaces of B, and H^- to be the direct sum of all negative eigenspaces. Thus both H^+ and H^- are G-invariant, and $H_2(M;\mathbb{R}) = H^+ \oplus H^-$.

Definition 7. The G-signature of M with respect to this action of G is defined by

$$\sigma(M, g) = \operatorname{tr}(g|_{H^+}) - \operatorname{tr}(g|_{H^-}).$$

The intersection pairing $I : H_2(M;\mathbb{R}) \times H_2(M;\mathbb{R}) \to \mathbb{R}$ extends to a hermitian pairing $H_2(M;\mathbb{C}) \times H_2(M;\mathbb{C}) \to \mathbb{C}$ by setting $I(x \otimes \mu, y \otimes \lambda) = \lambda\bar{\mu}\, I(x, y)$ and extending by linearity (here we have made the identification $H_2(M;\mathbb{R}) \otimes \mathbb{C} \cong H_2(M;\mathbb{C})$). The G-action extends similarly to $g : H_2(M;\mathbb{C}) \to H_2(M;\mathbb{C})$ and thus induces an eigenspace decomposition

$$H_2(M;\mathbb{C}) = \bigoplus_{m=0}^{d-1} H_2(M;\mathbb{C})_{\omega^m},$$

where $H_2(M;\mathbb{C})_{\omega^m}$ denotes the ω^m-eigenspace of a generator $g \in G$. We then obtain from the definition of G-signature the relation

$$\sigma(M, g) = \sum_{m=0}^{d-1} \omega^m \sigma(H_2(M;\mathbb{C})_{\omega^m}).$$

The signature $\sigma(H_2(M;\mathbb{C})_{\omega^m})$ is the signaure of the restriction of the intersection pairing to the eigenspace $H_2(M;\mathbb{C})_{\omega^m}$. That this characterization coincides with that of Definition 7 follows from the observation that $H^\pm$ decomposes as $\bigoplus_\omega H^\pm_\omega$, and the term $\operatorname{tr}(g|_{H^-})$ corresponds exactly to the subtracted negative eigenvalues of $\sigma(H_2(M;\mathbb{C})_{\omega^m})$. It is this interpretation of the G-signature in terms of eigenspace signatures that will be most generally useful for our purposes. We remark too that the results of Theorem 4.3 hold in the generality of G-signatures, and reduce to the forms stated in that theorem when the group action is trivial.

The first step in proving the G-signature theorem is the observation that the G-signature depends in an appropriate sense only on the fixed-point structure of the G-action. We will then reduce the general case of the G-signature theorem to cases of manifolds with more easily handled fixed point sets. First a preliminary lemma is required.

Lemma 4.4. *Suppose that $G = C_m$, and that M is disconnected, such that the action of g fixes no component of M. Then $\sigma(M, g) = 0$.*

Proof. Let M_0 be one component of M. We can write

$$M = \coprod_{j=0}^{m-1} g^j M_0.$$

Decompose $H_n(M_0; \mathbb{C}) = H_0^+ \oplus H_0^- \oplus H_0^0$, where I is positive-definite on H_0^+, negative-definite on H_0^-, and zero on H_0^0. Then set

$$H^+ = \sum_{j=0}^{m-1} g_*^j H_0^+, \quad H^- = \sum_{j=0}^{m-1} g_*^j H_0^-, \quad H^0 = \sum_{j=0}^{m-1} g_*^j H_0^0.$$

This gives a decomposition $H_n(M; \mathbb{C}) = H^+ \oplus H^- \oplus H^0$. Since $\mathrm{tr}(g_*|_{H^+}) = \mathrm{tr}(g_*|_{H^-}) = 0$, it follows that $\sigma(M, g) = 0$. □

Theorem 4.5 ([19])**.** *Suppose that g generates a free C_m action on a closed 4-manifold M. Then $\sigma(M, g) = 0$.*

Proof. One may check that $\Omega_4(*) \to \Omega_4(X) \to H_4(X; \mathbb{Z})$ is exact, and so $\Omega_4(*) \to \Omega_4(BC_m)$ is surjective. Thus a free C_m action on M is bordant to the action $C_m \times (M/C_m)$. This is a disconnected space and the G-action fixes no components, so $\sigma(M, g) = 0$ for any $g \in G$. □

This result in fact holds much more generally, and is a consequence of the full G-signature theorem or of earlier work of Atiyah and Bott dealing with G-signatures for actions with only isolated fixed points. As a result of Theorem 4.5, to compute the G-signature, it is necessary only to consider contributions from each component of the locus of fixed points. This is analogous to the result of the Riemann-Hurwitz theorem, which relates the Euler characteristics (and thus the genera and signatures) of two Riemann surfaces to information about the behavior of branched coverings between them in terms only of behavior of these maps near the branch points.

Theorem 4.6. *Suppose that M and M' are two manifolds admitting a G-action and have the same fixed-point structure, in the sense that G-equivariant tubular neighborhoods of their fixed point loci are diffeomorphic. Then $\sigma(M, g) = \sigma(M', g)$.*

Proof. Let H be a component of the locus of points fixed by the G-action, and let $N \subset M$ be a tubular neighborhood of H, so that $N \to F$ is a D^3-bundle admitting a fiberwise G-action (obtained as the restriction of the action on M).

The existence of a tubular neighborhood admitting such an action is sometimes known as the equivariant tubular neighborhood theorem. We sketch the proof below in a slightly more general setting than that required.

Theorem 4.7 ([3])**.** *Suppose that a compact Lie group G acts smoothly on a compact manifold M. Then any G-invariant submanifold $N \subset M$ admits a G-invariant tubular neighborhood.*

Proof. We seek to construct a smooth vector bundle E over N with a G-action whose associated disk bundle F embeds into M, such that the 0 section of F is sent to $N \subset M$. It is first necessary to construct a G-invariant metric on M. Pick any metric $\langle \cdot, \cdot \rangle_1$ on TM and set

$$\langle v, w \rangle = \int_G \langle gv, gw \rangle_1 \; dg.$$

Then $\exp : S \to M$ is defined on some neighborhood of the zero section of TM, and satisfies $\exp(X) = \gamma(1)$ where γ is a geodesic with respect to the metric given earlier and satisfies $\gamma(0) = p$ and $\gamma'(0) = X$. The exponential map is equivariant in that $\exp(gX) = g \exp(X)$. Taking E to be the normal bundle of N we obtain an embedding $\exp : \mathring{D}_\epsilon(TM) \to M$, which is the required neighborhood. $\square$

We apply this in the case that G is a finite group and N is the set of fixed points of the action. Observe that by applying this to to a single point set, we conclude that the fixed point set of the G-action is indeed a smooth manifold. Moreover, it is then easy to see that the fixed point set has dimension 0 or 2 when M is 4-dimensional.

Write $M = (M - \mathring{N}) \cup N$. Performing this for each component of the set of fixed points, we obtain a manifold $\mathcal{N} = \bigcup_i N_i$. Then $M = \mathcal{M} \cup \mathcal{N}$, $\partial\mathcal{M} = \partial\mathcal{N}$, and G preserves the both components of the decomposition. Suppose that M' is another manifold the the same fixed-point structure, so that $M' = \mathcal{M}' \cup \mathcal{N}$ and $\partial\mathcal{M}' = \partial N$: the point is that $\sigma(M', g) = \sigma(M, g)$. This fact follows from Novikov additivity of the G-signature. The manifold $\mathcal{M} \cup -\mathcal{M}'$ obtained by gluing $\mathcal{M}$ and $\mathcal{M}'$ along common boundary admits a free G-action, and so $\sigma(\mathcal{M} \cup -\mathcal{M}', g) = \sigma(\mathcal{M}, g) - \sigma(\mathcal{M}', g) = 0$, whence $\sigma(\mathcal{M}', g) = \sigma(\mathcal{M}', g)$. But then

$$\sigma(M, g) = \sigma(\mathcal{M}, g) + \sigma(\mathcal{N}, g) = \sigma(\mathcal{M}', g) + \sigma(\mathcal{N}, g) = \sigma(\mathcal{M}' \cup \mathcal{N}, g) = \sigma(M', g) \qquad \square.$$

Therefore, in order to compute the G-signature of a manifold M, it is sufficient to compute the G-signature of a simpler manifold with the same fixed-point structure. By finding model manifolds for the possible fixed-point sets and making explicit computations of the G-signature, we will obtain the desired theorem.

As a consequence of the equivariant tubular neighborhood theorem, the fixed point sets are either 0-dimensional or 2-dimensional manifolds. We will assume that 2-dimensional branch sets are orientable, though the theorem holds even without this assumption. Around an isolated fixed point a regular neighborhood with a G action takes the form $(D_1^2, \theta_1) \times (D_2^2, \theta_2)$, where g acts by rotation by θ_1 and θ_2 on the two factors. Write $\theta_1 = 2\pi k_1/m$ and $\theta_2 = 2\pi k_2/m$.

We may give an explicit manifold $Q(\theta_1, \theta_2)$ whose isolated fixed points are all of the type (θ_1, θ_2). It is constructed as a product of two 2-manifolds, and we essentially require the 2-dimensional version of the G-signature theorem. First we construct a two-manifold $Q(\theta)$ whose fixed points are of the form $(D^2, 2\pi s/m))$ for a given s coprime to m. Suppose first that m is even. Let M_0 be the manifold pictured in Figure 4.2, formed by attaching m twisted bands between two disks D^2. Attach two disks along the boundary of M_0 to obtain a closed 2-manifold.

Let h be a counterclockwise rotation of the manifold M by $\exp(2\pi i/m)$ in Figure 4.2, so that it sends each band to a neighboring one. For any s coprime to m, the rotation

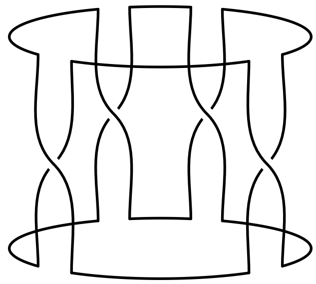

Figure 4.2: A 2-manifold with a G-action

h^s generates a C_m action on M, whose fixed points are two centers of the two disks on top and bottom and have neighborhoods of the desired form $(D^2, 2\pi s/m)$. Decompose $H_1(M;\mathbb{C}) = E_0 \oplus \cdots \oplus E_{m-1}$, where E_r is the ω^r eigenspace of h_*. This gives

$$\sigma(M, h_*^s) = \sum_{r=0}^{m-1} \omega^{rs}\epsilon_r,$$

where ϵ_r is the signature of the restriction to E_r. Let $\alpha \in H_1(M;\mathbb{C})$ be a path which goes clockwise through the disks on top and bottom and two neighboring bands, so that $h_*^s\alpha$ for $0 \leq s < m$ generate $H_1(M;\mathbb{C})$. The linking numbers may be seen to be

$$I(h_*^s\alpha, h_*^t\alpha) = \begin{cases} -1 & \text{if } s = t+1, \\ +1 & \text{if } s = t-1, \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$e_r = \sum_{s=0}^{m-1} \omega^{-rs} h_*^s \alpha.$$

This clearly lies in E_r. We see that $\dim H_1(M;\mathbb{C}) = m-2$, and that E_0 and $E_{m/2}$ are empty, and so E_r must be one-dimensional and generated by e_r for each other r. It follows that

$$\epsilon_r = \begin{cases} -1 & \text{if } 1 \leq r < \frac{m}{2}, \\ +1 & \text{if } \frac{m}{2} < r < m. \end{cases}$$

Then writing $\theta = 2\pi s/m$ for $\gcd(s, m) = 1$,

$$\begin{aligned} \sigma(M, h_*^s) &= -\sum_{r=1}^{\frac{m}{2}-1} \omega^{rs} + \sum_{r=\frac{m}{2}+1}^{m-1} \omega^{rs} = -2\sum_{r=1}^{\frac{m}{2}-1} \omega^{rs} \\ &= -2\left(\frac{1+\omega^s}{1-\omega^s}\right) = -2\cot\frac{\theta}{2}. \end{aligned}$$

Thus in the two-manifold case with fixed points of the form $(D^2, 2\pi s/m)$ with m even, the contribution of each fixed point of type (D^2, θ) to the signature of the cyclic action is $-\cot\frac{\theta}{2}$.

The case that m is odd is slightly more complicated: only one disk needs to be attached to the boundary of M_0 to obtain a closed manifold, and this disk contains a third fixed point. A similar computation establishes the result.

The product of two surfaces as described above then gives the necessary result for $Q(\theta_1, \theta_2)$. Let $Q(\theta_1, \theta_2) = -(M(\theta_1) \times M(\theta_2))$. This space admits a $\mathrm{lcm}(m_1, m_2)$-cyclic action, and the fixed points all have neighborhoods with G-actions of the form $(D^2, \theta_1) \times (D_2, \theta_2)$. That the G-signature contribution from such points is $-\cot\frac{\theta_1}{2}\cot\frac{\theta_2}{2}$ then follows from multiplicativity of the G-signature. We thus conclude that if all fixed points of the G action on M are isolated points of type (θ_1, θ_2), then $\sigma(M, g) = n \cdot \cot(\theta_1/2)\cot(\theta_2/2)$,, where n is the number of points fixed by the action. As a consequence of Theorem 4.6 if the fixed points of M are p_i of type (θ_1^i, θ_2^i), indexed over some set I, then

$$\sigma(M, g) = -\sum_i \cot(\theta_1^i/2)\cot(\theta_2^i/2).$$

It remains to handle the case of 2-dimensional fixed point sets. We assume that all 2-dimensional fixed point sets are smooth, oriented, closed manifolds, though the theorem still holds even for non-orientable fixed-point sets. There is an action of C_m on $\mathbb{CP}^2$ defined by

$$g([Z_0; Z_1; Z_2]) = [\omega^r Z_0; \omega^r Z_1; Z_2],$$

where g is the r^{th} power of a generator of G. The fixed point set of this action is points of the form $[Z_1, Z_2, 0]$ as well as $[0, 0, 1]$. The former set is a copy of $\mathbb{CP}^1 \cong S^2$. G acts on the normal bundle of S^2 by multiplication by ω^r. We see that $\sigma(\mathbb{CP}^2, g) = \sigma(\mathbb{CP}^2) = 1$, since g acts as the identity on $H^2(\mathbb{CP}^2; \mathbb{C})$.

