SOME CONSTRUCTIONS OF IRREDUCIBLE REPRESENTATIONS OF GENERIC HECKE ALGEBRAS OF TYPE A_n

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Contents

Table of Symbols in Order of Appearance	
S_n	Symmetric Group
$s_i \in S$	Simple Reflections
l	Length function
$\lambda \vdash n$	λ is a partition of n
T	Tableau
SYT_λ	Standard Young Tableaux of λ
$D(T)$	Descent Set of a Tableau
ρ_λ	Classical Representation
$\mathbb{C}[G]$	Group Algebra
$\mathrm{End}_k(V)$	Endomorphisms over k of V
Res_H^G	Restriction
W_J	Parabolic Subgroup
$\tilde{\mathcal{H}}_W(q)$	Generalized Hecke Algebra
W	Coxeter Group
T_w	Basis element of $\mathcal{H}_W(q)$
$\mathcal{H}_W(q)$	Hecke Algebra
Ω	The roots of 1 without 1 and with 0 in $\mathbb{C}$
$\mathrm{Mat}_\mu(k)$	Full matrix algebra over k of $\mu \times \mu$ matrices
T	Reflections
$<$	Bruhat Order (in addition to usual use with $\mathbb{R}$)
$w = s_1 \ldots s_r$	w as a reduced word
$\mathcal{D}_n$	Dihedral Group
ϵ_w	$(-1)^{l(w)}$
q_w	$q^{l(w)}$
$R_{x,w}(q)$	R-Polynomial
C_w	Kazhdan-Lusztig Basis element
$P_{x,w}(q)$	Kazhdan-Lusztig Polynomial
$\mu(x,w)$	coefficient of highest possible degree term of $P_{x,w}$
$x \prec w$	$\mu(x,w) \neq 0$
$D_L(w)$	Left Descent Set of w
$D_R(w)$	Right Descent Set of w
w_0	Longest element of W
$\leftrightarrow$	precursor to $\leq_L$
$\leq_L$	An order on W
$\sim_L$	Equivalence relation defining Left Cells
$\mathcal{C}$	Left Cell
$\Gamma_{\mathcal{C}}$	Subgraph of K-L graph of the cell $\mathcal{C}$
$\mathcal{K}_{\mathcal{C}}$	Left Cell Representation
$D_R(\mathcal{C})$	Right Descent set associated with $\mathcal{C}$
$\mu(x,y)$	$\mu(x,y)$ from before, extended a bit.
ϵ	Sign representation
χ	Character
DR_I^J	Elements with right descent sets containing I and inside J

T_{ξ_i}, $\bar{T}_{\xi_i}$	Particular sums of T_w's
1	Often the trivial representation
$w = x_1 \dots x_n$	Representation of w where $w(i) = x_i$
$P(w), Q(w)$	Tableau associated with w under the R-S correspondence
$\sim_K, \sim_{dK}$	Knuth equivalence, dual Knuth Equivalence
$KD(i)$	Knuth Descent set with involved indices at i
w'	A unique element in S_n associated with each $w \in KD(i)$
M, N	Semi-simple Algebra over an algebraically closed field
p_i, q_j	minimal central idempotents of the above algebras
M_{ij}, N_{ij}	sub algebras of the above algebras
Λ, λ_{ij}	Inclusion Matrix
K_0	K_0 group
$K(G)$	Groethendieck group of a group
Q_i	$(T_{s_i} + 1)/(q + 1)$
Ω_n	part of Ω
p	path on Bratteli Diagram
J_λ	Tower Construction of Representations
e_p	Basis vector of J_λ
V_λ	Vector spaces corresponding to some representation associated with λ
$d_p^i(q)$	Constant used to define the tower construction
$\mu \rightharpoonup \lambda$	λ can be obtained from μ through adding a box.

1. Introduction

1.1. **Introductory Comments.** The permutation groups S_n are arguably the most ubiquitous objects in mathematics. For starters, every group is a subgroup of one of them, almost any use of combinatorics invariably uses them, and their representations over $\mathbb{C}$ link naturally to the ideas of partitions of natural numbers.

Study of their representation has naturally led to consideration of the group algebra $\mathbb{C}[S_n]$. This, along with the development of the theory of Weyl groups and the more general Coxeter groups, has motivated study of the Iwahori-Hecke Algebra $\mathcal{H}_W(q)$, a deformation of this group algebra which is inextricably linked to representation theory. Hecke Algebras have also been used to discover the Jones Polynomial, a polynomial invariant of knots [11], and are intricately related to the study of towers of finite dimensional semi-simple algebras, where they appear as their quotients the Temperley-Lieb algebras (when these towers admit a Markov Trace of generic modulus). And through the Temperley-Lieb algebras they are related to statistical mechanics, quantum groups, and von Neumann algebras [6].

In this thesis, we will look at the Hecke Algebra $\mathcal{H}_{S_n}(q)$ and use it to formulate three different constructions of the irreducible representations of S_n. These three constructions will correspond to considerations from the classical theory of S_n, the theory of left cell representations of general Hecke Algebras over arbitrary Coxeter Groups, and the theory of inclusions of semi-simple algebras and path algebras of what we call "Bratteli Diagrams." We will study the relationships of these constructions to each other and show an example (S_3) displaying the different constructions. And through continuity as q varies through $\mathbb{C}$, the identifications made between the latter two representations will hold in fact hold for generic q.[1]

1.2. **Outline of Thesis.** In Section 2 we will set up the basic definitions of the permutation groups and Coxeter groups, and then introduce the language of partitions, which we will use extensively throughout the rest of the thesis as indices for our representations.

In Section 3 we will begin our discussions of representations by citing some basic results from the classical theory of representations ρ_λ of S_n.

In Sections 4 and 5 we will expand our discussion to the Iwahori-Hecke Algebra $\mathcal{H}_W(q)$, which we will refer to hereafter as simply the Hecke Algebra. To do this, we create in Section 4 what we term the "generalized Hecke Algebra," from which we produce the Hecke Algebra by a specialization. In Section 5, we make some general statements about the Hecke Algebra that we created. In particular, we prove that for almost all specializations, it is semisimple (Lemma 5.1), isomorphic to the group algebra $\mathbb{C}[W]$ (Theorem 5.7 and the following comments), and has characters given by certain polynomials which deform continuously as we change q (Lemma 5.6, Theorem 5.7).

In Sections 6, 7, and 8 we discuss our second approach to representations of the Hecke Algebra, the left cell representations. Sections 6 and 7 are preliminary, with Section 6 introducing the Bruhat Order, the R-Polynomials, and the Kazhdan-Lustzig Polynomials, and Section 7 using these definitions to define the left cell representations (subsections 7.3 and 7.4) via the left multiplication law (Theorem 7.1) and the theory of descent sets (subsection 7.2).

[1] What it means to be "generic" is also narrowed down considerably.

Section 8 identifies the left cell representations $\mathcal{K}_{\mathcal{C}}$ defined in Section 7 with the classical case explicated in Section 3, for the specialization $q = 1$. First we show that each left cell $\mathcal{C}$ can be associated with a right descent set $D_R(\mathcal{C})$ (Corollary 8.2). Then in subsection 8.2, we have a theorem identifying the representations of the cells $\mathcal{C}$, $\mathcal{C}w_0$, and $w_0\mathcal{C}w_0$ for w_0 the longest element of S_n, which is instrumental in particular to proving Corollary 8.12, the main result of subsection 8.3. Subsection 8.4 defines and states some facts about the Robinson-Schensted Correspondence ($w \leftrightarrow (P(w), Q(w))$), Knuth equivalences ($\sim_K$), and dual Knuth equivalences ($\sim_{dK}$). It is preliminary to subsection 8.5, which proves that $u \sim_{dK} v$ implies u and v are in the same left cell (Theorem 8.17), that $u \sim_K v$ implies that $Q(u)$ uniquely determines $Q(v)$, and creates a method of proof (Corollary 8.21) that will be used in subsection 8.6.

Finally, subsection 8.6 finishes off our discussion of left cell representations. We prove that each cell $\mathcal{C}$ can be associated with a tableau T by taking $Q(w)$ of any $w \in \mathcal{C}$ (Corollary 8.23). Then we prove that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are associated with tableaux that are both of the same partition λ, then $\mathcal{K}_{\mathcal{C}_1} \simeq \mathcal{K}_{\mathcal{C}_2}$, allowing us to identify left cell representations with partitions (Theorem 8.24). Using this identification (letting us define $\mathcal{K}_\lambda$), we finally prove $\rho_\lambda = \mathcal{K}_\lambda$, using similar occurances of "Young's Rule" from the classical theory and the left cell theory (Theorem 8.25).

In Sections 9 and 10 we discuss our third and final approach to representations of the Hecke Algebra, the tower representations. In Section 9, we start out with some preliminary theorems and musings, allowing us to set up the language of Bratteli Diagrams and inclusion matrices Λ for inclusions of semisimple algebras (subsection 9.3), and think of these inclusions in terms of representations (subsection 9.2). Then in Section 10 we define the tower representations J_λ (subsection 10.1, Theorem 10.1), show that they are irreducible and mutually inequivalent (Theorem 10.2), and reduce the set of possible q that make $\mathcal{H}_W(q)$ not semisimple (Theorem 10.3). Then we make the identification with the classical theory, $\rho_\lambda = J_\lambda$ (Theorem 10.4), by using the language introduced before.

In Section 11, we calculate all of these different kinds of representations for S_3, and demonstrate that the different constructions indeed produce the identifications proved in Theorem 10.4 and Theorem 8.25.

In Section 12, we make some concluding remarks, and show that the identification between $\mathcal{K}_\lambda$ and J_λ established through ρ_λ for the case $q = 1$ in fact extends to more arbitrary values of $q \in \mathbb{C}$ (Theorem 12.1).

2. Preliminaries and Notation

2.1. **The Permutation Group.** Let S_n be the symmetric group of permutations on n letters, with the presentation by the generators $s_1, \dots, s_{n-1}$ and relations

$$\begin{array}{rcll} s_i s_j & = & s_j s_i & \qquad |i-j| \geq 2, \\ s_i s_j s_i & = & s_j s_i s_j & \qquad |i-j| = 1, \\ s_i^2 & = & 1 & \qquad \forall i. \end{array}$$

The length function l is given on $w \in S_n$ as the minimum number of generators s_i needed to express w; an expression of w with $l(w)$ letters is called a reduced word. For example, the lengths of the transpositions $1 \leftrightarrow 2$ and $2 \leftrightarrow 3$ are 1, while the length of the transposition $1 \leftrightarrow 3$ is 3 as this is equal to $s_1 s_2 s_1$.

Recall [9] that a Coxeter Group W is defined as a group generated by the set of simple reflections $S = \{s_1, \dots, s_n\}$ with relations

$$(s_i s_j)^{n_{ij}} = 1,$$

where n_{ij} is an integer equal to 1 if $i = j$, greater than 1 if $i \neq j$, and satisfies $n_{ij} = n_{ji}$. This information can be categorized in a Coxeter graph, where a point is identified with every simple reflection and an edge is made between the points corresponding to s_i and s_j if $n_{ij} \geq 3$, labeled with n_{ij} if $n_{ij} > 3$.

Then S_n can [9] also be realized as a finite Coxeter group of type A_{n-1} with Coxeter Graph

$$\circ \text{———} \circ \text{———} \dots \text{———} \circ$$

and set of simple reflections $S = \{s_1, \dots, s_{n-1}\}$. It can be shown [9] that S_n has a unique element w_0 of longest length.
Finally, we state the Exchange Condition [9] for Coxeter groups,

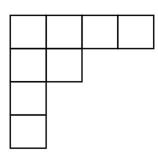
Theorem 2.1. *If $w = s_1 \dots s_r$ is a reduced expression for w and $l(sw) < l(w)$, then we have $sw = s_1 \dots s_{i-1} \hat{s}_i s_{i+1} \dots s_r$*

multiplying by s on both sides, this means that

Corollary 2.2. *If $l(sw) < l(w)$, then w has a reduced expression beginning with s.*

Note that these have a corresponding "right-hand" versions.

2.2. **Partitions.** Given a positive integer n, we define a partition λ of n (written $\lambda \dashv n$) to be a set of positive integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ where $\sum_i \lambda_i = n$. Each partition is associated with a Young diagram, a diagram of left justified boxes with k rows and λ_i boxes in each row, with the 1-st row being the one on the top and the k-th row being on the bottom. For example, a partition of 8 can be $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (4, 2, 1, 1)$ with the Young diagram



To any partition λ of n and its associated Young diagram, we say that a standard Young tableau of λ is a bijection between the boxes of the Young diagram and the natural numbers $\{1, \dots, n\}$ such that each column of boxes increases from top to bottom and each row increases from left to right. A standard tableau of the partition considered before is

1	3	5	8
2	4		
6			
7			

We let SYT_λ to be the set of standard tableaux T of the partition $\lambda \dashv n$, and given a tableau $T \in \mathrm{SYT}_\lambda$, we define the descent set $D(T)$ to be the set of $s_i \in \{s_1, \dots, s_n - 1\}$ such that $i + 1$ appears in a strictly lower row of T.