With this computation from $\mathbb{CP}^2$, we now decompose the fixed-point locus of any general 4-manifold with a G action and compute the G-signature. Let M be such a manifold, with fixed point locus N, such that G acts on the normal bundle of N by multiplication by ω^r. A lemma is required.

Lemma 4.8. *Suppose that M has fixed point set N, an orientable, connected 2-manifold, and that N has self-intersection number 0. Then $\sigma(M, g) = 0$.*

Proof. Since the self intersection number is 0, N admits a regular neighborhood $P \cong N \times D^2$. Pick a handlebody H such that $F = \partial H$, and define a 4-manifold W as $W = (M \setminus P) \cup_\partial (H \times S^1)$. W admits a free G action by rotation of S^1 by a root of unity and the given action on M, and these are compatible on the boundary. It follows that $\sigma(M \setminus \mathring{P}, g) + \sigma(H \times S^1, g) = \sigma(W, g) = 0$. Clearly $\sigma(H \times S^1, g) = 0$, and it follows that $\sigma(M, g) = \sigma(P, g) = 0$. $\square$

Suppose that M has only a fixed-point set of dimension 2. By taking the connected sum of M with copies of $\mathbb{CP}^2$, we may obtain a manifold admitting a G action whose fixed point locus consists only of isolated fixed points and 2-manifold fixed sets with 0 self-intersection number. Indeed, set $W = (M^4, \Sigma) \# (-[F]^2)(\mathbb{CP}^2, S^2)$. The connected sum is the connected sum of a pair, and is formed so that the G-actions on the boundary components coincide. This manifold has the desired fixed point locus, consisting of $-[F]^2$ isolated fixed points of type $(2\pi r/m, 2\pi r/m)$ arising from the isolated fixed point of the G-action on each copy of $\mathbb{CP}^2$, together with a two-dimension fixed set $\Sigma \# -[\Sigma]^2\mathbb{CP}^1$. The latter set has self-intersection number 0, and it follows that $\sigma(W, g) = -[F]^2(-\cot^2(\pi r/m))$. Since signature is additive under connected sum, it follows that

$$\sigma(W, g) = \sigma(M, g) - [\Sigma]^2\sigma(\mathbb{CP}^2, g) = \sigma(M, g) - [\Sigma]^2.$$

Thus since $1 + \cot^2\theta = \csc^2\theta$, we obtain

$$\sigma(M, g) = [\Sigma]^2 \csc^2(\pi r/m).$$

This completes the computations necessary for this case of the G-signature theorem. Combining the results for the cases of isolated fixed point sets and two-dimensional fixed point sets, in light of the result of Theorem 4.6.

Theorem 4.9 ([19])**.** *Suppose that M is a 4-manifold admitting an action by a cyclic group $G = C_d$. Let the isolated fixed points be p_i of type (θ_i^1, θ_i^2), and two dimensional branch sets Σ_j on which the action on fibers of a regular neighborhood is multiplication by ψ_j. Then*

$$\sigma(M, g) = -\sum_i \cot(\theta_i^1/2)\cot(\theta_i^2/2) + \sum_j [\Sigma_j]^2 \csc^2(\psi_j/2).$$

Our primary application of the G-signature theorem will be in relating signatures of cyclic branched covers to the signatures of the covered spaces, an analogue of the Riemann-Hurwitz formula.

4.4. Signatures of Branched Covering Spaces

Let $\widetilde{N} \to N$ be an m-fold branched cyclic covering of 4-manifolds, and let $\Sigma \subset N$ denote the branch locus, with preimage $\widetilde{\Sigma}$, both assumed to be orientable surfaces. Let τ be a covering translation, and observe that it is a rotation by $2\pi/m$ on the normal bundle, as in the preceding discussion. Let $\omega = \exp(2\pi i/m)$ and denote by E_r the ω^r eigenspace of τ. Denote by $\epsilon_r(\widetilde{N})$ the signature of the restriction of the intersection pairing to E_r. Each of the ϵ_r may be related to the signature of N.

Theorem 4.10 (Rohlin, [28])**.** *With notation as above,*

$$\epsilon_r(\widetilde{N}) = \sigma(N) - 2[\Sigma]^2 r(m-r)/m^2.$$

Proof. We follow the proof of Casson-Gordon [4]. Write $E_r = E_r^+ \oplus E_r^-$, where the intersection pairing I is positive (resp. negative) definition on E_r^+ (resp. E_r^-). This gives a decomposition $H = H_2(M; \mathbb{R}) = H^+ \oplus H^-$, where $H^\pm = E_0^\pm \oplus \cdots \oplus E_{m-1}^\pm$. This is exactly the decomposition into eigenspaces constructed earlier, and as we have seen,

$$\sigma(\widetilde{N}, \tau^s) = \operatorname{tr}(\tau^s|_{H^+}) - \operatorname{tr}(\tau^s|_{H^-}) = \sum_{r=0}^{m-1} \omega^{rs}\epsilon_r(\widetilde{N}).$$

There is a chain map $s : C_*(N) \to C_*(\widetilde{N})$ which sends x to $\sum_{i=1}^m \tau^i\widetilde{x}$, where $\widetilde{x}$ is some chain mapped to x under the map $\pi_\sharp$ induced by projection. This induces a map in homology, the transfer homomorphism $s_* : H_1(N; \mathbb{Q}) \to H_1(\widetilde{N}; \mathbb{Q})$ [17].

Lemma 4.11. *The map $s_* : H_2(N\ \mathbb{Q}) \to H_2(\widetilde{N};\mathbb{Q})$ is injective, with image $H_2(\widetilde{N};\mathbb{Q})^G$, the G-invariant subspace of $H_1(\widetilde{N};\mathbb{Q})$.*

Proof. The map $\pi_* \circ s_*$ is clearly given by multiplication by n, and since $\mathbb{Q}$ has characteristic 0 it follows that s is injective. On the other hand, $s \circ \pi_\sharp$ sends a simplex $\Delta^2 \to \widetilde{N}$ to the sum of all its images under the G-action. Thus $\mathrm{im}(s)$ lies in the G-invariant subspace of $H_2(\widetilde{N};\mathbb{Q})$; we claim that it is in fact surjective onto this subspace. Given $\alpha \in H_2(\widetilde{N};\mathbb{Q})^G$, we have

$$(s_* \circ \pi_*)(\alpha) = \sum_{g\in G} g\alpha = n\alpha,$$

and so $(s_* \circ \pi_*)(\frac{1}{n}\alpha) = \alpha$, whence s_* is surjective onto $H_2(\widetilde{N};\mathbb{Q})^G$, as claimed. □

Thus $\epsilon_0(\widetilde{N}) = \sigma(N)$, since ϵ_0 is the signature of the 1-eigenspace of the G-action. It follows that

$$\begin{aligned}
\sigma(\widetilde{N},\tau^s) - \sigma(N) &= \sum_{r=1}^{m-1} \omega^{rs}\epsilon_r(\widetilde{N}) \\
\frac{1}{m}\sum_{s=1}^{m-1}(\omega^{-rs}-1)(\sigma(\widetilde{N},\tau^s) - \sigma(N)) &= \frac{1}{m}\sum_{j=1}^{m-1}(\omega^{-rs}-1)\sum_{r=1}^{m-1}\omega^{rs}\epsilon_r(\widetilde{N}) \\
\epsilon_r(\widetilde{N}) &= \frac{1}{m}(\omega^{-rs}-1)(\sigma(\widetilde{N},\tau^s) - \sigma(N)) \\
&= \sigma(N) + \frac{1}{m}\sum_{s=1}^{m-1}(\omega^{-rs}-1)\sigma(\widetilde{N},\tau^s)
\end{aligned}$$

It follows from the version of the G-signature theorem proved above that $\sigma(\widetilde{N},\tau^s) = [\Sigma]^2\csc^2(\pi s/m)$.

We need an easy geometric lemma.

Lemma 4.12. *The self-intersection numbers are related by $[\widetilde{\Sigma}]^2 = [\Sigma]^2/m$.*

Proof. Let $L \to \Sigma$ and $\widetilde{L} \to \widetilde{\Sigma}$ be the normal bundles. These have the structure of holomorphic line bundles, so $[\Sigma]^2 = \deg(L)$ and $[\widetilde{\Sigma}]^2 = \deg(\widetilde{L})$. But the map π acts as $z \mapsto z^m$ on the disk bundle of $\widetilde{L}$, and so $\pi^*L = \widetilde{L}^{\otimes m}$. In particular $\deg(L) = m\deg(\widetilde{L})$, and the result follows. □

This yields

$$\epsilon_r(\widetilde{N}) = \sigma(N) + \frac{[F]^2}{m^2}\sum_{s=1}^{m-1}(\omega^{-rs}-1)\csc^2\frac{\pi s}{m}.$$

Observe that

$$\omega^{-rs} - 1 = \left(\cos\frac{2\pi rsi}{m} - 1\right) + i\sin\left(-\frac{2\pi rs}{m}\right) = -2\sin^2\frac{\pi rs}{m} - i\sin\frac{2\pi rs}{m}.$$

Next it is necessary to compute

$$\sum_{s=1}^{m-1}(\omega^{-rs}-1)\csc^2\frac{\pi s}{m} = -2\sum_{s=1}^{m-1}\sin^2\frac{\pi rs}{m}\csc^2\frac{\pi s}{m} - i\sum_{s=1}^{m-1}\sin\frac{2\pi rs}{m}\csc^2\frac{\pi s}{m}.$$

The imaginary part must vanish. To evaluate the real part, set $\xi = \exp(\pi i/m)$. This yields

$$\begin{aligned}\sum_{s=1}^{m-1}\sin^2\frac{\pi rs}{m}\csc^2\frac{\pi s}{m} &= \sum_{s=1}^{m-1}\left(\frac{\xi^{rs}-\xi^{-rs}}{\xi^s-\xi^{-s}}\right)^2\\ &= \sum_{s=1}^{m-1}(\xi^{s(r-1)}+\xi^{s(r-3)}+\cdots+\xi^{-s(r-1)})^2 = \sum_{s=1}^{m-1}P(\xi^s),\end{aligned}$$

where P is the polynomial which arises in the second line. We can see from the first expression above that $P(z) = P(z^{-1})$ and $\xi^{2m} = 1$. It follows that

$$\begin{aligned}\sum_{s=1}^{m-1}P(\xi^s) &= \frac{1}{2}\sum_{s=0}^{2m-1}P(\xi^s) - \frac{1}{2}(P(1)+P(-1))\\ &= \frac{1}{2}\left(2m\sum_t a_{2mt}\right) - r^2 = r(m-r),\end{aligned}$$

where a_{2mt} is the coefficient on z^{2mt} in the polynomial P. The only term adding to this coefficient is z^0, which arises r times. It follows that

$$\epsilon_r(\widetilde{N}) = \sigma(N) - 2[\Sigma]^2 r(m-r)/m^2,$$

as claimed. □

Corollary 4.13.

$$\sigma(\widetilde{N}) = m\sigma(N) - [\Sigma]^2\frac{m^2-1}{3m}.$$

Proof. This follows from a direct computation using Theorem 4.10.

$$\begin{aligned}\sigma(\widetilde{N}) &= \sum_{r=0}^{m-1}\epsilon_r(\widetilde{N}) = m\sigma(N) - 2[\Sigma]^2\sum_{r=1}^{m-1}\frac{r(m-r)}{m^2}\\ &= m\sigma(N) - 2[\Sigma]^2\left(\frac{1}{m}\sum_{r=1}^{m-1}r - \frac{1}{m^2}\sum_{r=1}^{m-1}r^2\right)\\ &= m\sigma(N) - 2[\Sigma]^2\left(\frac{1}{m}\frac{m(m-1)}{2} - \frac{1}{m^2}\frac{(m-1)(m)(2m-1)}{6}\right)\\ &= m\sigma(N) - 2[\Sigma]^2\left(\frac{3m(m-1)}{6m} - \frac{(m-1)(2m-1)}{6m}\right)\\ &= m\sigma(N) - [\Sigma]^2\frac{m^2-1}{3m}\end{aligned}$$

□

The following corollary is often useful.

Corollary 4.14. *Suppose that* $\widetilde{N} \to N$ *is an* m*-fold cyclic branched covering of four-manifolds whose fixed point set is a* 2*-dimensional orientable submanifold of self-intersection* 0*. Then* $\sigma(\widetilde{N}) = m\,\sigma(N)$*.*

In particular, the hypotheses of the corollary are met if $\widetilde{N} \to N$ is unbranched. This fact may be proved more directly [14].

5. The Casson-Gordon Approach

5.1. Introduction

The abelian invariant of algebraic sliceness defined in Section 2 is sufficient to distinguish slice knots in dimensions $n \geq 2$, and it was natural to ask whether algebraic sliceness is a complete invariant for detecting sliceness in the case $n = 1$. Thus, an open question for some time was whether there exists a knot K which is algebraically slice but not slice. This was settled in the positive by Casson and Gordon, who gave several families of such knots. Their method is to consider invariants which are detected by metabelian branched covers of (S^3, K) and which are finer than the abelian invariants considered previously, such as the Tristram-Levine signatures. We begin with the construction of a knot which is algebraically slice but not slice, making a computation akin in spirit to that of Casson and Gordon [5], but with the advantage of hindsight provided by knots and constructions which have more recently been important in the study of the concordance group [6]. Cochran, Harvey, and Leidy have proved results considerably more powerful than those of this section, and the approach here makes use of their constructions but avoids the analytic difficulties inherent in their more general approach. After making this computation to demonstrate that J(trefoil) is slice but not algebraically slice, we discuss the original methods and invariants given by Casson and Gordon in the course of their construction of a knot which is algebraically slice but not slice.