3. The Classical Theory of S_n

The results in this section are proved in [5], [8], and [14]. Given a symmetric group S_n, the classical theory gives us that the irreducible representations can be indexed by the conjugacy classes of S_n. But these conjugacy classes can be indexed by partitions $\lambda \dashv n$: a conjugacy class is characterized by the size of its "cycles," or its smallest sets of letters that are invariants under all $w \in S_n$. For example, the conjugacy class containing the trivial element has cycle sizes $(1, 1, \ldots, 1)$, and the class containing the cycle of all elements have cycle sizes (n). Considering this set of cycle sizes, arranged in nonincreasing order, we can identify conjugacy classes C with a partition of $\lambda \dashv n$. Taking the tranpose λ' of these partitions (i.e. taking the partition corresponding to the transposed Young Diagram of the original partition), we arrive at the indexing of irreducibles by partitions (so the trivial element is associated with $(1, 1, \ldots, 1)$). The classical theory says that the associated representations ρ_λ have dimension $|\mathrm{SYT}_\lambda|$, and Maschke's Theorem then gives us that

$$\mathbb{C}[S_n] = \bigoplus_{\lambda \dashv n} \mathrm{End}_k(\rho_\lambda).$$

Further, if we take one such ρ_λ, then the representation $\mathrm{Res}_{S_{n-1}}^{S_n} \rho_\lambda$ is given by the direct sum of all representations corresponding to partitions $\lambda' \dashv (n-1)$ obtained from λ by removing one box (note that these boxes have to be "corners"), each of these appearing with multiplicity one.

Finally, if $J \subset S$, we define the parabolic subgroup W_J of S_n to be the subset of S_n generated by the $s_j \in J$. Then we also have "Young's rule"

$$\mathrm{Ind}_{W_J}^{S_n}(1) \simeq \bigoplus_{\lambda \dashv n} |T \in \mathrm{SYT}_\lambda : D(T) \subseteq S \setminus J| \rho_\lambda,$$

where the left hand side is induction from W_J to S_n of the trivial representation of S_n.

4. Hecke Algebras

We now define a kind of deformation of the group algebra $\mathbb{C}[S_n]$, the Hecke Algebra with parameter $q \in \mathbb{C}$. In fact, we will define this deformation group for arbitrary finite Coxeter Group W.

For now, let the base ring be $A = \mathbb{C}[q^{1/2}, q^{-1/2}]$. We define the "generalized" Hecke Algebra $\tilde{\mathcal{H}}_W(q)$ of a finite Coxeter Group W with simple reflections S to be an algebra with linear basis $\{T_w\}_{w \in W}$ over this ring and relations

$$T_s T_w = \begin{cases} T_{sw} & l(sw) > l(w), \\ (q-1)T_w + qT_{sw} & l(sw) < l(w), \end{cases} \tag{1}$$

for $s \in S$ and $w \in W$.[2] Note that T_1 acts as the identity.

We define the Hecke Algebra $\mathcal{H}_W(q)$ to be this algebra under a specialization $A \to \mathbb{C}$. In particular, if we specialize to $q = 1$ we have $T_s T_w = T_{sw}$ for all cases, so that

$$\mathcal{H}_W(1) = \mathbb{C}[W].$$

[2] It shall be clear from our discussion of the Bruhat Order below that no cases are missed by the above definition.

More generally, for $q \neq 0$ (which we will assume for the rest of this thesis), we have that all T_s are invertible: as $s^2 = 1$, we have

$$T_s T_s = (q-1)T_s + qT_1,$$
$$T_s\left(\frac{T_s - (q-1)T_1}{q}\right) = T_1.$$

In fact, this means that all the T_w are invertible, because if $w = s_1 \dots s_r$ is a reduced word, then

$$(T_w)^{-1} = T_{s_r}^{-1} \dots T_{s_1}^{-1}.$$

We now come to the first theorems that we will prove, which will more explicitly state the relationship between $\mathbb{C}[W]$ and the $\mathcal{H}_W(q)$.

5. The Tits Deformation Theorem and the relations of $\mathcal{H}_W(q)$ to $\mathbb{C}[W]$

We first note that in this thesis, an algebra is said to be semisimple if it is finite-dimensional over its base field and its ideal of nilpotent elements is $\{0\}$.

We will show that for all but a finite number of $q \in \mathbb{C}$, $\mathcal{H}_W(q)$ and $\mathbb{C}[W]$ are in fact isomorphic. As $\mathbb{C}[W]$ is semisimple, this will require that for these q, $\mathcal{H}_W(q)$ is at least semisimple.

Lemma 5.1. *$\mathcal{H}_W(q)$ is semisimple for all but a finite number of $q \in \mathbb{C}$*

Proof. In general, a a finite algebra X with basis x_i over a field is semisimple if and only if

$$\det(\langle x_i, x_j \rangle) \neq 0,$$

where $\langle x_i, x_j \rangle$ is the trace of the map $\rho(x_i x_j) : X \to X$ given by right multiplication by $x_i x_j$ and the determinant is taken of the matrix with these as entries. But by the rules defining the Hecke Algebras, the matrix entries are polynomials in q, so the determinant is also a polynomial in q. Since $\mathbb{C}[W] = \mathcal{H}_W(1)$ is semisimple, we have that the polynomial is not identically zero. Then since a nonzero polynomial in q has only a finite number of roots in $\mathbb{C}$, we conclude the result. $\square$

This begs the question of what the bad values of q are. We will show later that these bad values are in the set of 0 and the nontrivial roots of 1, which we will call Ω. For now, let us just give the simple example of $W = S_2$.

S_2 has one generator, s_1, and $\mathcal{H}_{S_2}(q)$ has two basis vectors over $\mathbb{C}$, T_1 and T_{s_1}. By definition, we have that in the basis $\{T_1, T_{s_1}\}$,

$$\rho(T_1) = \rho(T_1 T_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\rho(T_{s_1}) = \rho(T_1 T_{s_1}) = \rho(T_{s_1} T_1) = \begin{pmatrix} 0 & q \\ 1 & q-1 \end{pmatrix},$$

$$\rho(T_{s_1} T_{s_1}) = \rho((q-1)T_{s_1} + qT_1) = \begin{pmatrix} q & q(q-1) \\ q-1 & (q-1)^2 + q \end{pmatrix},$$

so that the deteriminant of our matrix of traces becomes

$$\det \begin{pmatrix} 2 & q-1 \\ q-1 & (q-1)^2 + 2q \end{pmatrix} = (q+1)^2.$$

As suspected, $\mathcal{H}_{S_2}(q)$ is semisimple for all $q \neq -1$.

Now we will prove the Tits Deformation Theorem which will show that for all these values $q \in \mathbb{C}$ such that $\mathcal{H}_W(q)$ is semisimple, we in fact have $\mathcal{H}_W(q) \simeq \mathbb{C}[W]$. Taking one such q, Maschke's Theorem gives us that

$$\mathcal{H}_W(q) \simeq \oplus_i \mathrm{End}_{\mathbb{C}}(V_i) \simeq \oplus_i \mathrm{Mat}_{\mu_i}(\mathbb{C}),$$

$$\mathbb{C}[W] \simeq \oplus_\lambda \mathrm{End}_{\mathbb{C}}(V_\lambda) \simeq \oplus_i \mathrm{Mat}_{\rho_\lambda(1)}(\mathbb{C}),$$

where the V_i are the irreducible representations of $\mathcal{H}_W(q)$ and the V_λ are the irreducible representations of W (ρ_λ refers to the character of V_λ). Thus it will suffice to show that we can make a bijection from the μ_i to the $\rho_\lambda(1)$. We will prove this more generally. We start with a couple algebraic lemmas

Lemma 5.2. *Let A be an integral domain with field of fractions F, let $\overline{F}$ be the algebraic closure of F, and let $\overline{A}$ be the integral closure of A in $\overline{F}$. Then we have that the integral closure of the polynomial ring $A[x]$ is just $\overline{A}[x]$.*

Proof. See [3]. $\square$

Corollary 5.3. *The integral closure of $A[t_1, \ldots, t_d]$ is $\overline{A}[t_1, \ldots, t_d]$*

Lemma 5.4. *Keeping the same definitions as in the previous lemma, any ring homomorphism f from A to a field k extends to a ring homomorphism $\overline{A} \to \overline{k}$, where $\overline{k}$ is the algebraic closure of k.*

Proof. Let $\mathfrak{p}$ be the kernel of $A \to k \hookrightarrow \overline{k}$. Since $\overline{k}$ is a field, we have that $\mathfrak{p}$ is a prime ideal (it is in fact maximal). It can be shown ([1] 5.10) that there is a prime ideal $\mathfrak{q} \subset \overline{A}$ such that $\mathfrak{q} \cap A = \mathfrak{p}$. The elements of $\overline{A}/\mathfrak{q}$ are integral over $A/\mathfrak{p}$: taking any element $x \in \overline{A}$ we can use integral closure to let x be a root of a monic polynomial $p(t)$ with coefficients in A, and modding these coefficients out by $\mathfrak{p}$, we have

$$\begin{array}{ccc} p(t) & \in & A[t] \\ \downarrow & & \downarrow \\ \overline{p}(t) & \in & (A/\mathfrak{p})[t] \end{array}$$

and $\bar{x} \in \overline{A}/\mathfrak{q}$ will go to zero under $\overline{p}(t)$ (defined in the picture above). So $Frac(\overline{A}/\mathfrak{q})$ is a finite algebraic extension of $Frac(A/\mathfrak{p})$.

Now the induced map $A/\mathfrak{p} \to k$ naturally extends to a map $Frac(A/\mathfrak{p}) \to k$, as any nonzero element of $A/\mathfrak{p}$ goes to an invertible element of k. As $Frac(\overline{A}/\mathfrak{q})$ is a finite algebraic extension and $k \hookrightarrow \overline{k}$, we can extend this map naturally to $Frac(\overline{A}/\mathfrak{q}) \to \overline{k}$. We just define the map by the composition,

$$\overline{A} \to \overline{A}/\mathfrak{q} \to Frac(\overline{A}/\mathfrak{q}) \to k \to \overline{k}$$

. $\square$

Keeping the same notation as above, suppose that M is an A-free algebra, and that $\overline{M} := \overline{F} \underset{A}{\otimes} M$, the extension of M to $\overline{F}$, is semisimple and decomposes as an algebra[3]

$$\overline{M} \simeq \underset{c \in C}{\oplus} \mathrm{Mat}_{d_c}(\overline{F}) \simeq \oplus \mathrm{End}_{\overline{F}}(V_c),$$

where the V_c are the irreducible $\overline{M}$-modules. We can in fact view each $\mathrm{Mat}_{d_c}(\overline{F})$ as a $\overline{M}$ left module which decomposes as

$$(V_c)^{d_c},$$

by in particular considering each column of $\mathrm{Mat}_{d_c}(\overline{F})$ as a copy of V_c. This gives us a decomposition of $\overline{M}$ as a left module over itself as

$$\overline{M} = \underset{c \in C}{\oplus} (V_c)^{d_c}.$$

So now consider an arbitrary element $m \in \overline{M}$, the map $\lambda(m) : \overline{M} \to \overline{M}$ given by left multiplication by m, and the corresponding characteristic polynomial of this map

$$\det\left(x 1_{\overline{M}} - \lambda(m)\right).$$

From the above decomposition of $\overline{M}$ we have a basis over $\overline{F}$ of $\{e_{ij}^c\}$, where i corresponds to the i-th copy of V_c and j corresponds to the j-th basis vector of that V_c. Thus we can write

$$m = \sum_{c,i,j} a_{ij}^c e_{ij}^c,$$

and the characteristic polynomial as

$$f(x) = \det\left(x 1_{\overline{M}} - \sum_{c,i,j} a_{ij}^c \lambda(e_{ij}^c) \right),$$

as a polynomial in x over the ring[4] $A[\dots a_{ij}^c \dots]$.

Theorem 5.5. *The characteristic polynomial $f(x)$ of left multiplication by an arbitrary element $m = \sum_{c,i,j} a_{ij}^c e_{ij}^c$ factorizes over $\overline{F}[\dots a_{ij}^c \dots]$ into irreducibles as*

$$f(x) = \prod_c (f_c)^{d_c},$$

where the $f_c \in \overline{A}[\dots a_{ij}^c \dots]$ are irreducible polynomials of degree d_c

Proof. From the decomposition of $\overline{M}$ above as a left module over itself

$$\overline{M} = \underset{c \in C}{\oplus} \oplus_{i=1}^{d_c} V_c,$$

we have that the matrix representing

$$x 1_{\overline{M}} - \sum_{c,i,j} a_{ij}^c \lambda(e_{ij}^c)$$

[3] Note that we use algebraic closedness of $\overline{F}$ to achieve this decomposition: look ahead to Theorem 9.1 to see the statement that this is possible

[4] Now and in the future, $R[\dots a_{ij}^c \dots]$ refers to the polynomial ring with variables $\{a_{ij}^c\}$ and coefficients in R

is block diagonal, with d_c identical blocks corresponding to each $c \in C$ (Note that the action of left-multiplication on the module formed by each column of $\mathrm{Mat}_{d_c}(\overline{F})$ is identical). Thus we have the decomposition

$$f(x) = \prod_c (f_c)^{d_c},$$

where the $f_c(x)$ are polynomials of degree d_c over $\overline{F}[\dots a_{ij}^c \dots]$.

To prove that the $f_c(x)$ are in fact irreducible polynomials over x, suppose that $f_c = gh$, and let us specialize our element m by setting $a_{j+1,j}^c = 1$, $a_{1,d_c}^c = a$, and all other $a_{ij}^c = 0$. Then we have

$$f_c(x) = x^{d_c} - a,$$

which is irreducible as a polynomial in x over $\overline{F}[a]$. Since a factorization of f must propagate down the specialization, and the irreducible below has degree d_c, we conclude that either g or h had to have degree 0, a contradiction.

Finally, to prove that the $f_c(x)$ are polynomials with coefficents in $\overline{A}[\dots a_{ij}^c \dots]$, note that $f(x)$ is monic with coefficients in $A[\dots a_{ij}^c \dots]$. Then note that all roots of $f_c(x)$ are roots of $f(x)$, all roots of $f_c(x)$ must be (by the definition of integral closure and Corollary 5.3) in $\overline{A}[\dots a_{ij}^c \dots]$. Thus as the coefficients of $f_c(x)$ are symmetric functions in these roots, we must have that these coefficients are in $\overline{A}[\dots a_{ij}^c \dots]$, as desired. $\square$

Now from the definition of the f_c, we have

Lemma 5.6. *If we define the polynomial $\chi^c \in \overline{A}[\dots a_{ij}^c \dots]$ by*

$$f_c(x) = x^{d_c} - \chi^c x^{d_c - 1} \dots,$$

the function $\chi_c : \overline{M} \to \overline{F}$ given by $m = \sum b_{ij}^c e_{ij}^c \mapsto \chi^c(\dots b_{ij}^c \dots)$ is the character of V_c.

Theorem 5.7. *(Tits Deformation) Now suppose that we have a ring homomorphism $\phi : A \to k$, where k is a field, with the associated map (given in Lemma 5.4) $\bar{\phi} : \overline{A} \to \overline{k}$. If $\overline{k} \underset{A}{\otimes} M$ is semisimple, then it has the same matrix invariants as $\overline{M}$, i.e.*

$$\overline{k} \underset{A}{\otimes} M \simeq \underset{c \in C}{\oplus} Mat_{d_c}(\overline{k}).$$

Further, if the characters for $\overline{M}$ are χ_c, then the characters for $\overline{k} \underset{A}{\otimes} M$ are $\bar{\phi} \circ \chi_c$

Proof. Let $f(x)$ be the characteristic polynomial of a generic element of M as before, so that $f(x) = \prod f_c^{d_c}(x)$. Taking the image of its coefficients under $\bar{\phi}$, i.e. going from $\overline{A}[\dots a_{ij}^c \dots][x] \to \overline{k}[\dots a_{ij}^c \dots][x]$, we obtain the polynomials $\bar{\phi}(f_c)$ and a factorization of $\phi(f)$

$$\phi(f) = \prod_{c \in C} (\bar{\phi}(f_c))^{d_c}.$$

Specializing as in the previous theorem, we have that the $\bar{\phi}(f_c)$ are still irreducible. But a basis of M goes over to a basis of $\overline{k} \otimes M$, so that the general characteristic polynomial of $\overline{k} \otimes M$ is just $\phi(f)$. And if $\overline{k} \otimes M$ is semisimple and

$$\overline{k} \underset{A}{\otimes} M \simeq \underset{d \in D}{\oplus} \mathrm{Mat}_{d_d}(\overline{k}),$$

the theorem above tells us that the must decompose into a product of $f_d^{d_d}$. So the two decompositions that we just made must coincide, implying the rest of the theorem. $\square$

How does this apply to our situation with the Hecke Algebras? We let $A = \mathbb{C}[q^{1/2}, q^{-1/2}]$ and $M = \tilde{\mathcal{H}}_W(q)$. Specializing to $q = 1$ gives us a semisimple algebra $\mathbb{C}[W]$, so that since specialization induces a map of polynomial rings, we must have that $\tilde{\mathcal{H}}_W(q)$ is also semisimple. Thus, the Tits Deformation theorem tells us that all the matrix invariants of the semisimple $\mathcal{H}_W(q)$ are the same as the matrix invariants of $\overline{\tilde{\mathcal{H}}_W(q)}$, establishing finally that for all but a finite number of q, we have $\mathcal{H}_W(q) \simeq \mathbb{C}[W]$. The character discussions in Lemma 5.6 and Theorem 5.6 give us that as we vary q in $\mathbb{C}$ (avoiding the q that make $\mathcal{H}_W(q)$ not semisimple), the characters of the irreducibles deform continuously, a fact that we will use at the end.

6. Kazhdan Lusztig Polynomials

6.1. **Introduction.** Now that we have defined the Hecke Algebras and showed a bit of their relation to the more natural group algebra, we shall construct the left cell representations of the Hecke Algebras of type A_n, and relate them to the representations of S_n given by the Classical Theory above. We will first define the Kazhdan-Lusztig Polynomials for general Coxeter Groups W.

6.2. **The Bruhat Order.** Let W be a Coxeter group, with simple reflections $S = \{s_1, \ldots, s_n\}$. We define the set of reflections (this terminology comes from the representation of W as a reflection group) of W to be $T = \bigcup_{w \in W} wSw^{-1}$.[5] Using this we define the Bruhat order, saying that $u < v$ if we can build v from u through a series of multiplications by reflections, with length increasing by one each time. More formally, we say that $u < v$ if $\exists t_1, \ldots, t_r \in T$ such that

$$v = ut_1 \ldots t_r,$$

and $\forall i \in \{1, \ldots, r-1\}$, $l(ut_1 \ldots t_i) < l(ut_1 \ldots t_{i+1})$. As will be used in later sections, this order can actually be characterized [9] by the use of "subexpressions:" if $w = s_1 \ldots s_r$ is a reduced word expression for w, then $x < w$ if and only if $x = s_{i_1} \ldots s_{i_q}$ where $1 \leq i_1 < i_2 < \cdots < i_1 \leq r$. In other words $x < w$ if and only if we have a reduced word expression for x which is a subexpression of w.

6.2.1. *Examples.* Let S_3 be the symmetric group on 3 letters (generated by s_1 and s_2). We have

$$T = \{s_1, s_2, s_1s_2s_1\}.$$

Through further calculation, we have the following "Bruhat graph," where an arrow $u \to v$ means that $ut = v$ with $t \in T$ and $l(v) = l(u) + 1$ (in other words, $u < v$ if and only if we

[5]This is abuse of notation, as T before referred to a tableux. But there should not be any confusion

can follow arrows from u to v):

$$
\begin{tikzcd}
 & s_1s_2s_1 & \\
s_2s_1 \arrow[ur] & & s_1s_2 \arrow[ul] \\
s_1 \arrow[u] \arrow[urr] & & s_2 \arrow[u] \arrow[ull] \\
 & 1 \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Now let $\mathcal{D}_n$ be the dihedral group of symmetries of a n-gon. As a Coxeter Group, it has a presentation with two generators s and t, with relations $(st)^n = 1$, $s^2 = t^2 = 1$. In other words, its Coxeter graph is $I_2(n)$

$$\circ \overset{n}{\text{———}} \circ$$

Non-identity elements of $\mathcal{D}_n$ are of the form $ststs\ldots$ or $tstst\ldots$. Since $s, t \in T$, we have that

$$s < st < sts < \ldots \qquad\qquad t < ts < tst < \ldots.$$

Further, because of considerations like $s(sts) = ts$ and $t(tst) = st$ and $sts, tst \in T$, we can conclude that its Bruhat graph looks like

$$
\begin{tikzcd}
\vdots & & \vdots \\
sts & & tst \\
st \arrow[u] \arrow[urr] & & ts \arrow[u] \arrow[ull] \\
s \arrow[u] \arrow[urr] & & t \arrow[u] \arrow[ull] \\
 & 1 \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

with top floors looking like

$$
\begin{tikzcd}
 & (st)^{n/2} & \\
(st)^{n/2-1}s \arrow[ur] & & t(st)^{n/2-1} \arrow[ul]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (st)^{(n-1)/2}s & \\
(st)^{(n-1)/2} \arrow[ur] & & (ts)^{(n-1)/2} \arrow[ul]
\end{tikzcd}
$$

n is even $\qquad\qquad$ n is odd

6.3. **Definition of the R-Polynomials.** Suppose that $w \in W$, and that $w = s_1 \dots s_r$ is a reduced word. Then from the defining relations (1) of Hecke Algebras, we have that

$$T_w = T_{s_1} \dots T_{s_r}.$$

Recall that since we are assuming $q \neq 0$ all the T_{s_i} are invertible; thus all the T_w are also invertible. But calculation of $(T_w)^{-1}$ by

$$(T_w)^{-1} = (T_{s_r})^{-1} \dots (T_{s_1})^{-1}$$

quickly becomes unmanageable. Using the Bruhat ordering, however, we can in fact reduce the possibilities some: the T_x coefficients of $(T_w)^{-1}$ can be nonzero only if $x \leq w$, and these coefficients are in fact polynomials in q. More precisely,

Theorem 6.1. *For $w \in W$, we have*

$$(T_{w^{-1}})^{-1} = \frac{\epsilon_w}{q_w} \sum_{x \leq w} \epsilon_x R_{x,w}(q) T_x,$$

where ϵ_w is $(-1)^{l(w)}$, $q_w = q^{l(w)}$, and $R_{x,w}$ is a polynomial of degree $l(w) - l(x)$ with $R_{w,w} = 1$

The proof of this theorem can be found in [9]. It actually gives the following way to calculate the R polynomials recursively:

Lemma 6.2. *Take $x, w \in W$, $w \neq 1$. If $x \nleq w$, we have $R_{x,w} = 0$. If $x < w$, then taking s such that $l(sw) < l(w)$ for $s \in S$, we have:*
1: If $sx < x$, then $R_{x,w} = R_{sx,sw}$.
2: If $sx > x$, we have $R_{x,w} = (q-1)R_{x,sw} + qR_{sx,sw}$.
3: If $ws < w$ and $xs < x$, then $R_{x,w} = R_{xs,ws}$.

Proposition 6.3. *Suppose that $x < w$. If $l(w) - l(x) = 1$, $R_{x,w} = q - 1$. If $l(w) - l(x) = 2$, $R_{x,w} = (q-1)^2$.*

Proof. If $l(w) - l(x) = 1$ and $x < w$, then the characterization of the Bruhat order by subexpressions gives us that if $w = s_1 \dots s_r$, then $x = s_1 \dots s_{i-1}\hat{s}_i s_{i+1} \dots s_r$. Repeated use of 1 and 3 above then implies that $R_{x,w} = R_{1,s_i}$. But from our formula for $T_s^{-1} = T_{s^{-1}}^{-1}$ above, we have that $R_{1,s} = q - 1$ for any $s \in S$.