5.2. Branched Covers of $(S^3, 9_{46})$

Before proceeding it is necessary to understand the structure of the homology of covers of S^3 branched over 9_{46}. These results, together with the infection construction of Section 2, makes possible an explicit realization of manifolds which bound covers of M_K for any knot K. Computations about K made on these manifolds will then furnish obstructions to $J(K)$ being slice. The destriptions of these homology groups is possible using a surgery technique due to Rolfsen which provides a convenient visualization of the cyclic covers of the knot complement [29].

First, observe that the knot 9_{46} as presented in Figure 2.3 may be transformed by isotopy to the configuration depicted in Figure 5.1, with the indicated image of the loops η_1 and η_2. This isotopy involves several steps and was verified by the author using shoelaces and dental floss.

Next, remove tubular neighborhoods of η_1 and η_2 and glue in solid tori with a "twist", so that a meridian of η_i is identified with a path that goes around a longitude and then a meridian (this is surgery with coefficient $+1$). This has the effect of twisting by one turn the strands of 9_{46} passing through the disk bounded by η_i. In the resulting space, 9_{46} is unknotted. By a homeomorphism of S^3, this space transformed as illustrated in Figure 5.2.

It is then possible to describe the covers of S^3 branched over 9_{46}, including computations of their homology groups and generators. The infinite cyclic cover of S^3 branched over 9_{46} is obtained through the surgery indicated in Figure 5.3; the covering translation τ is realized by shifting all the components of the link in the diagram to the corresponding components to their right.

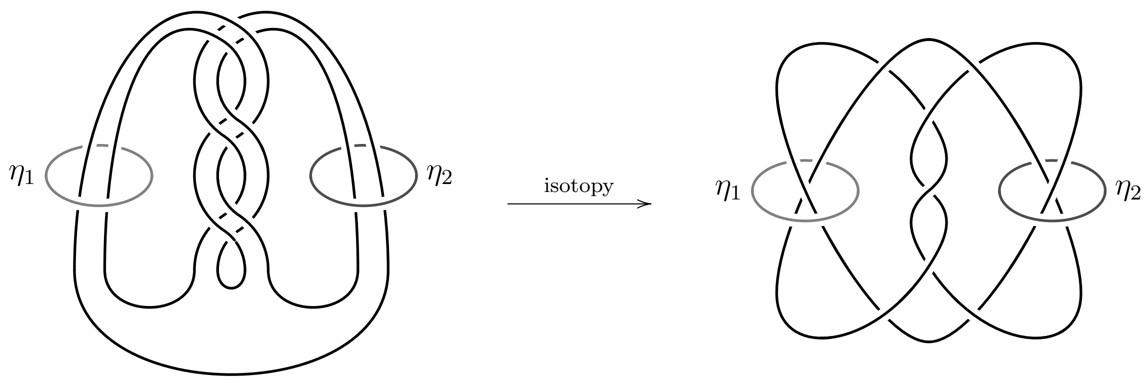

Figure 5.1: Another view of 9_{46}

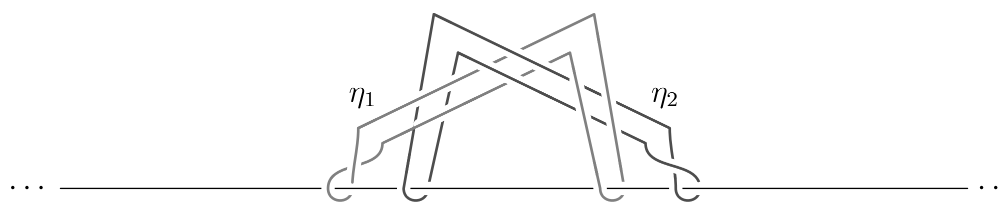

Figure 5.2: 9_{46} unknotted by surgery

Let α denote a meridian of a lift of η_1, and β a meridian of a lift of η_2, on two components of the surgered link which are intertwined in Figure 5.3. The homology $H_1(L_K^\infty;\mathbb{Z})$ is generated by the set of $\tau^i\alpha$ and $\tau^i\beta$ for $i \in \mathbb{Z}$. From the surgery description we read off relations

$$\alpha = (1-2\tau)\beta, \quad \beta = (1-2\tau^{-1})\alpha,$$

where τ is the indicated generator of the group of covering translations. This first makes possible a simple computation of the Alexander polynomial, for in the infinite cover, $\beta = (1-2\tau^{-1})(1-2\tau)\beta$, and so $\Delta_K(\tau) = -2\tau^{-1} + 5 - 2\tau$.

The space L_K^k is obtained as the quotient of L_K^∞ be the action of τ^k, and the homology $H_1(L_K^k)$ is generated by $\{\alpha, \beta, \ldots, \tau^{k-1}\alpha, \tau^{k-1}\beta\}$. Since $\alpha = (1-2\tau)\beta$ in fact it suffices to take $\{\beta, \ldots, \tau^{k-1}\beta\}$ as a generating set. In light of the relations computed above, a matrix of relations for the module $H_1(L_K^k;\mathbb{Z})$ is given by

$$M = \begin{pmatrix} -5 & 2 & 0 & & 0 & 2 \\ 2 & -5 & 2 & & & 0 \\ 0 & 2 & -5 & 2 & & \\ & & \ddots & \ddots & \ddots & \\ 0 & & & 2 & -5 & 2 \\ 2 & 0 & & & 2 & -5 \end{pmatrix}$$

A computation shows that the Smith normal form of this matrix with respect to an appro-

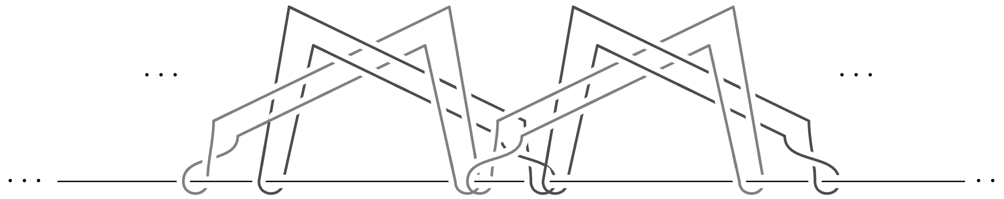

Figure 5.3: The infinite cyclic cover of $(S^3, 9_{46})$

priate generating set is diagonal:

$$M' = \begin{pmatrix} 2^k-1 & & & & 0 \\ & 2^k-1 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}$$

This means that $\left|H_1(L_K^k)\right| = (2^k-1)^2$. It is easy to find a basis with respect to which the homology satisfies the relations encoded by M'. We may explicitly compute the inverse of M, and observe that each entry is a rational number with denominator 2^k-1. Thus $(M^T)^{-1}(\tau^i\beta)$ does not have integer coefficients for any i. It follows that $\tau^i\beta$ is not in the integer span of the rows of M, and indeed and has order 2^k-1 in the module. A similar computation demonstrates that β and $\tau\beta$ generate $H_1(L_K^k)$ are each of order 2^k-1, and so these constitute a generating set. Since $\alpha = \beta - 2\tau\beta$, we may instead take $\{\alpha, \beta\}$ as a generating set for $H_1(L_K^k)$.

For the sake of concreteness, and to avoid the further proliferation of variables, we will focus on the case $k = 2$. Computations in this case are sufficient to demonstrate that $J(\text{trefoil})$ is not slice. Theorem 5.13 gives an indication of a slightly more general result that may be proved and describes the modification that must be made to the argument given.

In the case that $k = 2$, we have $2\tau\beta = \alpha - \beta$, whence $\tau\beta = (-2, 2)$ in the (α, β) basis. Similarly $\tau\alpha = \tau\beta + 2\alpha = (0, 2)$. Let $\chi : H_1(L_{J(U)}^2)$ be projection onto the β coordinate. The preceding discussion yields the following characterization of the lifts of η_1 and η_2 into $L_{J(U)}^2$, which we summarize as a lemma.

Lemma 5.1. *The map $\chi : H_1(L_{J(U)}^2) \to C_3$ given by projection onto the α coordinate is such that $\eta_1^1 \in \ker\chi$ and $\eta_1^2 \notin \ker\chi$, and similarly $\eta_2^1 \notin \ker\chi$ but $\eta_2^2 \in \ker\chi$.*

As a brief remark, we demonstrate that the result of Lemma 3.4 confirms this computation of the orders of the branched covers $|H_1(L_{J(U)}^k)| = (2^k-1)^2$. Recall that $\Delta_K(t) = -2t^{-1} + 5t - 2t = (1-2t)(1-2t^{-1})$. Since $\Delta_K(1) = 1$, we compute

$$\left|H_1(L_{J(U)}^k)\right| = \left|\prod_{j=1}^{k} \Delta_K(e^{2\pi ij/k})\right| = \left|\prod_{j=0}^{k}(1-2e^{2\pi ij/k})(1-2e^{-2\pi ij/k})\right| = \left|\prod_{j=0}^{k}(1-2e^{2\pi ij/k})\right|^2.$$

The factors in this product are exactly the roots of the polynomial $(x-1)^k - (-2)^k$. It follows that their product, up to sign, is the constant term of that polynomial, which is $2^k - 1$, establishing the result and confirming the above computation.

Moreover, it is easily verified that $\gcd(2^a-1, 2^b-1) = 2^{\gcd(a,b)}-1$. In particular, if k_1 and k_2 are coprime, then $H_1(L^{k_1}_{J(U)})$ and $H_1(L^{k_2}_{J(U)})$ are of coprime order. Thus as p_i ranges through the primes, there are arbitrarily large prime factors q_i of $2^{p_i}-1$, and projection following by multiplication by $(2^{p_i}-1)/q_i$ yields homomorphisms $\xi_i : H_1(L^{p_i}_K) \to C_{q_i}$, which are non-vanishing on at least one lift of η_1, $\eta_1 = (1,0)$ in the basis given above. It is this observation that makes the generalization of Theorem 5.13 possible.

5.3. J(trefoil) is not slice

Through explicit computations of signature invariants we will then show that the knot J(trefoil) is not a slice not, though it is algebraically slice. The rest of this section is devoted to proving this theorem.

Theorem 5.2. *The knot J(trefoil) is not slice.*

The first step is to associate to an arbitrary knot K an invariant $\tau_3(M_K, \phi)$. The sliceness of $J(K)$ will then place constraints on $\tau_3(M_K, \phi)$. By computing $\tau_3(M_K, \phi)$ in terms of the Tristram-Levine signatures of K, we will be able to show that J(trefoil) is not slice.

Choose manifolds $Z^{(r)}$ such that $\partial Z^{(r)} = 2^r \cdot M_K$ and such that $\phi : H_1(M_K) \to C_3$ extends to $\phi : H_1(Z^{(r)}) \to C_3$. Given this data, define the invariant

$$\tau_3(M_K, \phi) = \frac{1}{2^r}\left(\frac{1}{3}\sigma(\widetilde{Z}^{(r)}) - \sigma(Z^{(r)})\right),$$

where $Z^{(r)}$ is chosen to be a manifold with boundary 2^r copies of M_K, and $\widetilde{Z}^{(r)}$ is an induced (unbranched) triple cover whose boundary consists of 2^r copies of $\widetilde{M}_K$, the triple cover of M_K. A well-definedness computation is necessary.

Lemma 5.3. *$\tau_3(M_K, \phi)$ as defined is independent of r and the manifold $Z^{(r)}$.*

Proof. Let $Z_1^{(r_1)}$ and $Z_2^{(r_2)}$ be two such manifolds; assume without loss of generality that $r_2 \leq r_1$. Form $Z = Z_1^{(r_1)} \cup -2^{r_1-r_2} \cdot Z_2^{(r_2)}$ and the associated triple cover $\widetilde{Z} = \widetilde{Z}_1^{(r_1)} \cup 2^{r_1-r_2} \cdot -\widetilde{Z}_2^{(r_2)}$. Then $\widetilde{Z} \to Z$ is an unbranched 3-fold cover of 4-manifolds without boundary and so

$$\begin{aligned}\sigma(\widetilde{Z}) &= 3\sigma(Z)\\ \sigma(\widetilde{Z}_1^{(r_1)}) - 2^{r_1-r_2}\sigma(\widetilde{Z}_2^{(r_2)}) &= 3\left(\sigma(Z_1^{(r_1)}) - 2^{r_1-r_2}\sigma(Z_2^{(r_2)})\right)\\ \sigma(\widetilde{Z}_1^{(r_1)}) - \frac{1}{3}\sigma(Z_1^{(r_1)}) &= 2^{r_1-r_2}\left(\sigma(\widetilde{Z}_2^{(r_2)}) - \frac{1}{3}\sigma(Z_2^{(r_2)})\right)\end{aligned}$$

□

The approach to prove that the trefoil is not slice will be to compute $\tau_3(M_K, \phi)$ in two different ways. First, we show that it must vanish if $J(K)$ is slice, using techniques akin to those of Casson and Gordon [5] and a construction of Cochran, Harvey, and Leidy [6]. Second, we use the G-signature theorem to relate it to signatures of a branched triple cover (D^4, Δ) for K, giving a result in terms of the Tristram-Levine signatures of K. That this result does not vanish for K the trefoil then implies that J(trefoil) is not slice.