If $l(w) - l(x) = 2$ and $x < w$, then we can use the method above to reduce to the case $w = s_1 \dots s_r$ and $x = s_2 \dots s_{r-1}$. But letting $s = s_1$, 2 applies, so that we have

$$R_{x,w} = (q-1)R_{x,sw} + qR_{sx,sw}.$$

But sx and sw have the same length $r - 1$ and are unequal, so that $sx \nleq sw$ and $R_{sx,sw} = 0$. Applying 1 repeatedly gives us that $R_{x,sw} = R_{1,s_r} = q - 1$. Thus we have $R_{x,w} = (q-1)^2$. $\square$

6.3.1. *Example with S_3.* From Proposition 6.3, there is only one case left to compute, $R_{1,s_1 s_2 s_1}$. Then Lemma 6.2 part 2, with $s = s_1$, gives us that

$$R_{1,s_1 s_2 s_1} = (q-1)R_{1,s_2 s_1} + qR_{s_1,s_2 s_1} = (q-1)^3 + q(q-1).$$

6.4. **Definition of the Kazhdan-Lusztig Polynomials.** We define an involution ι on $\tilde{\mathcal{H}}_W(q)$, given by $q^{1/2} \mapsto q^{-1/2}$ and $T_w \mapsto T_{w^{-1}}^{-1}$. Kazhdan and Lusztig [13] showed

Theorem 6.4. *There exists a basis $\{C_w\}$ of $\tilde{\mathcal{H}}_W(q)$ such that $\iota(C_w) = C_w$ for all $w \in W$. This basis can expressed in terms of polynomials $P_{x,w} \in \mathbb{Z}[q]$ (note that these polynomials have terms of only integer coefficients and integer degrees) by*

$$C_w = \epsilon_w q_w^{1/2} \sum_{x \leq w} \epsilon_x q_x^{-1} P_{x,w}(q^{-1}) T_x, \tag{2}$$

where the polynomials $P_{x,w}$ have q degree $\leq (l(w) - l(x) - 1)/2$ for $x \leq w$, $P_{w,w} = 1$, and $P_{x,w} = 0$ otherwise.

The polynomials $P_{x,w}$ are referred to as the Kazhdan-Lusztig Polynomials.

If $(l(w) - l(x) - 1) \in 2\mathbb{Z}_{\geq 0}$, we let $\mu(x, w)$ be the coefficient of the $q^{(l(w)-l(x)-1)/2}$ term in $P_{x,w}$ and write $x \prec w$ if $\mu(x, w) \neq 0$, in other words, $x \prec w$ if $x < w$ and $P_{x,w}$ is a Kazhdan-Lusztig polynomial of the maximum possible degree. It is shown in [9] that the $P_{x,w}$ can be recursively calculated: let $s \in S$ such that $l(sw) < l(w)$ and let $v = sw$. Let $c = 0$ if $x < sx$ and $c = 1$ if $sx < x$. Then we have

$$P_{x,w} = q^{1-c} P_{sx,v} + q^c P_{x,v} - \sum_{z \prec v \text{ and } sz < z} \mu(z, v) q_z^{-1/2} q_w^{1/2} P_{x,z}. \tag{3}$$

Two other useful identities are (in the second equation, note the arguments of the $P_{x,w}$)

$$q_w q_x^{-1} P_{x,w}(q^{-1}) - P_{x,w}(q) = \sum_{x < y \leq w} R_{x,y} P_{y,w}(q), \tag{4}$$

$$C_w = C_s C_v - \sum_{z \prec v \text{ and } sz < z} \mu(z, v) C_z. \tag{5}$$

Proposition 6.5. *For all $x \leq w$, $P_{x,w}(0) = 1$.*

Proof. Plugging $q = 0$ into equation 2, we obtain that for general $x \leq w$, we have

$$P_{x,w}(0) = 0^{1-c} P_{sx,v} + 0^c P_{x,v}.$$

Thus in either case, induction on $l(w)$ gives us the result. □

Corollary 6.6. *If $l(w) - l(x) \leq 2$, then $P_{x,w} = 1$.*

6.4.1. *Running Examples.* S_3: Corollary 6.6 tells us that the only case we have to calculate is $P_{1,s_1s_2s_1}$. So we use equation (4), obtaining

$$q^3 P_{1,s_1s_2s_1}(q^{-1}) - P_{1,s_1s_2s_1}(q) = 2(q-1) + 2(q-1)^2 + (q-1)^3 + q(q-1) = q^3 - 1$$

Now $(l(s_1s_2s_1) - l(1) - 1)/2 = 1$, so that $P_{1,s_1s_2s_1}$ is of the form $1 + aq$. Thus the left hand side becomes $q^3 + aq^2 - 1 - aq$, so that we conclude that $a = 0$, i.e. $P_{x,w} = 1$ for all $x, w \in S_3$ such that $x \leq w$.

Since we will be doing the example of S_3 in section 11, let us calculate the C_w by equation (2) for completeness

$$C_1 = T_1,$$

$$C_{s_1} = -q^{1/2}(T_1 - q^{-1}T_{s_1}) \qquad C_{s_2} = -q^{1/2}(T_1 - q^{-1}T_{s_2}),$$

$$C_{s_1s_2} = q(T_1 - q^{-1}T_{s_1} - q^{-1}T_{s_2} + q^{-2}T_{s_1s_2}) \qquad C_{s_2s_1} = q(T_1 - q^{-1}T_{s_1} - q^{-1}T_{s_2} + q^{-2}T_{s_2s_1}),$$

$$C_{s_1s_2s_1} = -q^{3/2}(T_1 - q^{-1}T_{s_1} - q^{-1}T_{s_2} + q^{-2}T_{s_1s_2} + q^{-2}T_{s_2s_1} - q^{-3}T_{s_1s_2s_1}).$$

Reams of calculations will show that these C_w are indeed fixed by ι.

$\mathcal{D}_n$: We also claim that $P_{x,w} = 1$ for all $x, w \in \mathcal{D}_n$ such that $x \leq w$. Looking at our Bruhat graph, we have that $x < w$ if and only if $l(x) < l(w)$. By induction, we assume that for all v such that $l(v) < l(w)$, $P_{x,v} = 1$ for all $x \leq v$. Thus $z \prec v$ if and only if $l(v) - l(z) = 1$. Since there are only two z of this type, one of which begins with s and the other of which begins with t, we have that the right sum in equation (3) reduces to one term, which corresponds to a z where $l(w) - l(z) = l(w) - l(v) + l(v) - l(z) = 2$. And by induction we have

$$P_{x,w} = q^{1-c} + q^c - q,$$

so that in either case for c, we indeed have $P_{x,w} = 1$.

7. Left Cell Representations of Hecke Algebras

Now that we have defined the Kazhdan-Lusztig polynomials, we will use these and the basis $\{C_w\}$ to finally define the left cell representations of a Coxeter Group W. We first calculate the action of $T_s \in \mathcal{H}_W(q)$ on this basis by right multiplication.

7.1. **Left Multiplication.**

Theorem 7.1. *Take $s \in S$ and $w \in W$. Then we have*

$$T_sC_w = \begin{cases} -C_w & sw < w \\ qC_w + q^{1/2}C_{sw} + q^{1/2}\displaystyle\sum_{z \prec w \text{ and } sz<z} \mu(z,w)C_z & w < sw \end{cases}.$$

Proof. Let us start with the second case. Note that by definition we have

$$C_s = (-1)q^{1/2}(T_1 + (-1)q^{-1}T_s) = -q^{1/2}T_1 + q^{-1/2}T_s.$$

The assumption $w < sw$ means that we can consider equation 4, but with w taking the place of v and sw taking the place of w.

$$C_{sw} = C_sC_w - \sum_{z \prec w \text{ and } sz<z} \mu(z,w)C_z$$

$$C_{sw} = -q^{1/2}T_1C_w + q^{-1/2}T_sC_w - \sum_{z \prec w \text{ and } sz<z} \mu(z,w)C_z$$

$$qC_w + q^{1/2}C_{sw} + q^{1/2}\sum_{z \prec w \text{ and } sz<z} \mu(z,w)C_z = T_sC_w$$

as desired.

Now consider the first case. We have $sw < w$ so that in particular $w \neq 1$. We do induction on $l(w)$. The base case is $l(w) = 1$, i.e. $w = s$, so that

$$T_sC_s = T_s(-q^{1/2}T_1 + q^{-1/2}T_s) = -q^{1/2}T_s + (q^{1/2} - q^{-1/2})T_s + q^{1/2}T_1 = -q^{-1/2}T_s + q^{1/2}T_1 = -C_s.$$

Let us do the induction step. We have $sw < w$ implies that $sw < s(sw)$, so that we can apply the second part of the theorem with sw in the place of w to obtain

$$T_sC_{sw} = qC_{sw} + q^{1/2}C_w + q^{1/2}\sum_{z \prec sw \text{ and } sz<z} \mu(z,sw)C_z.$$

Solving for C_w, this gives

$$C_w = q^{-1/2}T_sC_{sw} - q^{1/2}C_{sw} - \sum_{z\prec sw \text{ and } sz<z} \mu(z,sw)C_z,$$

$$T_sC_w = q^{-1/2}T_s^2C_{sw} - q^{1/2}T_sC_{sw} - \sum_{z\prec sw \text{ and } sz<z} \mu(z,sw)T_sC_z.$$

Now since $z \prec sw$ implies that $z < sw$, $sz < z < sw < w$ for the terms in the sum. Thus we can use induction to obtain that $T_sC_z = -C_z$. After further simplification, we get

$$T_sC_w = q^{-1/2}(q-1)T_sC_{sw} + q^{1/2}T_1C_{sw} - q^{1/2}T_sC_{sw} + \sum_{z\prec sw \text{ and } sz<z} \mu(z,sw)C_z,$$

Now plugging in our expression for T_sC_{sw} from before, we obtain

$$T_sC_w = -C_w$$

as desired. $\square$

There is an interesting corollary to this which will become useful to us later. Let us have $x < w$, and suppose that we have an $s \in S$ such that $sw < w$ and $sx > x$. Corollary 2.2 gives us that w has a reduced expression beginning with s. Then if x is a subexpression of sw, then sx becomes a subexpression of w, so that $sx \leq w$. On the other hand, if a reduced expression for x begins with s, then sx will be a subexpression of w, so that again, $sx \leq w$. So in both cases, $sx \leq w$. Applying the first part of the previous theorem and using the definition of the C_w, we conclude

$$T_s\left(\epsilon_w q_w^{1/2}\sum_{x'\leq w}\epsilon_{x'}q_{x'}^{-1}P_{x',w}(q^{-1})T'_x\right) = -\epsilon_w q_w^{1/2}\sum_{x'\leq w}\epsilon_{x'}q_{x'}^{-1}P_{x',w}(q^{-1})T'_x.$$

$sx \leq w$ implies that the coefficient of T_{sx} on the right hand side is $-\epsilon_w q_w^{1/2}\epsilon_{sx}q_{sx}^{-1}P_{sx,w}(q^{-1})$. On the left hand side, T_{sx} occurs twice in the expansion, once from T_sT_x and once from T_sT_{sx}. The total coefficient on the left is thus

$$\epsilon_w q_{w^{1/2}}\left(\epsilon_x q_x^{-1}P_{x,w}(q^{-1}) + (q-1)\epsilon_{sx}q_{sx}^{-1}P_{sx,w}\right),$$

and equating the two, we obtain $P_{sx,w}(q^{-1}) = P_{x,w}(q^{-1})$ giving us the result

Corollary 7.2. *If $x < w$ and $s \in S$ such that $sw < w$ and $sx > x$, then $P_{x,w} = P_{sx,w}$*

7.2. **Descent Sets.** For any $w \in W$, let $D_L(w)$ be the left descent set of w,

$$D_L(w) := \{s \in S | l(sw) < l(w)\}.$$

As the case that we are most interested in, the symmetric group S_n, is finite, we explore the properties of $D_L(w)$ in the finite case. As stated in "Preliminaries," we have a unique word of maximal length w_0. It can be shown [2],[9], that for all $w \in W$,

$$l(ww_0) = l(w_0) - l(w). \tag{6}$$

Since there is only the identity is of length zero, this has the corollary that $w_0^2 = 1$. Now take $s \in S$. We have the two equations

$$l(sww_0) = l(w_0) - l(sw),$$

$$l(ww_0) = l(w_0) - l(w).$$

Subtracting, we obtain $l(sww_0) - l(ww_0) = l(w) - l(sw)$ which implies that $D_L(ww_0) = S \setminus D_L(w)$.

Looking again at equation 6 and letting $w = w_0 w$, we obtain (using $l(w) = l(w^{-1})$)

$$l(w_0ww_0) = l(w_0) - l(w_0w) = l(w_0) - l(w^{-1}w_0) = l(w_0) + l(w^{-1}) - l(w_0) = l(w).$$

Thus we have $D_L(w_0ww_0) = w_0 D_L(w) w_0$.

Similarly, we have $l(w_0w) = l(w^{-1}w_0) = l(w_0) - l(w)$. Thus the same method as above shows that

$$l(w_0sw) - l(w_0w) = l(w) - l(sw),$$
$$l(w_0sw_0(w_0w)) - l(w_0w) = l(w) - l(sw).$$

Thus, we have $D_L(w_0w) = w(S \setminus D_L(w))w_0 = S \setminus (w_0 D_L(w) w_0)$. So collecting our results, we have proved

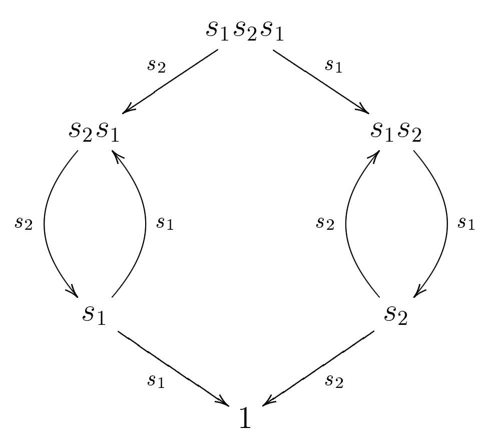
Proposition 7.3. .
$D_L(ww_0) = S \setminus D_L(w)$.
$D_L(w_0ww_0) = w_0 D_L(w) w_0$.
$D_L(w_0w) = w(S \setminus D_L(w))w_0 = S \setminus (w_0 D_L(w) W_0)$.

We also define right descent sets as $D_R(w) := \{s \in S | l(ws) < l(w)\}$.

7.3. **Left Cells.** Finally, we define the left cells of a Coxeter Group. We write $x \leftrightarrow w$ if either $x \prec w$ or $w \prec x$, and $x \leq_L w$ if there exists a set $x = x_1, x_2, \ldots, x_r = w$ such that $x_i \leftrightarrow x_{i+1}$ and $D_L(x_i) \nsubseteq D_L(x_{i+1})$ for $i < r$. We say that $x \sim_L w$ if both $x \leq w$ and $w \leq x$ (or if $x = w$). We call the resulting equivalence classes of W the left cells $\mathcal{C}$.

Another way to store this information is by using a "Kazhdan-Lusztig" graph, which is a biweighted directed multigraph with one vertex for each $w \in W$ and arrows $x \to w$ if $x \leftrightarrow w$ and $D_L(x) \nsubseteq D_L(w)$. For such a pair x, w, we put one arrow for each $s \in (D_L(x) \setminus D_L(w))$, and weight it with both s and $\mu(x, w)$ if $x \prec w$, or both s and $\mu(w, x)$ if $w \prec x$. In addition to these arrows, we also put arrows $x \to x$ for each $s \in S$ weighted with s and 1 if $s \notin D_L(x)$ and s and -1 if $s \in D_L(x)$. The left cells $\mathcal{C}$ then correspond to the vertices in each strongly connected component $\Gamma_{\mathcal{C}}$ of our graph.

7.3.1. *Examples.* S_3: All the Kazhdan-Lusztig polynomials are 1, so that $x \prec w$ if and only if $x < w$ and $l(w) - l(x) = 1$, i.e. all the lines on the Bruhat Graph. The left descent sets of all the elements are easily calculated, giving us the Kazhdan-Lusztig Graph (leaving out the self loops and the μ's (for $x \neq w$ the μ's are all 1))



Thus, the left cells of S_3 are $\{1\}, \{s_1, s_2s_1\}, \{s_2, s_1s_2\}$, and $\{s_1s_2s_1\}$.

$\mathcal{D}_n$: We calculated that $P_{x,w} = 1$ for $x \le w$, so that we again have $x \prec w$ if and only if $l(w) - l(x) = 1$. Further, for $x \neq 1, w_0$, $D_L(x) = \{s\}$ ($\{t\}$) if the unique reduced word for x begins with an s (t). From equation (6), $D_L(w_0) = \{s, t\}$. Combining these, we obtain 4 cells: $\{1\}, \{$ elements ending in $s\}, \{$ elements ending in $t\}, \{w_0\}$.

7.4. **Left Cell Representations.** Now we can combine the left cells with the left multiplication law to obtain representations of $\mathcal{H}_W(q)$.

Lemma 7.4. *Left multiplication by T_s takes C_w to an element in the subspace generated by C_x, $x \le_L w$*

Proof. If $sw < w$, then Theorem 6.1 immediately proves the result. If $w < sw$, then $l(sw) - l(w) = 1$, so that Corollary 6.6 shows that $w \prec sw$. But $s \in D_L(sw)$ and $s \notin D_L(w)$, so that $D_L(sw) \not\subseteq D_L(w)$ and $sw \le_L w$. Further, any element in the sum has $z \prec w$ and $sz < z$, so that again $s \in D_L(z)$, and $z \le_L w$ □

Now we define the left cell representations of $\mathcal{H}_W(q)$. Let $\mathcal{C}$ be a left cell of $\mathcal{H}_W(q)$, and $V_{\mathcal{C}}$ be the vector space generated by C_w, $w \le_L \mathcal{C}$ [6]. Let $W_{\mathcal{C}}$ be the vector space generated by C_w for $w \le_L \mathcal{C}$ and $w \notin \mathcal{C}$. Then we define the left cell representation of $\mathcal{C}$ to be

$$\mathcal{K}_{\mathcal{C}} = V_{\mathcal{C}}/W_{\mathcal{C}},$$

with the action of $\mathcal{H}_W(q)$ given by left multiplication above.

These are not necessarily irreducible: For the case of $\mathcal{D}_n$ above, if we specialize to $\mathbb{C}[\mathcal{D}_n]$ by setting $q = 1$, we find four representations of $\mathcal{D}_n$ whose dimensions sum to $|\mathcal{D}_n|$. But $\mathcal{D}_n$ has no irreducible representations with dimension greater than two, for all positive integers n. [15]

For the S_n, however, these will indeed turn out to be irreducible, and to match up naturally with the classical theory. This will be the content of the next section.

Another way to think of these representations is to let $\mathcal{I}_{\mathcal{C}}$ to be the vector space generated in $\mathcal{H}_W(q)$ (W is here finite) by C_w, $w \in \mathcal{C}$. We have

$$\mathcal{H}_W(q) = \oplus \mathcal{I}_{\mathcal{C}}$$

as a vector space. Ordering the cells $\mathcal{C}_j$ so that if $\mathcal{C}_i \le_L \mathcal{C}_j$ then $i < j$, the transformation matrix by multiplication by any T_s will be block-upper triangular, from Theorem 6.1 and our thoughts above. Specializing to $q = 1$ (i.e. to $\mathbb{C}[W]$) a version of Maschke's Theorem below shows [12] that we can in fact change bases so that all our matrices are nonzero only in the blocks, and the blocks are unchanged. In other words,

$$\mathbb{C}[W] = \mathcal{H}_W(1) \simeq \oplus \mathcal{K}_{\mathcal{C}}.$$

[6] this is well defined, because if w is $\le_L$ to one element in $\mathcal{C}$, then it is necessarily $\le_L$ to all elements of $\mathcal{C}$

Theorem 7.5. *(A version of Maschke's Theorem reproduced from Lederman page 21) Over a field of characteristic 0, given a representation ρ of a finite group G such that*

$$\rho(g) = \begin{pmatrix} C(g) & 0 \\ E(g) & D(g) \end{pmatrix},$$

we can change bases so that

$$\rho(g) = \begin{pmatrix} C(g) & 0 \\ & D(g) \end{pmatrix}.$$

8. The relationship between the Cell Representations and the Classical Theory of S_n

8.1. **Preliminaries.** We first show that each cell can be associated with a *right* descent set [7]

Lemma 8.1. *If $x \leq_L w$ then $D_R(w) \subset D_R(x)$.*

Proof. It is enough to consider the case $x \leftrightarrow w$ with $D_L(x) \not\subseteq D_L(w)$. There are then two possible situations

$w \prec x$: Suppose that $s \in D_L(x) \setminus D_L(w)$. Then by Corollary 7.2, we have $P_{w,x} = P_{sw,x}$. Now $sw > w$ implies $l(sw) > l(w)$, which implies

$$\frac{l(x) - l(sw) - 1}{2} < \frac{l(x) - l(w) - 1}{2}.$$

If $sw \neq x$, then as $P_{sw,x} \neq 0$ we have $sw < x$ and we can apply the degree requirement on $P_{sw,x}$ from definition 6.4 and conclude

$$\deg(P_{w,x}) = \deg(P_{sw,x}) \leq (l(x) - l(sw) - 1)/2 < (l(x) - l(w) - 1)/2,$$

which is a contradiction to $w \prec x$. Therefore we have $x = sw$ and $w < x$, and using the understanding of the Bruhat order from subexpressions, we conclude $D_R(w) \subset D_R(x)$.

$x \prec w$: Supposing we have a $s \in D_R(w) \backslash D_R(x)$, we use the argument above to show (using the parallel statement of corollary 7.2) that $w = xs$, which would imply that $D_L(x) \subset D_L(w)$, a contradiction. □

Thus, as our cells are characterized by the equivalence relation generated by $\leq_L$, we have that for $x, y \in \mathcal{C}$, $D_R(x) = D_R(y)$, so that

Corollary 8.2. *Each left cell $\mathcal{C}$ can be associated with a right descent set which we call $D_R(\mathcal{C})$*

Now recall that all the information for a left cell's representation is stored in the Kazhdan-Lusztig graph, so that to gain information about the similarities between cells, it is nice to look at their subgraphs $\Gamma_{\mathcal{C}}$. Some corollaries of Proposition 7.3 will then be useful. But first, we state a lemma [2] for finite Coxeter Groups:

Lemma 8.3. *If W is finite with longest element w_0, then we have $\mu(u, v) = \mu(w_0 v, w_0 u)$ and $P_{u,v} = P_{w_0 u w_0, w_0, v, w_0}$*

From this lemma and Proposition 7.3, we conclude:

[7] This association is in general not injective

Proposition 8.4. *(a) If $x \to y$ with $x \neq y$ is an arrow on the Kazhdan Lusztig graph weighted with s and $\mu(x,y)$, then we also have the arrows*

$yw_0 \to xw_0$ weighted with s and $\mu(x,y)$,

$w_0y \to w_0x$ weighted with w_0sw_0 and $\mu(x,y)$,

$w_0xw_0 \to w_0yw_0$ weighted with w_0sw_0 and μ.

(b) If $x \to x$ is an arrow weighted with ϵ and s, then so is $xw_0 \to xw_0$ weighted with $-\epsilon$ and s

Proposition 8.5. *$\Gamma_{\mathcal{C}w_0}$ is isomorphic as a graph to $\Gamma_{\mathcal{C}}$ by switching directions of non-loop arrows and switching μ weights of loop arrows, and identifying x with xw_0.*

Proof. The first assertion of the proposition allows us to identify $\Gamma_{\mathcal{C}}$ with $\Gamma_{\mathcal{C}w_0}$ in a natural way: taking each x, we identify it with xw_0; taking each arrow $x \to y$ $(x \neq y)$ in $\Gamma_{\mathcal{C}}$, we identify it with the arrow $yw_0 \to xw_0$ $(yw_0 \neq xw_0)$ in $\Gamma_{\mathcal{C}w_0}$; taking each arrow $yw_0 \to xw_0$ in $\Gamma_{\mathcal{C}w_0}$ we identify it with the arrow $x \to y$ in $\Gamma_{\mathcal{C}}$. The composition of both these maps, in either order, is the identity as $w_0^2 = 1$, so that each map is an isomorphism of arrows as well as vertices. The proposition also gives us that the weights of all non-loops are preserved. Finally, the comment at the end means that we identify loops about x with loops about xw_0 by switching the μ weights from 1 to -1 and vice versa. $\square$

Similar assertions exist for the other parts of Proposition 8.4.

8.2. **A first look at the Representations.** Let us specialize to $q = 1$, so that $\mathcal{H}_W(1) = \mathbb{C}[W]$, but still let W be an arbitrary finite Coxeter Group. The left multiplication in Theorem 6.1 simplifies considerably. From now on, we refer to the non s weight of an arrow from $x \to w$ as $\mu(x,w)$ so that[8]

$$T_s C_w = \sum_x \mu(x,w) C_x,$$

where the sum is over the set of $x \in W$ such that there is an arrow from x to w with S weight s

From the definition of Coxeter groups, we have a natural representation of W, which we call the sign representation ϵ, in which $w \mapsto (-1)^{l(w)} \in \mathbb{C} = \mathrm{End}_{\mathbb{C}}(\mathbb{C})$. We claim:

Theorem 8.6. *If $w_0 \in W$ is the longest element and $\mathcal{C}$ is a left cell in W, we have*

$$\mathcal{K}_{\mathcal{C}w_0} \simeq \epsilon \otimes \mathcal{K}_{\mathcal{C}},$$

$$\mathcal{K}_{w_0\mathcal{C}w_0} \simeq \mathcal{K}_{\mathcal{C}},$$

$$\mathcal{K}_{w_0\mathcal{C}} \simeq \epsilon \otimes \mathcal{K}_{\mathcal{C}},$$

where the tensor product is as a tensor product of representations over $\mathbb{C}$.