The first step in carrying out the former computation is to construct a cobordism E between M_K, $M_{J(K)}$, and $M_{J(U)}$, using a method of Cochran, Harvey, and Leidy [6]. Covers

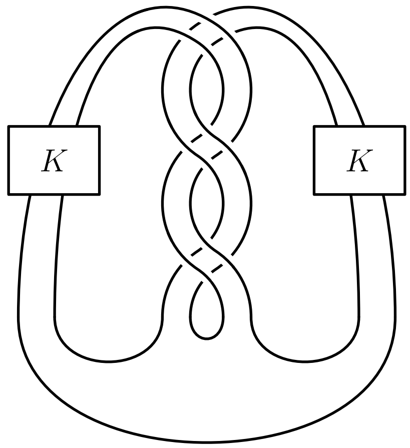

Figure 5.4: The knot $J(K)$ obtained by infection of 9_{46}.

of this space will eventually provide the manifolds $Z^{(r)}$. Form the union $M_R \times [0,1] \amalg -M_{K_1} \times [0,1] \amalg -M_{K_2} \times [0,1]$, where K_1 and K_2 are two copies of K, and identify $\eta_1 \times D^2$ and $\eta_2 \times D^2$ in $M_R \times \{1\}$ with tubular neighborhoods of K_1 and K_2 in $M_1 \times \{0\}$ and $M_2 \times \{0\}$ respectively. Recalling from Section 2.4 the construction of $J(K)$, it is clear that the resulting manifold has four boundary components, given exactly as

$$\partial E = M_{J(K)} \cup M_{J(U)} \cup -M_{K_1} \cup -M_{K_2}.$$

The cobordism E is diagrammed in Figure 5.5. The dashed arcs represent the solid tori $\eta_i \times D^2$.

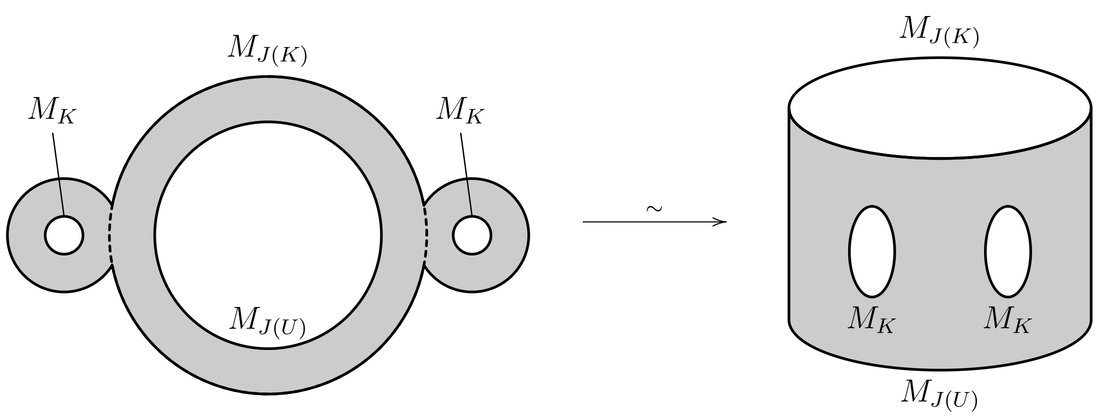

Figure 5.5: The cobordism E

We require some facts about the homology of E, captured in the following lemma.

Lemma 5.4 ([6])**.** *The inclusions of the boundary components into* E *induce isomorphisms* $H_1(M_{J(U)};\mathbb{Z}) \to H_1(E;\mathbb{Z})$, $H_1(M_{J(K)};\mathbb{Z}) \to H_1(E;\mathbb{Z})$, $H_2(M_{J(U)}) \oplus_i H_2(M_{K_I}) \to H_2(E)$, *and* $H_2(M_{J(K)}) \oplus_i H_2(M_{K_i}) \to H_2(E)$.

Proof. Observe that E deformation retracts to the space $\overline{E}$ obtained as $M_{J(U)} \cup \eta_1 \times D^2 \cup \eta_2 \times D^2$, where these solid tori are attached to $M_{J(U)}$ by gluing along the boundary to a tubular neighborhood of η_1 and η_2 in $M_{J(U)}$. Thus $\overline{E}$ is obtained from $M_{J(U)}$ by adjoining 2 two-cells and 2 three-cells with the two-cells as boundary. The isomorphisms $H_1(M_{J(U)}; \mathbb{Z}) \to H_1(E; \mathbb{Z})$ and $H_2(E) \cong H_2(M_L) \oplus_i H_2(M_{K_I})$ are then immediate.

For the other half, consider the Mayer-Vietoris sequence for the construction of E, which gives an exact sequence

$$H_2(M_R \times I) \oplus_i H_2(M_i \times I) \longrightarrow H_2(E) \longrightarrow H_1(\amalg \eta_i \times D^2) \longrightarrow H_1(M_R \times I) \oplus_i H_1(M_i \times I)$$

The key observation is that each of the maps $H_1(\eta_i \times D^2) \to H_1(M_i)$ is injective. This follows from the fact that the first homology of M_i is generated by a meridian of K. $\square$

Suppose now that $J(K)$ is slice. Then the slice knots $M_{J(K)}$ and $M_{J(U)}$ bound 4-manifolds $W_{J(K)}$ and $W_{J(U)}$, which may be taken as the complements in D^4 of neighborhoods slice disks for these knots. Let $Z = E \cup_\partial -W_{J(K)} \cup_\partial -W_{J(U)}$, so $\partial Z = M_K \cup M_K$.

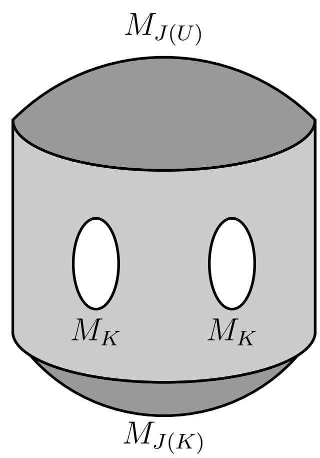

Figure 5.6: Z obtained by capping the boundary of E

Consider the map $\phi_r : H_1(E; \mathbb{Z}) \to C_{2^r}$. This induces a 2^r-fold covering space $E^{(r)}$ of E. Since $i_* : H_1(M_{J(U)}) \to H_1(W_{J(U)})$ is an isomorphism, the covering map extends to a cover $W^{(r)}_{J(U)} \to W_{J(U)}$ and $W^{(r)}_{J(K)} \to W_{J(K)}$ with boundary $M^{(r)}_{J(U)} \to M_{J(U)}$ and $M^{(r)}_{J(K)} \to M_{J(K)}$ by the first part of Lemma 5.4.

Gluing $E^{(r)}$, $-W^{(r)}_{J(U)}$, and $-W^{(r)}_{J(K)}$ along the common boundaries $M^{(r)}_{J(U)}$ and $M^{(r)}_{J(K)}$, we obtain a 2^r-fold cover $Z^{(r)} \to Z$, with $\partial Z^{(r)} = 2^{r+1} \cdot M_K$. Now, each of the maps $H_1(M_{K_i}) \to H_1(E)$ is zero, and so the boundary of $Z^{(r)}$ consists of 2^{r+1} copies of M_K. The spaces $E^{(r)}$ may be constructed straightforwardly. Let $M^{(r)}_{J(U)}$ denote the 2^r-fold cover of $M_{J(U)}$. The curves η_1 and η_2 have linking number 0 with $J(U)$, and so have 2^r lifts each in $M^{(r)}_{J(U)}$. Attach copies of M_K along tubular neighborhoods of these lifts along $M^{(r)}_{J(U)} \times [0, 1]$, as before. The resulting space is exactly the manifold $E^{(r)}$, with the covering map to E induced by the map $M^{(r)}_{J(U)} \to M_{J(U)}$.

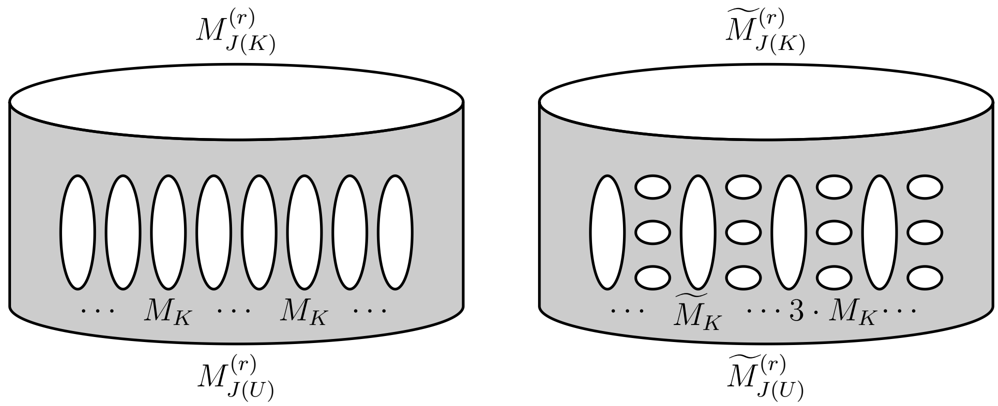

Figure 5.7: The spaces $E^{(r)}$ and $\widetilde{E}^{(r)}$

Let $\pi : H_1(M^{(1)}_{J(U)}) \to C_3$ be the projection map of Theorem 5.1. The compositions $\pi^{(r)} : H_1(M^{(r)}_{J(U)}) \to H_1(M^{(1)}_{J(U)}) \to C_3$ induce 3-fold covers $\widetilde{E}^{(r)}$ of $E^{(r)}$. The spaces $\widetilde{E}^{(r)}$ too may be constructed from covers of M_K. Let $\widetilde{M}^{(r)}_{J(U)}$ be the 3-fold cover of M_K induced by $\pi^{(r)}$ above, and form $\widetilde{M}^{(r)}_{J(U)} \times [0,1]$. Now, by the description of the map π computed in Section 5.2, exactly half of the 2^r lifts of η_1 in $M^{(r)}_{J(U)}$ lie in the kernel of $\pi^{(r)}$, and half of the lifts of η_2 in $M^{(r)}_{J(U)}$ do. Denote these by $\eta_1^1, \ldots, \eta_1^{2^{r-1}}$ and $\eta_2^1, \ldots, \eta_2^{2^{r-1}}$, and the lifts not in the kernel of $\xi_1^1, \ldots, \xi_1^{2^{r-1}}$ and $\xi_2^1, \ldots, \xi_2^{2^{r-1}}$. The lifts of η_1 and η_2 which lie in the kernel of $\pi^{(r)}$ each have three lifts in $\widetilde{M}^{(r)}_{J(U)}$. Attach a copy of $-\widetilde{M}_K$ along each $\xi_i^j \times \{1\}$, and three copies of $-M_K$ along the three lifts of $\eta_i^j \times \{1\}$ lying in the kernel of $\pi^{(r)}$. Thus the boundary of $\widetilde{E}^{(r)}$ consists of $3 \cdot 2^r$ copies of M_K, and 2^r copies of $\widetilde{M}_K$ as well as the components $\widetilde{M}^{(r)}_{J(U)}$ and $\widetilde{M}^{(r)}_{J(K)}$. The covering spaces $E^{(r)}$ and $\widetilde{E}^{(r)}$ are schematically illustrated in Figure 5.7.

To see that these covers extend to the caps $W^{(r)}_{J(U)}$ it is first necessary to describe the homology of the cobordisms $E^{(r)}$ and $\widetilde{E}^{(r)}$.

Lemma 5.5 ([6])**.** *The inclusions of the boundary components into $\widetilde{E}$ induce isomorphisms*

$$H_1(\widetilde{M}_{J(U)}) \to H_1(\widetilde{E}), \qquad H_2(\widetilde{M}_L) \oplus_i H_2(\widetilde{M}_{K_I}) \to H_2(E),$$
$$H_1(\widetilde{M}_{J(K)}) \to H_1(\widetilde{E}), \qquad H_2(\widetilde{M}_R) \oplus_i H_2(\widetilde{M}_{K_i}) \to H_2(E).$$

Similarly, the boundary inclusions induce isomorphisms

$$H_1(\widetilde{M}^{(r)}_{J(U)}) \to H_1(\widetilde{E}^{(r)}), \qquad H_2(\widetilde{M}^{(r)}_L) \oplus_i H_2(\widetilde{M}^{(r)}_{K_I}) \to H_2(\widetilde{E}^{(r)}),$$
$$H_1(\widetilde{M}^{(r)}_{J(K)}) \to H_1(\widetilde{E}^{(r)}), \qquad H_2(\widetilde{M}^{(r)}_R) \oplus_i H_2(\widetilde{M}^{(r)}_{K_i}) \to H_2(\widetilde{E}^{(r)}).$$

Proof. The proof is essentially the same as the one employed earlier. $E^{(r)}$ and $\widetilde{E}^{(r)}$ deformation retract onto $M^{(r)}_{J(U)}$ and $M^{(r)}_{J(K)}$ respectively, together with solid tori glued along the appropriate boundaries. A Mayer-Vietoris argument completes the proof. $\square$

From the construction in Theorem 2.3 it is apparent that after performing surgery along $\alpha_2 \in H_1(F)$, loop η_2 becomes homologically trivial in $D^4 \setminus \bar{\Delta}$. It follows that $H_1(W_{J(U)})$ is generated by a lift of η_1. It follows by Lemma 3.9 that $H_1(W_{J(U)})$ is of order 3 and is precisely the image under inclusion of the C_3 summand of $H_1(M^{(r)}_{J(U)})$ which does not lie in $\ker \pi$.