Proof. We will prove this on the level of characters. Recall that the character of $\rho_1 \otimes \rho_2$ is the multiplication of the characters $\chi_1\chi_2$.

Take $w = s_1 \dots s_r \in W$. From the definition of the representation and the revised multiplication law above, lets calculate the trace $\chi_{\mathcal{C}}$ of $\mathcal{K}_{\mathcal{C}}(w)$. Then for each $x \in \mathcal{C}$,

$$T_w T_x = T_{s_1} \dots T_{s_r} T_x.$$

[8] Before, we only had this nonzero if $x \prec w$. Now we have it nonzero if either $x \prec w$ or $w \prec x$

The right hand side is of course a giant sum; but as we are only looking for the trace of left multiplication by T_w, we need only find the coefficient of T_x. Looking at the new multiplication law, we conclude that this is the sum of the products of the μ weights of all circuits from x to x inside the subgraph given by $\mathcal{C}$ that have s weight sequence $s_1, s_2, \dots, s_r$. The trace of the representation $\chi_{\mathcal{C}}(w)$ is then the sum of all of these for all the basis vectors T_y, $y \in \mathcal{C}$, of $\mathcal{K}_{\mathcal{C}}$, i.e. its the sum of the products of the μ weights over all paths from a $y \in \mathcal{C}$ to itself in $\mathcal{C}$ that have weight sequence $s_1, \dots, s_r$.

Let us start with the first assertion, and as usual, take $w = s_1 \dots s_r \in W$. As asserted in the last paragraph, $\chi_{\mathcal{C}w_0}(w)$ is the sum of the product of the μ weights over all the paths in $\Gamma_{\mathcal{C}w_0}$ from a y back to itself with weights $s_1, \dots, s_r$. By Proposition 8.5, this is the same as the sum of the product of the μ weights over all paths in $\Gamma_{\mathcal{C}}$ of paths from an x to an x with weights $s_r \dots s_1$ (note that the order is reversed), with possible changes in signs because of loops. But for $x \neq y$ there is an arrow from $x \to y$ only if $x \prec y$ or $y \prec x$, meaning that we need that $|l(y) - l(x)|$ to be odd. Thus as every path starts and ends in the same place, each path must have an even number of non-loop components, and the sign change in all paths is just ϵ_x. We conclude (using that $s_r \dots s_1 = w^{-1}$),

$$\chi_{\mathcal{C}w_0}(w) = \epsilon_w \chi_{\mathcal{C}}(w^{-1}).$$

But by basic group representation theory [15] we have that $\mathcal{K}_{\mathcal{C}}(w^{-1}) = \overline{\mathcal{K}_{\mathcal{C}}(w)}$ (recall that we have specialized so that these are just representations of groups). As the P's have integer coefficients, the μ's are in particular real, so that the matrix corresponding to $\mathcal{K}_{\mathcal{C}}$ is real and $\overline{\mathcal{K}_{\mathcal{C}}(w)} = \mathcal{K}_{\mathcal{C}}(w)$. Thus we have

$$\chi_{\mathcal{C}w_0}(w) = \epsilon_w \chi_{\mathcal{C}}(w),$$

as desired.

Now we note that from Proposition 8.4 and the same method of proof as 8.5, $\Gamma_{w_0\mathcal{C}w_0}$ is obtained from $\Gamma_{\mathcal{C}}$ by applying the operator $x \mapsto w_0 x w_0$ to all nodes and S weights, and keeping the μ weights unchanged. As we have $D_L(w_0 w w_0) = w_0 D_L(w) w_0$ (Proposition 7.3), this applies even to loops. Thus we have

$$\chi_{\mathcal{C}}(w) = \chi_{w_0\mathcal{C}w_0}(w_0 w w_0).$$

Using that $\chi_{w_0\mathcal{C}w_0}$ is a class function and $w_0^2 = 1$, we conclude

$$\chi_{\mathcal{C}}(w) = \chi_{w_0\mathcal{C}w_0}(w),$$

as desired.

Finally as $w_0\mathcal{C} = w_0\mathcal{C}w_0w_0 = w_0(\mathcal{C}w_0)w_0$, we use the previous parts to conclude

$$\chi_{w_0\mathcal{C}}(w) = \chi_{w_0(\mathcal{C}w_0)w_0}(w) = \chi_{\mathcal{C}w_0}(w) = \epsilon_w \chi_{\mathcal{C}}(w).$$

$\square$

8.3. **A Second Look at Representations of $\mathcal{H}_W(q) = \oplus\mathcal{K}_\mathcal{C}$ through the Parabolic Subgroups.** We start out with a couple definitions: For $I \subset J \subset S$, let $\mathcal{DR}_I^J$ be the set of elements $w \in W$ that have *right* descent sets "in between" I and J. In other words,

$$\mathcal{DR}_I^J = \{w \in W | I \subset D_R(w) \subset J\}.$$

For $J \subset S$, define the parabolic subgroup $W_J \subset W$ (defined before for S_n only) to be the group generated by s_j, $j \in J$. It can be shown [2] that if we divide up W into its W_J cosets, then each coset wW_J has a unique maximal representative (under the Bruhat Order), and that the set of these representatives is indeed $\mathcal{DR}_J^S$. Further, if we map $\pi : W \to \mathcal{DR}_J^S$ by taking w to the maximal representative of wW_J, this map is order preserving.

Lemma 8.7. *Let $y \in \mathcal{DR}_J^S$ and $x \leq y$. Then $a \leq y$ for all $a \in xW_J$*

Proof. Since π is order preserving, we have that for all $a \in xW_J$, $\pi(a) \leq \pi(y) = y$ by the comments above. Thus as a is $\leq$ to the $\pi(a)$ and $a \in xW_J$, we have $a \leq \pi(a) = \pi(x) \leq \pi(y) = y$ $\square$

Corollary 8.8. *The set of elements $\leq y$ is a union of left cosets xW_J, for $y \in \mathcal{DR}_J^S$*

Lemma 8.9. *If we are in the situation of two corollaries ago, $P_{a,y} = P_{x,y}$*

Proof. Let $z = \pi(x)$. Then as $x, a \in xW_J$, we have $a, x \leq z$. As $z \in xW_J$ also, we have $s_1, \ldots, s_r \in J$ such that $z = xs_1 \ldots s_r$. Finally, as $z \in \mathcal{DR}_J^S$, we have that by the dual of the Corollary 7.2, $P_{x,y} = P_{z,y}$. And by the same thinking, $P_{a,y} = P_{z,y}$. $\square$

Let us return for a moment to the old T_w basis of $\mathcal{H}_W(1)$. Remember that because of our specialization $T_wT_x = T_{wx}$, so that we can consider the action of W as $wT_x = T_{wx}$.

Let $J \subset S$, with the corresponding parabolic subgroup W_J. Let $\{\xi_i\}$ be a system of left cosets of W_J in W, and define T_{ξ_i} to be

$$T_{\xi_i} := \sum_{x \in \xi_i} T_x.$$

By definition, $\mathrm{Ind}_{W_J}^W(1)$ can be realized by a vector space with basis $\{T_{\xi_i}\}$ and action of W by left multiplication as above.

Letting $\bar{T}_{\xi_i} = \sum_{\xi_i} \epsilon_x T_x$, we similarly have a realization of $\mathrm{Ind}_{W_J}^W(\epsilon)$ by $w\bar{T}_{\xi_i} = \epsilon_w w\bar{T}_{\xi_i}$ acting on the vector space $\mathcal{E}_J$ generated by the $\bar{T}_{\xi_i}$, where ϵ is the sign representation for Coxeter Groups stated before.

Finally, basic representation theory [15, ch. 7] gives us that $\mathrm{Ind}_{W_J}^W(\epsilon) = \epsilon \otimes \mathrm{Ind}_{W_J}^W(1)$.

Lemma 8.10. *For $y \in \mathcal{DR}_J^S$, $C_y \in \mathcal{E}_J$, where the C_y are elements of the Kazhdan Lusztig Basis. In fact, the C_y form a basis for $\mathcal{E}_J$.*

Proof. From the Corollary 8.8 and Lemma 8.9, we have that if xW_J is a left coset of W_J, then for all $u, v \in xW_J$, $P_{u,y} = P_{v,y}$. Letting $\{a_i\}$ be reprentatives of the left coset of W_J, and using the fact that $q = 1$, we can use the definition

$$C_y = \epsilon_y q_y^{1/2} \sum_{x \leq y} \epsilon_x q_x^{-1} P_{x,y}(q^{-1}) T_x$$

$$= \epsilon_y \sum_{x \leq y} \epsilon_x P_{x,y}(1) T_x$$

$$= \epsilon_y \sum_{\xi_i} P_{a_i,y}(1) \bar{T}_{\xi_i} \in \mathcal{E}_J.$$

Further, the equality above can be seen as a map from the C_y to the T_{ξ_i}. As mentioned before, we have an bijection between $\mathcal{DR}_J^S$ and the W_J cosets. So for starters, this matrix is square. Making this identification, we have that the coefficient of $\bar{T}_y$ is just ϵ_y, and that the coefficients are nonzero only if $\xi_i \leq y$, so that if we order the y's (and through our identification, the ξ_i's) so that $y_i < y_j$ implies $i < j$, then the matrix of transformation between C_y and T_{ξ_i}'s is upper triangular with ϵ's on the diagonal. Thus this matrix invertible so that we obtain that the C_y's are also a viable basis. □

Theorem 8.11. *Identifying cells with their right descent sets by Corollary 8.2, we have, for W_J a parabolic subgroup in W,*

$$Ind_{W_J}^W(\epsilon) = \bigoplus_{J \subset D_R(\mathcal{C})} \mathcal{K}_{\mathcal{C}}.$$

Proof. Let X be the collection of cells $\mathcal{C}$ satisfying $J \subset D_R(\mathcal{C})$. Since $x \leq_L y$ for $x, y \in \mathcal{C}$ implies $x' \leq_L y'$ for all $x, x' \in \mathcal{C}$ and $y, y' \in \mathcal{C}'$, we can consider the induced ordering on cells by $\leq_L$, and from this induce an ordering on X.

As discussed before, $\mathrm{Ind}_{W_J}^W(\epsilon)$ is realized by $\mathcal{E}_J$, and the C_y, for $y \in \mathcal{DR}_J^S$ are a basis for $\mathcal{E}_J$. Further, as the action of $x \in W$ on $\mathcal{E}_J$ coincides with the natural action by left multiplication in $\mathcal{H}_W(1)$ (with sign changes corresponding to ϵ_x), we have that Theorem 7.1 implies that xC_y in $\mathcal{E}_J$ is a sum of C_z with $z \leq_L y$. Ordering the cells so that $\mathcal{C}_j \leq \mathcal{C}_i$ implies $j < i$, we have that the matrix corresponding to action by x is upper block triangular, where each block on the diagonal corresponds to a cell $\mathcal{C}$ and the matrix inside this block corresponds to the action by x on $\mathcal{K}_{\mathcal{C}}$, so that by the augmented version of Maschke's theorem Theorem 7.4 we have the desired decomposition. □

Corollary 8.12.

$$Ind_{W_J}^W(1) \simeq \bigoplus_{J \subset D_R(\mathcal{C})} \epsilon \otimes \mathcal{K}_{\mathcal{C}} \simeq \bigoplus_{D_R(\mathcal{C}) \subset S \setminus J} \mathcal{K}_{\mathcal{C}}$$

Proof. Since $\mathrm{Ind}_{W_J}^W(\epsilon) = \epsilon \otimes \mathrm{Ind}_{W_J}^W(1)$ [13] and $\epsilon \otimes \epsilon = 1$ as representations, we tensor the statement of Theorem 8.11 with ϵ to obtain

$$\mathrm{Ind}_{W_J}^W(1) \simeq \bigoplus_{J \subset D_R(\mathcal{C})} \epsilon \otimes \mathcal{K}_{\mathcal{C}}.$$

For the second statement, we refer to Theorem 8.6 and Proposition 7.3 to see that we have

$$\bigoplus_{J \subset D_R(\mathcal{C})} (\epsilon \otimes \mathcal{K}_{\mathcal{C}}) \simeq \bigoplus_{J \subset D_R(\mathcal{C})} \mathcal{K}_{\mathcal{C}w_0} \simeq \bigoplus_{D_R(\mathcal{C}) \subset S \setminus J} \mathcal{K}_{\mathcal{C}}.$$

□

8.4. **Some thoughts on Young Diagrams.**

8.4.1. *The Robinson-Schensted Correspondence.* We now state the Robinson-Schensted Correspondence, along with some miraculous facts about it. Since the proofs of these theorems are largely combinatorial, we will leave them out due to space considerations.