These remarks demonstrate that the maps $\pi^{(r)}$ induce 3-fold covers of $\widetilde{W}^{(r)}_{J(U)}$ and $\widetilde{W}^{(r)}_{J(K)}$ of $W^{(r)}_{J(U)}$ and $W^{(r)}_{J(K)}$. Attaching these to $\widetilde{E}^{(r)}$ along the common boundary components $\widetilde{M}^{(r)}_{J(U)}$ and $\widetilde{M}^{(r)}_{J(K)}$ we obtain covering spaces $\widetilde{Z}^{(r)} \to Z^{(r)}$. $Z^{(r)}$ is a manifold with boundary $2^{r+1} \cdot M_K$, and $\widetilde{Z}^{(r)}$ has $3 \cdot 2^r \cdot M_K$, and $2^r \cdot \widetilde{M}_K$. Let Q be a 4-manifold with boundary $-M_K$, and form $Y^{(r)}$ and $\widetilde{Y}^{(r)}$ by attaching Q to $Z^{(r)}$ and $\widetilde{Z}^{(r)}$ along the common boundary. Then $\partial \widetilde{Y}^{(r)} = 2^r \cdot \widetilde{M}_K$ and $\partial Y^{(r)} = 2^r \cdot M_K$. Since $E^{(r)}$ and $\widetilde{E}^{(r)}$ have only second homology from the boundary, it follows that $\sigma(\widetilde{E}^{(r)}) = \sigma(E^{(r)}) = 0$. Then by Novikov Additivity,

$$\begin{aligned}
\sigma(Y^{(r)}) &= \sigma(E^{(r)}) + \sigma(W^{(r)}_{J(U)}) + \sigma(W^{(r)}_{J(K)}) + \sigma(Q) \\
&= \sigma(W^{(r)}_{J(U)}) + \sigma(W^{(r)}_{J(K)}) + \sigma(Q) \\
\sigma(\widetilde{Y}^{(r)}) &= \sigma(\widetilde{E}^{(r)}) + \sigma(\widetilde{W}^{(r)}_{J(U)}) + \sigma(\widetilde{W}^{(r)}_{J(K)}) + 3 \cdot 2^r \sigma(Q) \\
&= \sigma(\widetilde{W}^{(r)}_{J(U)}) + \sigma(\widetilde{W}^{(r)}_{J(K)}) + 3 \cdot 2^r \sigma(Q).
\end{aligned}$$

Thus by definition,

$$\begin{aligned}
\tau_3(M_K, \phi) &= \frac{1}{2^r}\left(\frac{1}{3}\sigma(\widetilde{Y}^{(r)}) - \sigma(Y^{(r)})\right) \\
&= \frac{1}{3}\sigma(\widetilde{W}^{(r)}_{J(U)}) + \frac{1}{3}\sigma(\widetilde{W}^{(r)}_{J(K)}) - \sigma(W^{(r)}_{J(U)}) - \sigma(W^{(r)}_{J(K)}.
\end{aligned}$$

Now, by Lemma 3.7, $\dim H_2(V^{(r)}_{J(U)}; \mathbb{Q}) = 0$, and so $\dim H_2(W^{(r)}_{J(U)}; \mathbb{Q}) = 1$, and similarly $\dim H_2(W^{(r)}_{J(K)}; \mathbb{Q}) = 1$. Thus in order to show that $\sigma(Z^{(r)})$ is bounded in r, it would be sufficient to demonstrate that $\operatorname{rank} H_2(\widetilde{W}^{(r)}_{J(U)})$ and $\operatorname{rank} H_2(\widetilde{W}^{(r)}_{J(K)})$ are bounded in r. In fact, we will not show that these ranks are bounded: instead, it will be easier to use a different pair of caps for $E^{(r)}$ obtained through surgery on the interiors of $W^{(r)}_{J(U)}$ and $W^{(r)}_{J(K)}$. The proof of this fact mimics the proof of boundedness of an invariant $\sigma_r(M, \phi)$ given by Casson and Gordon [5].

Theorem 5.6. *Suppose that K is slice. Then $\tau_K(M_K, \phi) = 0$.*

Proof. Let $\Delta \subset D^4$ be a slice disk for K, and let $V^{(r)}_K$ denote the 2^r-fold cover of of D^4 branched over Δ. We have seen in Lemma 3.7 that $\widetilde{H}_*(V^{(r)}_K; \mathbb{Q}) \cong 0$.

Then consider $i_{r*} : H_1(L^{(r)}_K) \to H_1(V^{(r)}_K)$ induced by inclusion, where $L^{(r)}_K$ is the 2^r-fold cover of S^3 branched over K. Then $G = \ker i_{r*}$ satisfies $|G|^2 = |H_1(L^{(1)}_K)| = |TH_1(M^{(1)}_K)|$ by Lemma 3.9.

Now, our computations on branched covers of $(S^3, 9_{46})$ demonstrated that there is a map $\psi : H_1(V^{(r)}_K) \to \mathbb{Z}_3$ which commutes with the map ϕ described in the discussion of covers. Composition with the maps on homology induced by the coverings gives rise to $\phi_r : H_1(L_K^{(r)}) \to C_3$:

$$\begin{tikzcd}
H_1(L_K^{(r)};\mathbb{Z}) \arrow[r] \arrow[d] \arrow[dd, bend right=60, "\phi_r"'] & H_1(V_K^{(r)};\mathbb{Z}) \arrow[d] \arrow[dd, bend left=60, "\psi_r"] \\
H_1(L_K^{(1)};\mathbb{Z}) \arrow[r] \arrow[d, "\phi"] & H_1(V_K^{(1)};\mathbb{Z}) \arrow[d, "\psi"] \\
C_3 \arrow[r, equal] & C_3
\end{tikzcd}$$

Now, set $d_n = \dim H_1(V_K^{(r)};\mathbb{Z}_p)$. Performing surgery along representatives in the interior of $V_K^{(r)}$ for generators of the corresponding homology classes, we obtain $X_K^{(r)}$ satisfying $H_1(X_K^{(r)};\mathbb{Z}_3) \cong \mathbb{Z}_3$ and

$$\begin{tikzcd}
H_1(L_K^{(r)};\mathbb{Z}) \arrow[r, "i'_n*"] \arrow[d, "\phi_r"] & H_1(X_K^{(r)};\mathbb{Z}) \arrow[d, "\psi'_n"] \\
\mathbb{Z}_3 \arrow[r, equal] & \mathbb{Z}_3
\end{tikzcd}$$

Now, ψ'_n induces a 3-fold branched cover $\widetilde{X}_K^{(r)}$ of $X_K^{(r)}$, with boundary $\widetilde{L}_K^{(r)} \to \widetilde{L}_K^{(r)}$.

Since $V_K^{(r)}$ is a rational homology ball it has $\chi(V_K^{(r)}) = 1$. Then $\chi(W_K^{(r)}) = \chi(V_K^{(r)}) + 2(d_n - 1) = 2d_n - 1$. Thus $\chi(\widetilde{X}_K^{(r)}) = 3(2d_n - 1)$. Then by exactness of $H_1(\widetilde{X}_K^{(r)};\mathbb{Q}) \to H_1(\widetilde{X}_K^{(r)}, \partial\widetilde{X}_K^{(r)};\mathbb{Q}) \to H_0(\partial\widetilde{X}_K^{(r)};\mathbb{Q}) \to H_0(\widetilde{X}_K^{(r)};\mathbb{Q})$, where the last map is an isomorphism, we see that $H_1(\widetilde{X}_K^{(r)}, \partial\widetilde{X}_K^{(r)};\mathbb{Q}) \cong 0$, whence $\dim H_3(\widetilde{X}_K^{(r)};\mathbb{Q}) = 0$ according to Poincaré duality. It follows that

$$\dim H_2(\widetilde{X}_K^{(r)};\mathbb{Q}) = 2d_n - 2.$$

But note that $H_1(W_K^{(r)};\mathbb{Z}_3) = \operatorname{coker}(t^{2^n} - 1)$. So $d_n \leq 1 + \dim H_1(\widetilde{W}_K^{\infty};\mathbb{Z}_3) = d + 1$, for some finite d by Theorem 3.12. In particular, these are bounded, and so $\sigma(X_K^{(r)})$ are bounded as r increases. Form

$$\begin{aligned}
T^{(r)} &= E^{(r)} \cup 2^r \cdot Q \cup X_{J(K)} \cup X_{J(U)} \\
\widetilde{T}^{(r)} &= \widetilde{E}^{(r)} \cup 3 \cdot 2^r \cdot Q \cup \widetilde{X}_{J(K)} \cup \widetilde{X}_{J(U)}
\end{aligned}$$

This is the same as $Y_K^{(r)}$, but with the caps $W_K^{(r)}$ replaced by the surgered caps $X_K^{(r)}$, which have the same boundary. Then $\partial(\widetilde{X}_K^{(r)} \to X_K^{(r)}) = 2^r \cdot (\widetilde{M}_K \to M_K)$, and we obtain as before

$$\begin{aligned}
\tau_3(M_K, \phi) &= \frac{1}{2^r}\left(\frac{1}{3}\sigma(\widetilde{T}^{(r)}) - \sigma(T^{(r)})\right) \\
&= \frac{1}{2^r}\left(\frac{1}{3}\sigma(\widetilde{X}_{J(U)}^{(r)}) + \frac{1}{3}\sigma(\widetilde{X}_{J(K)}^{(r)}) - \sigma(X_{J(U)}^{(r)}) - \sigma(X_{J(K)}^{(r)})\right).
\end{aligned}$$

We have seen that all of these signatures are bounded in r, and passing to the limit as r goes to infinity, we obtain $\tau_3(M_K, \phi) = 0$. $\square$

Thus $\tau_3(M_K, \phi)$ vanishes if the knot $J(K)$ is slice. On the other hand, it is possible to compute $\tau_3(M_K, \phi)$ directly in terms of the Tristram-Levine signatures of K. We seek to relate $\tau_3(M_K, \phi)$ to the Tristram-Levine signatures of the knot K. The following result relates the signature of covers of covers of D^4 branched over a Seifert surface, and is due to work of Kauffman [18]. Let $\omega = \exp(2\pi i/3)$.

Lemma 5.7. *Let K be a slice knot.*

$$\tau_3(M_K, \phi) = \sum_{i=0}^{2} \sigma_{\omega^i}(K),$$

The approach is to compute the signature of a cover of D^4 branched over the push-in of a Seifert surface for K, in terms of the Tristram-Levine signatures of K. This computation is possible for any K, not necessarily assumed slice, and is due to a paper of Kauffman [18]. We then apply the G-signature theorem to relate this signature to the signature of a cover of D^4 branched over a slice disk for K, which in term may be used to compute the invariant $\tau_3(M_K, \phi)$.

Let $K \subset S^3 \subset D^4$ be a knot with Seifert surface F. Slide F into the interior of D^4 to obtain a copy of F near the boundary of D^4. Join the boundary of this copy to K; after smoothing, we obtain $\hat{F} \subset D^4$ with $\hat{F} \cap \partial D^4 = K$ and $\hat{F}$ diffeomorphic to F. Let G_K^a be the a-fold of D^4 branched over $\hat{F}$. In a slight abuse of notation, we will generally write F for the push-in $\hat{F}$ when no confusion is possible. The result of this computation is the following.

Lemma 5.8. *Let K be any knot, and G_K^a be constructed as above from any Seifert surface for K. Then*

$$\sigma(G_K^a) = \sum_{i=0}^{a-1} \sigma_{\omega^i}(K)$$

As a first observation, we verify that the signature of G_K^a does not depend on the choice of Seifert surface F: this will also follow once the computation is complete.

Proposition 5.9. *The signature $\sigma(G_K^a)$ does not depend on the choice of Seifert surface F.*

Proof. Suppose that $F' \subset D^4$ is another spanning surface. Pick $G \subset D^5$ such that $\partial G = F \cup -F' \subset D^4 \cup D^4 = S^4$. Construct a double cover of D^5 branched over G. Then $\partial G = G_{K,F}^a \cup -G_{K,F'}^a$. By the Novikov addition theorem and the fact that the signature of a boundary is 0, we have $\sigma(G_{K,F}^a) = \sigma(G_{K,F'}^a)$, establishing the required well-definedness. $\square$

To make this computation, it is useful to have an explicit construction of G_K^a. One way at this is constructing it by cutting and pasting, as we constructed branched covers of S^3 in Section 3, along the lines employed by Rolfsen [29]. Let $N : \mathring{F} \times (-1, 1) \to D^4$ be the push-in of a bicollar neighborhood for F in S^3, so that $\mathring{F} = N(\mathring{F} \times \{0\})$. Let W, W_- and W_+ denote these submanifolds of S^3:

$$\begin{aligned} W &= N(\mathring{F} \times (-1, 1)) \\ W_- &= N(\mathring{F} \times [0, 1)) \\ W_+ &= N(\mathring{F} \times (-1, 0]) \end{aligned}$$

There is an involution T which satisfies $T(W_\pm) = W_\mp$. Take balls $D_1, \ldots, D_a \cong D^4$. The ball D_i has subspaces W_-^i and W_+^i of its boundaries S_i^3. Form the disjoint union $D_1 \amalg \cdots D_a$ and identify W_-^i with W_+^{i+1} $(i = 1, \ldots, a-1)$ and W_-^{a-1} with $W + -^1$ under inclusion. In each case set $p \equiv \tau T p$, where T is the reflection defined earlier and τ sends a point to the corresponding point in the other ball. The branch locus is exactly the fixed set of T, i.e. $W_+ \cap W_- = W_0 = F \subset S^3$. There is a map $\tau : G_K^a \to G_K^a$ which sends a point in D_j to the corresponding point in D_{j+1}.

We now consider the homology structure of G_K^a. Let cF^j be the join of F with the center of D_j; this is the cone of the copy of F in D_j with the center of that ball. Consider $G_K^{a\prime} = CF^1 \cup \cdots CF^a \subset G_K^a$, and it is easy to see that G_K^a deformation retracts onto $G_K^{a\prime}$ and is thus homotopy equivalent to it. Note too that $\bigcap_{i=1}^a cF^j = F$.

Let $\alpha_1, \ldots, \alpha_{2g}$ generate $H_1(F; \mathbb{Z})$. Choosing representatives, one may regard these as embedded circles on F and form cones $c\alpha_i \subset D_1, \ldots, \tau^{a-1} c\alpha_i \subset D_a$. Define $\lambda : C_j(F) \to C_{j+1}(M)$ to send a cycle $\alpha_i \in H_1(F_a, \mathbb{Z})$ in to the associated sphere $\lambda(\alpha_i) = c\alpha_i - \tau c\alpha_i \subset G_K^a$. It is evident that $\lambda\alpha_i$ generate the homology group $H_2(G_K^{a\prime})$ and thus $H_2(G_K^a)$ since these spaces are homotopy equivalent.

Now define $\rho(\lambda\alpha) = c\alpha_+ - \tau c\alpha_-$. This is simply the result of perturbing the two cones translating the paths on its equator along the normal field to F. This clearly defines a cycle, for $\rho(\lambda\alpha)$ is homologous to $\lambda\alpha$ by the family $\rho_t(\lambda\alpha) = cN_{-t}\alpha - \tau cN_{+t}\alpha$.

It is then possible to compute directly the intersection numbers and see that these coincide with the Seifert pairing. Let $I : H_2(G_K^a) \times H_2(G_K^a) \to \mathbb{Z}$ denote the intersection form.

$$\begin{aligned} I(\lambda\alpha, \lambda\beta) &= I(\lambda\alpha, \rho(\lambda\beta)) = I(c\alpha - \tau c\alpha, c\beta_- - \tau c\beta_+) \\ &= I(c\alpha, c\beta_-) + I(c\alpha, c\beta_+) = \operatorname{lk}(\alpha, \beta_-) + \operatorname{lk}(\alpha, \beta_+) \\ &= \theta(\alpha, \beta) + \theta(\beta, \alpha). \end{aligned}$$

Similarly,

$$\begin{aligned} I(\lambda\alpha, \tau\lambda\beta) &= I(\rho(\lambda\alpha), \tau\lambda\beta) = I(c\alpha_-, \tau c\alpha_+, \tau c\beta - \tau^2 c\beta) \\ &= -I(\tau c\alpha_+, \tau c\beta) = -I(c\alpha_+, c\beta) = -\operatorname{lk}(\alpha_+, \beta) \\ &= -\operatorname{lk}(\alpha_+, \beta) = \theta(\alpha, \beta). \end{aligned}$$

At least we may compute $I(\tau\lambda\alpha, \lambda\beta) = -\theta(\beta, \alpha)$. Taken together, these computations establish that

$$I(\tau^i\lambda\alpha, \tau^j\lambda\beta) = \begin{cases} \theta(\alpha, \beta) + \theta(\beta, \alpha) & \text{if } i \equiv j \bmod a \\ -\theta(\alpha, \beta) & \text{if } i \equiv j - 1 \bmod a \\ -\theta(\beta, \alpha) & \text{if } i \equiv j + 1 \bmod a \end{cases}$$

We are primarily interested in relating the signatures of these covers to the signatures of K. Now, the covering transformation $\tau : H_2(G_K^a) \to H_2(G_K^a)$ is an isometry. Consider the complexification of I, a hermitian form obtained by defining $I(\lambda a, \mu b) = \lambda\bar{\mu}\, I(a, b)$. As usual, we consider the eigenspaces of the action and seek to understand the signatures on eigenspaces of the complexified intersection pairing I. Given $e \in H_2(G_K^a)$ and ω_a an a^{th} root of unity, define an eigenvector

$$V(\omega, e) = \sum_{j=0}^{a-1} \tau^j \bar{\omega}^j e.$$

This satisfies

$$\tau V(\omega, e) = \tau \left(\sum_{j=0}^{a-1} \tau^j \bar{\omega}^j e \right) = \sum_{j=0}^{a-1} \tau^{j+1} \bar{\omega}^j e$$
$$= \bar{\omega}^{a-1} + \sum_{j=0}^{a-2} \tau^{j+1} \bar{\omega}^j e = \omega \left(\sum_{j=0}^{a-1} \tau^j \bar{\omega}^j e \right),$$

as required. Moreover, the linear independence of the basis $\{\alpha_j\}$ for $\lambda H_1(F) \subset H_2(G_K^a)$ implies that $\{V(\omega, \alpha_j)\}$ is a linearly independent set of vectors. Therefore given ω an a^{th} root of unity, define

$$V(\omega) = \{(1 + \tau\bar{\omega} + \cdots + \tau^{a-1}\bar{\omega}^{a-1})e : e \in FH_1(F)\}.$$

This gives an decomposition of $H_2(G_K^a; \mathbb{C})$ into eigenspaces as

$$H_2(G_K^a; \mathbb{C}) \cong \bigoplus_{j=1}^{a-1} V(\omega^k).$$

It is then possible to compute directly the intersection numbers and see that these coincide with the Seifert form. The point is that the signatures of I on its eigenspaces are related to the Tristram-Levine signatures of the knot K. The following lemma makes this explicit.

Lemma 5.10. *The restriction of f to the eigenspace $V(\omega^k)$ is given by the matrix $a((1 - \omega^k)A + (1 - \omega^k)A^T)$.*

Proof. Let $\alpha, \beta \in \lambda H_1(F)$. By our earlier characterization of the homology of G_K^a, we can assume that $\alpha = \lambda A$ and $\beta = \lambda B$. Then using the relation between the intersection pairing and the Seifert form [29],

$$\begin{aligned}
I(V(\omega, \alpha), V(\omega, \beta)) &= I \left(\sum_{j=0}^{a-1} \tau^i \bar{\omega}^i \alpha, \sum_{j=0}^{a-1} \tau^j \bar{\omega}^j \beta \right) \\
&= \sum_{j=0}^{a-1} I(\tau\omega\alpha, \tau\omega\beta) + \sum_{j=0}^{a-1} I(\tau^j \bar{\omega}^j \alpha, \tau^{j+1} \bar{\omega}^{j+1} \beta + \sum_{j=0}^{a-1} I(\tau^{j+1} \bar{\omega}^{j+1} \alpha, \tau^j \bar{\omega}^j \beta) \\
&= a(I\alpha, \beta) + \omega I(\alpha, \tau\beta) + \bar{\omega} I(\tau\alpha, \beta)) \\
&= a(\theta(A, B) + \theta(B, A) - \omega\theta(A, B) - \bar{\omega}\theta(B, A) - \bar{\omega}\theta(B, A)) \\
&= a((1 - \omega)\theta(A, B) + (1 - \bar{\omega})\theta(A, B)).
\end{aligned}$$

Recall that the Tristram-Levine signatures are defined by

$$\sigma_\omega(K) = \sigma((1 - \omega)A + (1 - \bar{\omega})A^T).$$

Thus we have proved

$$\sigma(G_K^a) = \sum_{j=0}^{a-1} \sigma_{\omega^j}(K).$$

□

This $G_K^a \to D^4$ is a branched cover with boundary $L_K^a \to S^3$. These computations are valid for an arbitrary knot K; suppose now that K is slice and let $V_K^a \to D^4$ be the a-fold cover of D^4 branched over a slice disk Δ. Form the union $G_K^a \cup_\partial -V_K^a$, such that the two boundaries L_K^a are identified. This space is a cover of S^4 branched over $F \cup -\Delta$. Both F and Δ are bicollared and it follows that $F \cup -\Delta$ has self-intersection 0. Then by Corollary 4.14 of the G-signature theorem we obtain

$$\sigma(G_K^a \cup_\partial -V_K^a) = a \cdot \sigma(S^4) = 0.$$

In particular

$$\sigma(V_K^a) = \sigma(G_K^a) = \sum_{j=0}^{a-1} \sigma_{\omega^i}(K).$$

Now, form W_K^a by removing a regular neighborhood of the slice disk Δ from D^4, and its preimage $\widetilde{\Delta} \subset V_K^a$ under the covering map. The resulting space is an unbranched cover of $D^4 \setminus \Delta$. A Mayer-Vietoris argument demonstrates that $H_2(W_K^a) \cong H_2(V_K^a)$. In particular, $\sigma(V_K^a) = \sigma(W_K^a)$. Now, W_K^3 is a 3-fold cover of M_K, with boundary $\widetilde{M}_K \to M_K$, and thus may be employed in the computation of the invariant $\tau_3(M_K, \phi)$. By definition of that invariant, we have computed

$$\tau_3(M_K, \phi) = \sum_{j=0}^{2} \sigma_{\omega^i}(K).$$

Corollary 5.11. *Suppose that $\sigma(K) + \sigma_\omega(K) + \sigma_{\omega^2}(K) \neq 0$. Then $J(K)$ is not slice.*

Proof. We have already seen that $\tau_3(M_K, \phi) = 0$ if K is slice. On the other hand,

$$\tau_3(M_K, \phi) = \sum_{i=0}^{2} \sigma_{\omega^i}(K).$$

Thus if $J(K)$ is slice it must be that $\sigma(K) + \sigma_\omega(K) + \sigma_{\omega^2}(K) = 0$, where $\omega = \exp(2\pi i/3)$. $\square$

Corollary 5.12. *J(trefoil) is not slice.*

Proof. In light of the computations given earlier, the knot $K =$ trefoil has $\sigma_1(K) + \sigma_\omega(K) + \sigma_{\omega^2}(K) = 0 - 2 - 2 = -4$. But if J(trefoil) were slice, then $\tau_3(M_K, \phi)$ would vanish. $\square$

By similar means, one may recover a particular of a much more general theorem due to Cochran, Harvey, and Leidy [6].

Theorem 5.13. *Suppose that $\rho_0(K) \neq 0$. Then $J(K)$ is not slice.*

Proof. The computations of Section 5.2 show that $|H_1(L^r_{J(U)})| = (2^r - 1)^2$. Then there exist infinitely many pairs of primes (p, q) such that $q \mid |H_1(L^p_{J(U)})|$. We can then apply the arguments of the preceding section, with 2^r-fold covers replaced by p^r-fold ones, and the 3-fold covers of these replaced by q-fold ones. These are induced by the maps $H_1(L^p_{J(U)}) \to C_{2^p-1} \to C_q$. Although it will no longer be the case that exactly half the lifts of η_1 and η_2 lie in the kernel of this map, at least some positive proportion of them do, for lifts of each are the generators used in the description of $H_1(L^p_{J(U)})$; we cap off the rest with the manifold Q as before. By the same arguments, it will follow that if K is slice, it must be that $\tau_q(M_K, \phi) = 0$. But the computations about covers branched over Seifert surfaces are still valid, and so if

$$\tau_q(M_K, \phi) = \sum_i \sigma_{\omega^i}(K) \neq 0,$$

where ω is a q^{th} root of unity, it follows that the knot is not slice. On the other hand, we see that

$$\left| \rho_0(K) - \sum_i \sigma_{\omega^i}(K) \right| \leq \frac{NC}{q},$$

where N is the number of roots of the Alexander polynomial of K on the unit circle in $\mathbb{C}$, and $C = \sup_\omega \sigma_\omega(K)$. This follows from the fact that the signature σ_ω is piecewise constant on S^1 and changes at most N times. Noting that q may be taken arbitrarily large then proves the desired theorem. □

The result of Cochran, Harvey, and Leidy is much stronger: in fact, none of the knots $J(K)$, $J(J(K))$, $J(J(J(K)))$, ..., is slice, even though these have vanishing signature and Casson-Gordor invariants [6].

5.4. The Casson-Gordon Invariant

Through a computation along the lines of the one above, Casson and Gordon were able to prove that certain knots are algebraically slice but not slice. They went on to introduce an invariant $\tau(K, \chi)$ which is able to detect obstructions of this sort. We will briefly outline the construction and main results on this invariant. In the computations of the preceding result, we see that all necessary information is already captured in $M_K^{(1)}$, for the invariant $\tau_3(M_K, \phi)$ can be calculated from this alone. Passing to the covers $M_K^{(r)}$ is merely a computational tool. It is this observation that leads to the consideration of $\tau(K, \chi)$.

Let K be a knot and M_K be the result of nulhomologous surgery on S^3 along K. Then $H_*(M_K; \mathbb{Z}) \cong H_*(S^1 \times S^2; \mathbb{Z})$. Let M_K^n denote the n-fold cyclic cover of M associated to the unique homomorphism $f : \pi_1(M) \to H_1(M; \mathbb{Z}) \cong \mathbb{Z} \to C_n$, and let M_∞ denote the infinite cyclic cover. Similarly let L_K^n denote the n-fold branched cover of S^3 with branch locus K, and let $\chi : H_1(L_K^n) \to \mathbb{C}_m$ be a homomorphism, where $\mathbb{C}_m = \mathbb{C}(e^{2\pi i/m})$.

Since $H_1(M_K^n; \mathbb{Z}) \cong H_1(L_K^n; \mathbb{Z}) \oplus \mathbb{Z}$, the character χ induces m-fold coverings $\widetilde{M_K^n}$ of M_K^n for each n. Similarly there is an induced m-fold covering $\widetilde{M_K^\infty}$ of M_K^∞, with covering group $C_m \times C_\infty$ over M_K^n.

Now, the bordism group $\Omega_3(\mathbb{Z} \times C_m)$ is finite [8], and so there exists a 4-manifold W_K^n with $\partial W_K^n = r M_K^n$ for some finite r. The various coverings in question commute:

$$r \left(\begin{array}{ccc} \widetilde{M_K^\infty} & \longrightarrow & \widetilde{M_K^n} \\ \downarrow & & \downarrow \\ M_K^\infty & \longrightarrow & M_K^n \end{array} \right) = \partial \left(\begin{array}{ccc} \widetilde{W_K^\infty} & \longrightarrow & \widetilde{W_K^n} \\ \downarrow & & \downarrow \\ W_K^\infty & \longrightarrow & W_K^n \end{array} \right)$$

In the computations of the preceding section, the spaces $T^{(r)}$ realized the necessary bounding manifolds to define this invariant, but without the assumption that $J(K)$ is slice, these spaces are not very computationally useful.