For this section, we specialize to $W = S_n$. Earlier we represented each $w \in S_n$ with reduced word expressions $w = s_1 \dots s_r$. Another way to represent permutations, however, is by simply writing where each letter is sent. For example, we write $w = x_1 \dots x_n$, where $w(i) = x_i$.

Now we associate to each $w \in S_n$ a pair $(P(w), Q(w))$ of tableaux, where both $P(w)$ and $Q(w)$ are tableaux of the same partition. To do this, let $w = x_1 \dots x_n$, and let us construct $(P(w), Q(w))$ recursively. Supposing that the $i-1$-th step has already been completed, the i-th step goes as follows:[9]

(1) Consider x_i and the P that has been constructed so far.
(2) Compare x_i with the elements of the first row, from left to right.
(3) If x_i is greater than all the elements of the row, create a box at the end of the row, and put x_i into it.
(4) If not, then let the first box that x_i is less than have p in it. Put x_i in the box that p was in, and start this process over again considering p and now going to the second row.
(5) Continue this process until there are no rows left.
(6) In Q, place a new box with i in it in the location that a new box was created in P.

Let us do an example: let $w = 43125 \in S_5$. The process goes as follows, with P on the left and Q on the right.

$$\begin{array}{|c|}\hline 4\\\hline\end{array} \qquad \begin{array}{|c|}\hline 1\\\hline\end{array}$$

$$\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array} \qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$$

$$\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array} \qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 2\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1} \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 4\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1} \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}$$

$$P(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1} \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 4 & 5\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1} \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array} = Q(w)$$

Theorem 8.13. *(The Robinson Schensted Correspondence [2]) This association creates an bijection between S_n and pairs of same shape standard Young Tableaux. In other words,*

$$S_n \simeq \bigsqcup_{\lambda \dashv n} (SYT_\lambda \times SYT_\lambda).$$

[9]In the description below, if a certain row is empty, then we automatically consider any number to be greater than everything in that row, and insert in our element. Also note that this construction creates standard Young Tableaux.

8.4.2. *Descent Sets and the Robinson Schensted Correspondence.* Recall the descent sets defined earlier, $D_R(w) = \{s \in S | l(ws) < l(w)\}$, $D_L(w) = \{s \in | l(sw) < l(w)\}$. Now let $w = x_1 \dots x_n \in S_n$. It can be shown [11]

Proposition 8.14. *If $Inv(w)$ is the set of inversions of w, i.e. the set of (i, j), $i < j$, such that $x_i > x_j$, then $l(w) = |Inv(w)|$. In particular, $s_i \in D_R(w)$ if and only if $x_i > x_{i+1}$.*

Now recall the definition of descent set of a standard Young Taleaux T as stated in section 2:

$$D(T) = \{s_i | \ i + 1 \text{ appears in a strictly lower row than } i \ \}.$$

As might be guessed, there is a connection between these kinds of descent sets: [2]

Theorem 8.15. *For $w \in S_n$*

$$D_L(w) = D(P) \qquad\qquad D_R(w) = D(Q).$$

8.4.3. *Knuth and Dual-Knuth equivalence.* We say that $u, v \in S_n$ are Knuth similar if, for $u = x_1 \dots x_n$ and $v = y_1 \dots y_n$, $x_i = y_i$ for all but two adjacent i, and that next to these there is a number between the two. For example, if $u = 213456$ and $v = 231456$, then u and v are Knuth similar.

We define Knuth equivalence $\sim_K$ to be the equivalence relation generated by Knuth similarity. In other words, $u \sim_K v$ if we have a sequence $u = a_0, a_1, \dots, a_m = v$ where a_i and a_{i+1} are Knuth similar for all $i \in \{1, \dots, m-1\}$.

Now we say that u and v are dual Knuth similar if u^{-1} and v^{-1} are Knuth equivalent. Equivalently [2], u and v are dual Knuth similar if both have representations as $x_1 \dots x_n, y_1 \dots y_n$ such that $x_i = y_i$ for all but two i, that these x_i differ by 1, and letting them be j and $j+1$, that $j+2$ or $j-1$ occurs between these x_i. For example, $u = 143562$ and $v = 243561$ are dual Knuth Similar.

We define the dual Knuth equivalence $\sim_{dK}$ to be the one generated by this. We now have the most amazing fact [2]:

Theorem 8.16. *If $u, v \in S_n$ are associated with $(P(u), Q(u))$ and $(P(v), Q(v))$, then $P(u) = P(v)$ if and only if u and v are Knuth equivalent, and $Q(u) = Q(v)$ if and only if the two are dual Knuth equivalent.*

8.5. **Knuth equivalences, Robinson-Schensted Correspondence, and Kazhdan-Lusztig Graphs.**

Theorem 8.17. *If $u \sim_{dK} v$, then $u \sim_L v$.*

Proof. It is enough to show that if u and v are dual Knuth similar, then there are antiparallel arrows between u and v in the Kazhdan Lusztig graph of W.

This might seem random, but suppose for a moment that we have s, s' such that

$$su < u < s'u = v < sv.$$

Then since $(l(v) - l(u) - 1)/2 = 0$, $P_{u,v}$ must have degree zero Corollary 6.6 implies that this constant is 1 and $u \prec v$. Further, we have $s \in D_L(u) \setminus D_L(v)$, and $s' \in D_L(v) \setminus D_L(u)$, so that there are antiparallel arrows between u and v.

Thus it is actually enough to show the above statement, for some $s, s' \in S$ (possibly switching u and v). Dualizing and relabeling, this means that it is enough to show that $u, v \in S_n$ being Knuth similar implies that there exist s and s' such that

$$su^{-1} < u^{-1} < s'u^{-1} = v^{-1} < sv^{-1}$$

(with u and v possibly switched). Since $x < y$ if and only if $x^{-1} < y^{-1}$ (Humphreys 5.9), this is equivalent to

$$us < u < us' = v < vs.$$

So suppose that u and v are Knuth similar, and their representations differ in positions i and $i+1$. Then we have two cases:

(a) If in either u or v, $x_i < x_{i-1} < x_{i+1}$, i.e. the intermediate value is before, then without loss of generality let $u = x_1 \dots x_n$ (possibly switching the labels of u and v). Then by Proposition 8.14, we have $us_{i-1} < u < us_i = v < us_{i-1}$, as desired.

(b) If the intermediate value is after, then the same logic, again possibly switching the labels u and v, we have $us_i < u < us_{i-1} = v < us_i$ $\square$

It is proved in [2] that

Theorem 8.18. *If $u \sim_K v$, then $Q(u)$ uniquely determines $Q(v)$. In particular, if u and v are Knuth similar, then $Q(u)$ and $Q(v)$ differ by a transposition of labels.*

Now we define the Knuth descent set of an index i,

$$KD(i) = \{w = x_1 \dots x_n \in S_n | x_{i-1}x_i x_{i+1} \text{ is not monotonically increasing or decreasing}\}.$$

In other words, $KD(i)$ is the set of $w \in S_n$ that are Knuth similar to some w' with the three involved indices being $x_{i-1}x_i x_{i+1}$ (the intermediate neighbor is considered one of the "involved indices.") [10] For the time being, we will keep this notation: if $w \in KD(i)$, then the unique element of S_n that is Knuth Similar to it with involved indices centered about i, is w'.

There is the following lemma [2].

Lemma 8.19. *If $u, v \in KD(i)$ such that $u \prec v$ or $v \prec u$ with Kazhdan-Lusztig Polynomial coefficient μ, we have $u' \prec v'$ or $v' \prec u'$ also with Kazhdan-Lusztig coefficient μ.*

Now suppose that $u \in KD(i)$ and $u \sim_L v$. Then by definition, we have a loop on the Kazhdan-Lusztig graph from u to v and back to u again. From Corollary 8.2, we have $D_R(u) = D_R(v)$. But from the definition of $KD(i)$ and Proposition 8.14, this means that $v \in KD(i)$ as well. But everything in this path is also, by the same argument, in $KD(i)$. But then by Lemma 8.19 we can construct a corresponding path from u' to v' and back to u', if we can verify that the Left Descent sets give us that these arrows exist (i.e. if we have an arrow $a \to b$ on the first circular path, we have s such that $D_L(a) \not\subseteq D_L(b)$). From the amazing fact 8.16 and the fact that by definition w and w' are Knuth similar, we have that $P(w) = P(w')$ for all w in this path, so that the amazing fact 8.15 gives us that $D_L(w) = D(P(w)) = D(P(w'))D_L(w')$ for the path, so that in fact these arrows exist, and we conclude

Corollary 8.20. *If $u \in KD(i)$ and $u \sim_L v$, then we also have $u' \sim_L v'$*

[10] Note that this w' is unique, as only one of x_{i-1} and x_{i+1} can be intermediate to the other two.

This corollary also has a corollary. Suppose that $u \sim_L v$ and that we have a chain of Knuth similarities $u = u_0, u_1, \ldots, u_r$ with the three involved indices between u_i and u_{i+1} being centered about j_i. Then u_0 and u_1 are Knuth similar with $u_0 \in KD(j_0)$, and Corollary 8.20 combined with the definition of $KD(j_0)$ gives us that $v \in KD(i)$ and

$$u_1 = u' \sim_L v' =: v_1$$

Iterating this process, we can construct chain of Knuth similarities $v_0, \ldots v_r$ with $u_i \sim_L v_i$. Marking KS as a Knuth similarity we have essentially constructed

$$\begin{tikzcd}
u_0 \arrow[r, dash, "KS"] \arrow[d, dash, "L"'] & u_1 \arrow[r, dash, "KS"] \arrow[d, dotted, "L"'] & u_2 \arrow[d, dotted, "L"'] & \ldots & u_r \arrow[d, dotted, "L"'] \\
v_0 \arrow[r, dotted, "KS"] & v_1 \arrow[r, dotted, "KS"] & v_2 & \ldots & v_r
\end{tikzcd}$$

Making the endpoint u_r move about the set of elements Knuth Equivalent to u and using Corollary 8.2, we have

Corollary 8.21. $\{D_R(x) | x \sim_K u\} = \{D_R(y) | y \sim_K v\}$

8.5.1. *Recap.* Let us stop for a moment to take stock of the previous section. We first defined $P(w)$ and $Q(w)$ for an element $w \in S_n$. Then we asserted that $D_L(w) = D(P(w))$ and $D_R(w) = D(Q(w))$, and that two elements were Knuth equivalent (dual Knuth Equivalent) if and only if they corresponded to the same P (Q). It turned out that $\sim_{dK} \implies \sim_L$, and that $w \sim_K w' \implies Q(w)$ determines $Q(w')$. Finally, we introduced the sets $KD(i)$ which allowed us to associate each w in it with a w', and introduced a method of proof in Corollary 8.21 that will be used extensively in the future.

8.6. **Conclusion of this Approach.** We now come to the final part of our discussion of left cell representation of Hecke Algebras of type A_{n-1}: the connection to the classical theory. We start with a few preparatory theorems.

Theorem 8.22. *$u \sim_L v$ implies that $Q(u) = Q(v)$*

Proof. From Corollary 8.21, we have that

$$E_u := \{D_R(x) | x \sim_K u\} = \{D_R(y) | y \sim_K v\} =: E_v,$$

and from 8.15 and 8.16, we have that (as $P(x) = P(u)$ implies that $Q(x)$ and $Q(u)$ correspond to the same partition)

$$\{D_R(x) | x \sim_L u\} = \{D(Q(x)) | \ Q(x) \text{ and } Q(u) \text{ correspond to the same partition}\}.$$

In particular, if the partition is $\lambda = (\lambda_1, \ldots, \lambda_k)$ both E_u and E_v have the descent set $\{\lambda_1, \lambda_1 + \lambda_2 \ldots\}$ corresponding to what we call the superstandard Young tableau. In each, this descent set corresponds to only one element, as an element is uniquely determined by its P and Q. (which we call u_A and v_A, respectively).

Now letting u be connected to u_A by a chain of Knuth similarities $u = u_0, \ldots, u_A$, we can use the method of Corollary 8.21 to give us an associated chain of Knuth similarities $v = v_0, \ldots, v_z$.

We claim $v_z = v_A$. To see this, $D_R(v_z) = D_R(v_A)$ by construction, and since V_A is the unique element in E_v that has that descent set, we conclude $v_z = v_A$.