Let $C_*(\widetilde{V}_\infty)$ be the simplicial chain complex of $\widetilde{V}_\infty$. This has the structure of a module over $\mathbb{Z}[C_m \times C_\infty] \cong \mathbb{Z}[\omega_m][t, t^{-1}]$. Adopting the notation of Casson-Gordon, write $k = \mathbb{Q}(\omega_m)$ and let $H_*(V_n; k(t))$ denote the homology of the twisted complex $C_*(\widetilde{V}_\infty) \otimes_{\mathbb{Z}[C_m \otimes C_\infty]} k(t)$. Since $k(t)$ is flat as a $\mathbb{Z}$-module,

$$H_*(W_K^n; k(t)) \cong H_*(\widetilde{W}_K^\infty; \mathbb{Z}) \otimes_{\mathbb{Z}[C_m \times C_\infty]} k(t).$$

Now, the intersection pairing

$$H_2(W_K^n; k(t)) \times H_2(W_K^n; k(t)) \to k(t)$$

is hermitian, and so defines an element of the Witt group of $k(t)$:

Definition 8. Let R be a ring. Then the *Witt group* $L_0(R)$ is the set of non-singular, hermitian forms on finite-dimensional R-modules, with two forms regarded as equivalent if one can be obtained from the other by addition or subtraction of copies of the hyperbolic pairing $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

This $k(t)$-valued intersection pairing on $H_2(V_n; k(t))$ then defines an element of $L_0(k(t))$ as long as it is nonsingular. If $m = p^k$ is a prime-power, then it in fact is [5]. Let $t(V_n)$ denote this element. Define $t_0(V_n)$ to be the untwisted intersection pairing on $H_2(V_n; \mathbb{Q})$, which can also be regarded as an element of $L_0(k(t))$. Then the Casson-Gordon invariant is defined as

$$\tau(K, \chi) = \frac{1}{r}\left(t(V_n) - t_0(V_n)\right)) \in L_0(k(t)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

With this definition, the main theorem giving obstructions to sliceness describes the vanishing of $\tau(K, \chi)$ on certain submodules of $H_1(L_K^n)$.

Theorem 5.14 ([4])**.** *Let K be a knot and L_K^n the n-fold branched cover of (S^3, K). If K is slice then there is a submodule $G \subset H_1(L_K^n)$ on which the torsion linking pairing vanishes and such that $\tau(K, \chi) = 0$ for every character χ of prime-power order which vanishes on G.*

Proof. (sketch) The vanishing of the torsion linking pairing was observed in Theorem 3.5. Let $\chi : H_1(L_K^n) \to \mathbb{C}^*$ be a character with order m, vanishing on G. Then one checks that this extends to a character $\chi' : H_1(V_K^n) \to \mathbb{C}^*$ order order m^ℓ for some ℓ, such that the following commutes.

$$\begin{tikzcd}
H_1(M_K^n) \arrow[r] \arrow[d] & H_1(L_K^n) \arrow[r] \arrow[d] & \mathbb{C}^* \\
H_1(W_K^n) \arrow[r] & H_1(V_K^n) \arrow[ur] &
\end{tikzcd}$$

Then χ' induces m-fold covers $\widetilde{V}_K^n$ and $\widetilde{W}_K^n$. The point is that the manifold W_K^n can be used in the computation of $\tau(K, \chi)$ even though its boundary is a disconnected covering space. Suppose M_K^n bounds $W_K^{n\prime}$, and let $\widetilde{W}_K^{n\prime}$ and $\widetilde{W}_K^{\infty\prime}$ be the induced m^ℓ-fold covers. Then

$$t(W_K^n) - t(W_K^{n\prime}) = t(W_K^n \cup -W_K^{n\prime}) = t_0(W_K^n \cup -W_K^{n\prime})$$
$$t(W_K^n) - t_0(W_K^n) = t(W_K^{n\prime}) - t_0(W_K^{n\prime}).$$

This computation is in the Witt ring $L_0(\bar{k}(t))$ with $\bar{k} = \mathbb{Q}(C_{m^\ell})$, where $\bar{k}(t)$ is a degree $m^{\ell-1}$. There exists a map $L_0(k(t)) \to L_0(\bar{k}(t))$ given by $\cdot \otimes_{k(t)} \bar{k}(t)$, and $t(W_K^{n\prime}) - t_0(W_K^{n\prime})$ is the image of $\tau(K,\chi)$ under this map. A quick algebraic verification shows that the map is injective.

If m is a prime power, then $\widetilde{W}_K^\infty$ is a m^ℓ-fold cover of W_K^∞, which is an infinite cyclic cover of $D_4 \setminus \bar{\Delta}$. The results of Section 3.4 show that $H_2(W_K^\infty;\mathbb{Q})$ is of finite dimension over $\mathbb{Q}$ and thus torsion over $\mathbb{Z}[t,t^{-1}]$. It follows that

$$H_2(W_K^n; k(t)) \cong H_2(\widetilde{W}_K^\infty;\mathbb{Z}) \otimes_{\mathbb{Z}[C_m \times C_\infty]} k(t) = 0.$$

So $t(W_K^n) = 0$, and since $H_2(W_K^n;\mathbb{Q}) = 0$ it follows that $t_0(W_K^n) = 0$. Thus $\tau(K,\chi) = 0$. $\square$

To prove that a knot is slice but not algebraically slice, one must compute $\tau(K,\chi)$, and show too that it is non-vanishing on the appropriate submodules. The usual way to detect nonzero elements of the Witt group is by computing signatures of these hermitian forms. Indeed, for $\xi \in \mathbb{C}$ with $|\xi| = 1$, define

$$\sigma_\xi(\tau(K,\chi)) = \sigma(\tau(K,\chi)(\xi)).$$

In the case that $\tau(\xi)$ is singular, one instead wishes to consider the average of the two one-sided limits of this signature. Casson and Gordon demonstrate the computation of an approximation of $\sigma_1\tau(K,\chi)$ by surgery techniques. By means of an explicit description of the spaces L_n, they show that certain twisted doubles of the unknot are slice but not algebraically slice.

The example of J(trefoil) and similar ones obtained by the infection construction have figured prominently in a number of examination of the concordance group [25],[12]. The first approach to this class of knots was due to Litherland, who proved the following result relating the Casson-Gordon invariants of an infected knot to the Casson-Gordon invariants of the original knot and the Tristram-Levine signatures of the infecting knot K.

Let η be unknotted in $S^3 \setminus K$. Then η has q' lifts to L_q, where $q' = \gcd(q, \mathrm{lk}(\eta,K))$.

Theorem 5.15 (Litherland [25])**.** *The signature of the infected knot is given by*

$$\sigma(J(K),\chi') = \sigma(K(U),\chi) + \sum_{i=1}^{q} \sigma_{\chi(U_i)/p}(K).$$

To apply this to the situation of J(trefoil), we observe that the branched double cover of $(S^3, J(U))$ has homology $\mathbb{Z}_3 \oplus \mathbb{Z}_3$; let h_1 and h_2 be generators. Our description in Section 5.2 of the lifts of the paths η_1 and η_2, together with Theorem 5.14 then implies that either $\sigma(K,\chi_1)$ or $\sigma(K,\chi_2)$ vanishes, and so

$$0 = \sigma(J(K),\chi') = \sigma(J(U),\chi) + 2\sigma_{\exp(2\pi i/3)}(K),$$

Thus $2\sigma_{\exp(2\pi i/3)}(K) = 0$. By the computation illustrated in Figure 4.1, this is not the case: we conclude that J(trefoil) is not slice.

6. Afterword

6.1. Introduction

The construction of non-slice but algebraically slice knots by Casson and Gordon revealed that algebraic sliceness was not equivalent to sliceness in dimension $n = 1$, and the next question was naturally whether there exist knots which are not slice but have vanishing Casson-Gordon invariants. The next important work on the concordance group was the construction of a filtration of the concordance group $\mathcal{C}_1^3$ by Cochran, Orr, and Teichner. We seek here to give a brief introduction to the construction, geometric characterization, and main results of this filtration, at the cost of detailed treatments of the techniques employed. The exposition of this section draws from the original works of the authors as well as the discussion of Friedl and Livingston [7],[11],[26]. The methods of this section apply to the topological knot concordance group, in which the slice disk Δ is required only to be locally flat, rather than smooth; the methods of Casson and Gordon may similarly be extended to this context using a version of the G-signature theorem valid in the topological category.

6.2. The Cochran-Orr-Teichner Filtration

A classical result equivalent to the Levine condition arises in the context of the Cappell-Shaneson surgery program.

Theorem 6.1 ([7]). *A knot K is algebraically slice if and only if $M = M_K$ bounds a compact spin manifold W such that the inclusion induces an isomorphism $i_* : H_1(M;\mathbb{Z}) \to H_1(W;\mathbb{Z})$ in homology and that the intersection form λ_1 defined above has a submodule on which λ_1 vanishes and whose image in $H_2(W;\mathbb{Z})$ is a Lagrangian submodule.*

Consider the lower derived series of a group G defined by $G^{(0)} = G$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$. A group is called (n)-solvable if $G^{(n)}$ is trivial. Given a manifold W, let $W^{(n)}$ denote the covering space arising from the subgroup $\pi_1(W)^{(n)}$. There exists an intersection form

$$\lambda_n : H_2(W^{(n)}) \times H_2(W^{(n)}) \to \mathbb{Z}[\pi_1(W)/\pi_1(W)^{(n)}]$$

and a related pairing, the self-intersection form

$$\mu_n : H_2(W^{(n)}) \times H_2(W^{(n)}) \to \mathbb{Z}[\pi_1(W)/\pi_1(W)^{(n)}].$$

A submodule $L \subset H_2(W^{(n)})$ is called a *Lagrangian submodule* if both μ_n and λ_n vanish on L, and $i_{n*}(L) \subset H_2(W)$ is a submodule on which the intersection pairing $\lambda_0 : H_2(W) \times H_2(W) \to \mathbb{Z}$ vanishes. Two Lagrangian submodules are called *dual* if the the pairings λ_n and μ_n on the two are nonsingular, and the descend to a generating set of $H_2(W;\mathbb{Z})$.

The filtration of the concordance group is then defined by

Definition 9. A knot K is (n)-solvable if it bounds a spin 4-manifold W such that the map $H_1(M_K) \to H_1(W)$ induced by inclusion is an isomorphism and there are two dual (n)-Lagrangians on W. K is called $(n.5)$-solvable if there is an (n)-Lagrangian in $H_2(W^{(n)})$ which is dual to the image of some Lagrangian in $H_2(W^{(n+1)})$.

Two observations follow immediately from this definition.

Theorem 6.2. *A slice knot is* (n)*- and* $(n.5)$*-solvable for all* n*.*

Proof. The complement N_Δ of a neighborhood of the slice disk Δ has the homology type of $S^1 \times S^3$. In particular, $H_2(N_\Delta) = 0$ and so N_Δ is an (n)- and $(n.5)$-solution of K for all n. $\square$

Theorem 6.3. *If* K *is* (h)*-solvable, then it is* (k)*-solvable for all* $k < h \in \frac{1}{2}\mathbb{Z}$*.*

Proof. This is a consequence of the naturality of covering spaces and homology: the image of an (k)-Lagrangian in $H_2(W^{(h)})$ is an (h)-Lagrangian. $\square$

This filtration corresponds neatly to many classical concordance invariants.

Theorem 6.4.

1. *K is (0)-solvable if and only if it has vanishing Arf invariant, or equivalently, the Alexander polynomial* $\Delta_K(t) \equiv \pm 1 \bmod 8$.

2. *K is (0.5)-solvable if and only if it is algebraically slice.*

3. *If K is (1.5)-solvable then it has vanishing Casson-Gordon invariants. However, there exist knots with vanishing Casson-Gordon invariants which are not (1.5)-solvable.*

The geometric interpretation of the Cochran-Orr-Teichner filtration arises from a construction known as a *grope* [7]. The motivation for this construction is simple: a map $\phi : S^1 \to M$ represents a class in $\pi_1(M)$, and is trivial if and only if the map extends to a map $D^2 \to M$. Similarly, the map lies in the commutator subgroup of $\pi_1(M)^{(1)}$ if and only if ϕ extends to an embedding of a surface with a disk removed: this is evident from the fact that the commutator subgroup is generated by the elements $aba^{-1}b^{-1}$, the product of g of which is the word along the boundary of a g-gon whose quotient is a surface. Generally, a grope is constructed to be a space such that ϕ extends to a grope of height h if and only if there the corresponding element of $\pi_1(M)$ lies in $\pi_1(M)^{(h)}$.

A grope is then mostly simply defined inductively. A grope of height 1 is just a surface with a disk removed. Given a grope of height h, a grope of height $h+1$ is formed by attaching surfaces (with disks removed) along each element of a symplectic basis for the grope homology of the height h grope. The full complex thus obtained is not a manifold, but has simple singularities. The slogan motivating this construction, and its applications in topology, is "if you are looking for a disk, try to find a grope first" [31].

This construction underlies the geometrical interpretation of the Teichner-Cochran-Orr filtration.

Theorem 6.5 ([7])**.** *A knot is slice if it bounds a grope of height* 0*, and more generally it lies in the graded piece* $\mathcal{F}_n$ *if and only if it bounds a grope of height* n *embedded in the interior of* D^4 *and with boundary* K*.*

6.3. Detecting Sliceness Obstructions

Just as the G-signature proved to be a powerful analytic tool making possible the computations of Casson and Gordon, so other invariants with their roots in analysis are at the heart of the work of Cochran, Orr, and Teichner. Associated to a three-manifold M with a homomorphism $\phi : \pi_1(M) \to \Gamma$, there is a real-valued *von Neumann ρ-invariant* defined by Cheeger and Gromov. In the case that (M, ϕ) is the boundary of a pair (W, ψ) for a compact, oriented 4-manifold W and $\psi : \pi_1(W) \to \Gamma$, then $\rho(M, \phi) = \sigma_\Gamma^{(2)}(W, \psi) - \sigma(W)$, where $\sigma_\Gamma^{(2)}$ is the L^2 *signature* intersection form on the $H_2(W; \mathbb{Z}\Gamma)$ [11].

Cochran, Orr, and Teichner work in the case that Γ is a *poly-torsion free abelian* group, in which case it may be verified that Γ has a skew field of fractions $\mathcal{K}\Gamma$. The utility of such coefficient systems lies in the following theorem, which allows the detection of slice knots.

Theorem 6.6 ([7], Theorem 4.2)**.** *Suppose that K is topologically slice, that Γ is a PTFA group, and that $\phi : \pi_1(M_K) \to \Gamma$ is a homomorphism. If ϕ extends to $\psi : \pi_1(N_\Delta) \to \Gamma$, then $\rho(M_K, \phi) = 0$.*

Just as Casson and Gordon made use of the torsion linking pairing, and Letsche of the Blanchfield pairing, to detect sliceness obstructions, so Cochran, Orr, and Teichner consider intersection forms on so-called *rationally universal n-solvable knot groups*, taking images in generalizations of the quotients $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{Q}(t)/\mathbb{Q}[t, t^{-1}]$ in which these other pairings reside. These generalized Blanchfield pairings are constructed inductively. $\mathcal{A}_0 = H_1(M; \mathcal{R}_0)$ is the usual Alexander module, with dual

$$\mathcal{A}_0^\sharp = \mathrm{Hom}_{\mathcal{R}_0}(\mathcal{A}_0, \mathcal{K}_0/\mathcal{R}_0).$$

The Blanchfield form is $Bl : \mathcal{A}_0 \times \mathcal{A}_0 \to \mathcal{K}_0/\mathcal{R}_0$. Cochran, Orr, and Teichner demonstrate that there is a bijective correspondence between $\mathcal{A}_0^\sharp$ and $\mathrm{Rep}^*_{\Gamma_0}(\pi_1(M), \Gamma_1)$, where these are a particular class of representations of the knot group into Γ_n. Then one sets $\mathcal{A}_1 = \mathcal{A}_1(a_0) = H_1(M; \mathcal{R}_1)$, and with this module in hand one constructs $Bl_1 : \mathcal{A}_1 \to \mathcal{A}_1^\sharp = \mathrm{Hom}_{\mathcal{R}_1}(\mathcal{A}_1, \mathcal{K}_1/\mathcal{R}_1)$. This is turn induces a correspondence $\mathcal{A}_1 \longleftrightarrow \mathrm{Rep}^*_{\Gamma_1}(\pi_1(M), \Gamma_2)$. Inductively one constructs a Blanchfield pairing $Bl_{n-1} : \mathcal{A}_{n-1} \to \mathcal{A}_{n-1}^\sharp$. The key application of these pairings is in detecting nonslice knots, as in the approach of Casson and Gordon. The general approach to obstructing (n)-solvability is that if a knot K is (n)-solvable, with W an (n)-solution, then $1/2^n$ of the representations of $\pi_1(M_K) \to \Gamma_n$ may be factored through $\pi_1(W)$.

The following properties of the von Neumann invariants are proved in Sections 4 and 5 of [7].

Theorem 6.7 ([6])**.**

1. *If ϕ is trivial then $\rho(M, \phi) = 0$.*
2. *If ϕ factors as $\phi : \pi_1(M) \to \Gamma' \to \Gamma$ for some subgroup $\Gamma' \subset \Gamma$, then $\rho(M, \phi) = \rho(M, \phi')$.*
3. *If $M = M_K$ and $\phi : \pi_1(M_K) \to \mathbb{Z}$ is the abelianization, then*

$$\rho(M_K, \phi) = \rho_0(K) = \int_{S^1} \sigma_\omega(K)\, d\omega.$$

It is useful to understand how the invariants $\rho(M_K, \phi)$ behave under the infection operation. Cochran, Harvey, and Leidy prove the following result, along the lines of a theorem of Litherland for the Tristram-Levine signatures [25]. Using this technique, they proceed to demonstrate that all gradings $\mathcal{F}_n/\mathcal{F}_{n.5}$ are infinitely generated.

First it is necessary to understand how a map $\phi : \pi_1(M_L) \to \Gamma$ extends to the other components of the boundary of the cobordism E. Suppose that L is obtained by infection of R by K_1, K_2 along η_1 and η_2. Suppose $\phi : \pi_1(M_L) \to \Gamma$ is a map such that the longitudes ℓ_1 and ℓ_2 lie in the kernel. This restrictions to $\phi : \pi_1(S^3 \setminus K_1)$, and since ℓ_1 is in the kernel of ϕ, induces a map $\phi_1 : M_{K_1} \to \Gamma$, and similarly for K_2. Moreover, ϕ induces a map on $\pi_1(M_R \setminus (\eta_1 \amalg \eta_2))$ and so $\phi_R : M_R \to \Gamma$.

Theorem 6.8 ([6])**.** *The maps described above satisfy*

$$\rho(M_L, \phi) - \rho(M_R, \phi_R) = \rho(M_{K_1}, \phi_1) + \rho(M_{K_2}, \phi_2).$$

Proof. Let E be the cobordism of Section 5 and $\bar{E}$ the manifold described there. By assumption on ϕ it extends to $\bar{\phi} : \pi_1(\bar{E}) \to \Gamma$ and thus $\bar{\phi} : \pi_1(E) \to \Gamma$). The restrictions to the boundary components coincide with ϕ_1 and ϕ_2. Thus we have

$$\rho(M_L, \phi) - \rho(M_R, \phi_R) - \rho(M_{K_1}, \phi_1) - \rho(M_{K_2}, \phi_2) = \sigma^{(2)}_\Gamma(E, \bar{\phi}) - \sigma(E).$$

Arguments similar to those employed in Section 5 show that because all homology of E is from the boundary, these signatures are 0, implying the result. ☐

Cochran, Harvey, and Leidy go on to construct the *first-order signatures* of a knot K. Let $\mathcal{A}_0 = \mathcal{A}_0(K)$ denote the Alexander module, which may be regarded as

$$\mathcal{A}_0 = G^{(1)}/G^{(2)} \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Q}[t, t^{-1}].$$

The submodules $P \subset \mathcal{A}_0$ correspond to quotients $\phi_P : G \to G/\widetilde{P}$ by means of the correspondence

$$\widetilde{P} = \{x : x \in \ker(G^{(1)} \to G^{(1)}/G^{(2)} \to \mathcal{A}_0/P)\}.$$

Thus to each submodule $P \subset \mathcal{A}_0$ there is an associated von Neumann invariant $\phi(M_K, \phi_P : G \to G/\widetilde{P})$.

Definition 10. The invariant $\rho(M_K, \phi)$ is a *first-order signature* of K if ϕ factors through $G/G^{(2)}$ and $\ker(\phi) = \ker(G^{(1)} \to G^{(1)}/G^{(2)} \to \mathcal{A}_0/P)$ for some submodule $P \subset \mathcal{A}_0$ on which $P \subset P^\perp$ with respect to the Blanchfield pairing. In the case that $P = 0$ this invariant is $\rho(M_k, \phi : G \to G/G^{(2)}) = \rho^1(K)$.

The main result on these first-order signatures is that if K is slice, then there is a submodule P as above such that the associated first-order signature of K associated to P vanishes.

A genus one algebraically slice knot has only two metabolizers for the Seifert form, and thus three first-order signatures, which correspond to $\langle \alpha_1 \rangle$, $\langle \alpha_2 \rangle$, and 0. By the preceding lemma, we have $\rho(M_K, \phi_P) = \rho(M_R, \phi_P) + \epsilon_P^{\alpha_1} \rho_0(K_1) + \epsilon_P^{\alpha_2} \rho_0(K_2)$, where ϵ_P^α is 0 or 1 depending on whether $\phi_P(\alpha) = 1$ or not. In particular, we have $\phi_{P_{\alpha_i}}(\alpha_j) = \delta_{ij}$. Thus one finds that the first order signatures of this knot are $\rho_0(K_1)$, $\rho_0(K_2)$, and $\rho^1(R) + \rho_0(K_1) + \rho_0(K_2)$.

Cochran, Harvey, and Leidy then prove that if $J(K)$ is slice, then one of the first-order signatures of K vanishes. This implies in particular that if $J(J(K))$ is slice, then $\rho_0(K) \in \{0, -\frac{1}{2}\rho^1(J(U))$. Here $\rho^1(J(U))$ is some (yet-uncomputed) invariant of 9_{46}. They then more generally consider the sequence of knots J_n defined by $J_{n+1} = J(J_n)$ for some fixed J_0. If J_0 is slice, then each J_n is by Corollary 2.4. The problem of whether the J_n would ever be slice for fixed non-slice J_0 (and inparticular J_0 the trefoil) remained open until the advent of the methods of Cochran-Orr-Teichner. This problem was settled in the positive by the following theorem.

Theorem 6.9 ([6])**.** *There exists a constant C such that if $|\rho_0(J_0)| > C$, then each J_n is of infinite order in the concordance group.*

Moreover, the knot J_n lies in the component $\mathcal{F}_n$ of the Cochran-Orr-Teichner filtration. By considering the knots J_n for a variety of choices of J_0, and proving their independence in the concordance group, Cochran-Harvey-Leidy demonstrate the following.

Theorem 6.10 ([6])**.** *The quotient $\mathcal{F}_n/\mathcal{F}_{n.5}$ is infinitely generated for all $n \geq 0$.*

This is an improvement on the result of Cochran, Orr, and Teichner, which demonstrated only that this filtration was nontrivial, and relied on deep analytic results concerning the ρ-invariants.

Bibliography

[1] M. F. Atiyah and I. M. Singer. The index of elliptic operators. I. *Ann. of Math. (2)*, 87:484–530, 1968.

[2] M. F. Atiyah and I. M. Singer. The index of elliptic operators. III. *Ann. of Math. (2)*, 87:546–604, 1968.

[3] Glen E. Bredon. *Introduction to compact transformation groups.* Academic Press, New York, 1972. Pure and Applied Mathematics, Vol. 46.

[4] A. J. Casson and C. McA. Gordon. On slice knots in dimension three. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2*, Proc. Sympos. Pure Math., XXXII, pages 39–53. Amer. Math. Soc., Providence, R.I., 1978.

[5] A. J. Casson and C. McA. Gordon. Cobordism of classical knots. In *À la recherche de la topologie perdue*, volume 62 of *Progr. Math.*, pages 181–199. Birkhäuser Boston, Boston, MA, 1986.

[6] Tim D. Cochran, Shelly Harvey, and Constance Leidy. Knot concordance and higher-order Blanchfield duality. *Geometry and Topology*, 13(3):1419–1482, 2009.

[7] Tim D. Cochran, Kent E. Orr, and Peter Teichner. Knot concordance, Whitney towers and l^2-signatures. *Ann. of Math. (2)*, 157(2):433–519, 2003.

[8] P. E. Conner and E. E. Floyd. The SU-bordism theory. *Bull. Amer. Math. Soc.*, 70:670–675, 1964.

[9] S. K. Donaldson and P. B. Kronheimer. *The geometry of four-manifolds.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1990. Oxford Science Publications.

[10] Ralph H. Fox and John W. Milnor. Singularities of 2-spheres in 4-space and cobordism of knots. *Osaka J. Math.*, 3:257–267, 1966.

[11] Stefan Friedl. Eta invariants as sliceness obstructions and their relation to Casson-Gordon invariants. *Algebr. Geom. Topol.*, 4:893–934 (electronic), 2004.

[12] Patrick M. Gilmer. Slice knots in S^3. *Quart. J. Math. Oxford Ser. (2)*, 34(135):305–322, 1983.

[13] C. McA. Gordon. Some aspects of classical knot theory. In *Knot theory (Proc. Sem., Plans-sur-Bex, 1977)*, volume 685 of *Lecture Notes in Math.*, pages 1–60. Springer, Berlin, 1978.

[14] C. McA. Gordon. On the G-signature theorem in dimension four. In *À la recherche de la topologie perdue*, volume 62 of *Progr. Math.*, pages 159–180. Birkhäuser Boston, Boston, MA, 1986.

[15] Joshua Greene and Stanislav Jabuka. The slice-ribbon conjecture for 3-stranded pretzel knots. Preprint, 2007. math.GT/0706.3398.

[16] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1994.

[17] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[18] Louis H. Kauffman. Branched coverings, open books and knot periodicity. *Topology*, 13:143–160, 1974.

[19] Louis H. Kauffman. *On knots*, volume 115 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 1987.

[20] Michel A. Kervaire. Les nœuds de dimensions supérieures. *Bull. Soc. Math. France*, 93:225–271, 1965.

[21] Michel A. Kervaire and John W. Milnor. On 2-spheres in 4-manifolds. *Proc. Nat. Acad. Sci. U.S.A.*, 47:1651–1657, 1961.

[22] Carl F. Letsche. An obstruction to slicing knots using the eta invariant. *Math. Proc. Cambridge Philos. Soc.*, 128(2):301–319, 2000.

[23] J. Levine. Knot cobordism groups in codimension two. *Comment. Math. Helv.*, 44:229–244, 1969.

[24] Paolo Lisca. Lens spaces, rational balls and the ribbon conjecture. *Geom. Topol.*, 11:429–472, 2007.

[25] R. A. Litherland. Cobordism of satellite knots. In *Four-manifold theory (Durham, N.H., 1982)*, volume 35 of *Contemp. Math.*, pages 327–362. Amer. Math. Soc., Providence, RI, 1984.

[26] Charles Livingston. A survey of classical knot concordance. In *Handbook of knot theory*, pages 319–347. Elsevier B. V., Amsterdam, 2005.

[27] John Milnor. *Lectures on the h-cobordism theorem.* Notes by L. Siebenmann and J. Sondow. Princeton University Press, Princeton, N.J., 1965.

[28] V. A. Rohlin. Two-dimensional submanifolds of four-dimensional manifolds. *Funkcional. Anal. i Priložen.*, 5(1):48–60, 1971.

[29] Dale Rolfsen. *Knots and links*, volume 7 of *Mathematics Lecture Series.* Publish or Perish Inc., Houston, TX, 1990.

[30] Peter Teichner. Knots, von Neumann signatures, and grope cobordism. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pages 437–446, Beijing, 2002. Higher Ed. Press.

[31] Peter Teichner. What is…a grope? *Notices of the AMS*, 51:894–895, 2004.