Now by Theorem 8.18, we go back down the change to obtain $Q(u) = Q(v)$. ∎

Corollary 8.23. *Each cell $\mathcal{C}$ in S_n can be associated with a tableau T such that $\mathcal{C} = \{w \in S_n | Q(w) = T\}$*

Proof. From above we have that $u \sim_L v$ implies that $Q(u) = Q(v)$. From Theorem 7.15 and Theorem 7.16, we have that $Q(u) = Q(v)$ implies that $u \sim_L v$. The result follows. $\square$

From the above corollary, we can associate with any $\mathcal{C}$ a partition λ which is the underlying shape of the tableau T.

Theorem 8.24. *If $\mathcal{C}_1$ and $\mathcal{C}_2$ both correspond to the same partition λ, then their subgraphs $\Gamma_{\mathcal{C}_1}$ and $\Gamma_{\mathcal{C}_2}$ are isomorphic.*

Proof. From the above corollary, each cell can be realized as

$$\mathcal{C}_i = \{w \in S_n | Q(w) = T_i\}.$$

Now using the Robinson-Schensted correspondence, we identify the vertices w of the Kazhdan Lusztig graph of S_n with $(P(w), Q(w))$, allowing us to define a bijection of vertices $\Gamma_{\mathcal{C}_1} \to \Gamma_{\mathcal{C}_2}$ given by $(P(w), T_1) \mapsto (P(w), T_2)$. From Theorem 8.15, this map also preserves left descent sets. Thus, it only remains to show that the μ weights are also preserved. Note that the preservation of left descent sets also implies that we do not need to consider μ weights of loops $x \to x$ in the Kazhdan Lusztig graphs (as these are determined by the left descent Sets). In the following, we will say that two vertices are "associated" with weight μ if the arrows between them have weight μ.

We follow the method introduced in Corollary 8.21. Suppose we have an arrow $(P(u), T_1) \xrightarrow{\mu} (P(v), T_1)$ in $\Gamma_{\mathcal{C}_1}$, and consider the vertices $(P(u), T_2), (P(v), T_2) \in \Gamma_{\mathcal{C}_2}$. From Theorem 8.16, $(P(u), T_1)$ and $(P(u), T_2)$ are Knuth equivalent; let us connect them with a chain of Knuth similarites

$$(P(u), T_1) = (P(u), Q_0), (P(u), Q_1), \ldots, (P(u), Q_r) = (P(u), T_2),$$

where $(P(u), Q_i)$ is Knuth similar to $(P(u), Q_{i+1})$ with involved indices centered about j_i. We have that $(P(u), Q_0)$ and $(P(v), Q_0)$ are associated with weight μ, and as they both are associated with Q_i, then Theorem 8.15 implies that both have the same right descent sets, so that both are in $KD(j_0)$. Thus we can associate, by Lemma 8.19, $(P(u), Q_0)'$ and $(P(v), Q_0)'$ with weight μ. But by definition, $(P(u), Q_0)' = (P(u), Q_1)$, and by Theorem 8.18, $(P(v), Q_0)' = (P(v), Q_1)$. Thus we have associated $(P(u), Q_1)$ with $(P(v), Q_1)$ with weight μ. Iterating the process, we can associate $(P(u), T_2)$ with $(P(v), T_2)$ with weight μ as desired.

$$\begin{tikzcd}
(P(u),Q_0) \arrow[r, dash, "KS"] \arrow[d, dash, "\mu"'] & (P(u),Q_1) \arrow[r, dash, "KS"] \arrow[d, dotted, "\mu"'] & (P(u),Q_2)\ldots \arrow[d, dotted, "\mu"'] & (P(u),Q_r) \arrow[d, dotted, "\mu"'] \\
(P(v),Q_0) \arrow[r, "KS"] & (P(v),Q_1) \arrow[r, "KS"] & (P(v),Q_2)\ldots & (P(v),Q_r)
\end{tikzcd}$$

$\square$

Thus, we have $\mathcal{K}_{\mathcal{C}_1} \simeq \mathcal{K}_{\mathcal{C}_2}$ if $D_R(\mathcal{C}_1)$ and $D_R(\mathcal{C}_2)$ correspond to the same partition. So we group our representations by partitions and consider the representations $\mathcal{K}_\lambda$. Recalling the ρ_λ from the classical theory in section 2, we arrive at the final result of this section

Theorem 8.25. *$\mathcal{K}_\lambda = \rho_\lambda$ as representations of S_n*[11]

[11] Remember that for some time now we have specialized to $q = 1$

Proof. We will prove this on the level of characters. Recall "Young's Rule"

$$\mathrm{Ind}_{W_J}^{S_n}(1) \simeq \bigoplus_{\lambda \dashv n} |T \in \mathrm{SYT}_\lambda : D(T) \subseteq S \setminus J| \rho_\lambda,$$

and rewrite it

$$\mathrm{Ind}_{W_{S\setminus J}}^{S_n}(1) \simeq \bigoplus_{\lambda \dashv n} |T \in \mathrm{SYT}_\lambda : D(T) \subseteq J| \rho_\lambda$$

by interchanging $S \setminus J$ and J.

From Corollary 8.12, we have

$$\mathrm{Ind}_{W_{S\setminus J}}^{S_n}(1) = \bigoplus_{D_R(\mathcal{C}) \subset J} \mathcal{K}_{\mathcal{C}},$$

and by Corollary 8.23, Theorem 8.24, and Theorem 8.15, we can rewrite this as

$$\mathrm{Ind}_{W_{S\setminus J}}^{S_n}(1) = \bigoplus_{D(Q) \subset J} \mathcal{K}_{\mathcal{C}} = \bigoplus_{\lambda \dashv n} |Q \in SYT_\lambda : D(Q) \subset J| \mathcal{K}_\lambda,$$

so that writing T instead of Q we obtain Young's rule, except with $\mathcal{K}_\lambda$ in the place of ρ_λ.

$$\mathrm{Ind}_{W_{S\setminus J}}^{S_n}(1) = \bigoplus_{\lambda \dashv n} |T \in SYT_\lambda : D(T) \subset J| \mathcal{K}_\lambda.$$

Now consider these two versions of Young's Rule as matrices (these matrices have identical coefficients) from the vector space with basis elements $\mathrm{Ind}_{W_{S\setminus J}}^{S_n}(1)$ to the vector spaces with basis elements ρ_λ and $\mathcal{K}_\lambda$. We claim that if we trim the set of $S \setminus J$ appropriately, these matrices become an invertible square matrices, allowing us to write the characters ρ_λ and $\mathcal{K}_\lambda$ in terms of the characters of $\mathrm{Ind}_{W_{S\setminus J}}^{S_n}(1)$. And since the trimming will be identical on both matrices, this will show that the characters of ρ_λ and $\mathcal{K}_\lambda$ are the same, proving the theorem.

Now for all $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, we associate it with[12]

$$J_\lambda := (\lambda_1, \lambda_1 + \lambda_2, \ldots, \sum_1^{k-1} \lambda_i).$$

Order the λ lexicographically,[13] inducing an ordering of the J_λ and thus an ordering of the $S \setminus J_\lambda$ (we say that $(S \setminus J_1) < (S \setminus J_2)$ if $J_1 < J_2$). Consider the submatrix M of Young's rule with columns corresponding to the $S \setminus J_\lambda$, with both rows and columns with this ordering. We claim that this matrix is lower triangular with 1's on the diagonal, implying that M is invertible and thus the theorem. Remember that since the map $\lambda \rightsquigarrow J_\lambda$ is injective, this matrix is square.

Consider the diagonal elements $M_{\lambda, S\setminus J_\lambda} = |T \in SYT_\lambda : D(T) \subset J_\lambda|$. The tableau T_0 with first row $1, 2, \ldots, \lambda_1$, second row $\lambda_1 + 1, \ldots \lambda_1 + \lambda_2$, and so forth has $D(T) = J_\lambda$. If T is any other tableau of λ, consider the first row j in which T is different from T_0. There are two cases:

(1) The first element of this row is different. Then there is nowhere on T to place what should have been there, $\left(\sum_1^{j-1} \lambda_i\right) + 1$, as T is standard so increases to the right and going down.

---

[12]Note that this mapping is injective, i.e no J corresponds to more than one λ.

[13]i.e. $\lambda^1 < \lambda^2$ if for the first j such that λ_j^i are different, $\lambda_j^1 < \lambda_j^2$

(2) The first difference occurs at location i on the row, where $1 < i \leq \lambda_j$. Then $i-1 \in D(T)$, but $i-1 \notin J_\lambda$.

Thus we have $M_{\lambda, S\setminus J_\lambda} = 1$.

Now consider the elements above the diagonal, i.e. the elements $M_{\lambda, S\setminus J_{\lambda'}}$ with $\lambda' > \lambda$. Suppose that the lengths of the first rows are not equal, i.e. $\lambda_1 < \lambda'_1$. Then looking at the first row of any tableau T of λ, we must have at least one of $\{1, 2, \ldots, \lambda_1\}$ inside $D(T)$. But none of these elements are in $J_{\lambda'}$, so that $M_{\lambda, S\setminus J_{\lambda'}} = 0$. The same argument holds if the first length different occurs later, i.e. $\lambda_i = \lambda'_i$ for $i < j$, but $\lambda_j \neq \lambda'_j$.

So this matrix is invertible, and the theorem follows. □

Recap: We showed that each left cell can be associated with a partition λ by taking any element w in that cell and taking the partition corresponding to $Q(w)$. Then we proved that all cells corresponding to the same partition gave rise to the same representations, and made our beautiful isomorphism.

Corollary 8.26. *The representations $\mathcal{K}_\mathcal{C}$ are irreducible for $W = S_n$ and $q = 1$.*

9. A completely different aproach of reps

We now give a different approach to this same problem, that of constructing irreducible representations of $\mathcal{H}_{S_n}(q)$ (and thus of $\mathbb{C}[S_n]$). This approach will be fueled by a more fundamental approach to semi-simple algebras. First recall the most basic result:

Theorem 9.1. *If M is a semi-simple algebra over an algebraically closed field k, then we have a natural algebra decomposition into simple algebras*

$$M \simeq \bigoplus_i End_k(V_i) \simeq \bigoplus_i Mat_{\mu_i}(k) \simeq \bigoplus_i M p_i,$$

where the V_i are vector spaces over k of dimension μ_i, $Mat_\mu(k)$ is the full matrix algebra of rank μ over k, and the p_i form a set of minimal central idempotents of M.

While the next results will be stated and proved for the more general case of Theorem 9.1, note that Lemma 5.1 gives us that $\mathcal{H}_W(q)$ satisfies this requirement for all but finite $q \in \mathbb{C}$. Thus from now on, let M (and occasionally N) be semi-simple algebras over an algebraically closed field. To maintain the connection, we will again recall the more canonical version of Maschke's theorem

Theorem 9.2. *If the set of ρ_i are the irreducible representations for G with corresponding vector spaces V_i, we have*

$$\mathbb{C}[G] = \oplus End_k(V_i).$$

9.1. **A First way of Inclusion analysis.**

For simple algebras, we have the following proposition:

Proposition 9.3. *If $M = End_k(V)$ and $q \in M$ is an idempotent, $qMq \simeq End_k(qV)$*

Proof. We just take a basis $e_1, \ldots e_r$ of qV, and extend to a basis $e_1, \ldots e_s$ of V. As q is the projection onto qV, we have that qmq, by simple matrix multiplication, is just m in the upper left corner $r \times r$ block, and 0 everywhere else. Thus the claim follows. □

Now consider an algebra inclusion $N \subset M$, where the N and M decompose into sums of simple algebras as $N = \oplus N q_j$ and $M = \oplus M p_i$. Define

$$M_{ij} := p_i q_j M p_i q_j \qquad\qquad N_{ij} := p_i q_j N p_i q_j.$$

Proposition 9.4. $M_{ij} = End(q_j V_i)$

Proof. The p_i are central in M, and the $q_j \in N \subset M$, so that

$$(p_i q_j)^2 = p_i^2 q_j^2 = p_i q_j,$$

and $p_i q_j$ are idempotents in $p_i M$. By the previous proposition, if $M_i := M p_i$,

$$M_{ij} = q_j p_i M p_i q_j = q_j M_i q_j = q_j \mathrm{End}_k(V_i) q_j = \mathrm{End}_k(q_j V_i).$$

□

Now consider M as an N module. Since p_i is a central idempotent, we have that the map $q_j N \to M_{ij}$ given by $q_j n \mapsto p_i q_j n = p_i q_j n p_i q_j \in M_{ij}$ is an algebra homomorphism. In particular, this makes $q_j V_i$ into a representation of the simple algebra $q_j N$. Thus, if $q_j N = \mathrm{End}_k(W_j)$, the theory of modules over a simple algebra gives us that that $q_j V_i = \oplus W_j$ as an $N q_j$ module, and that the representation is given by the diagonal action on this set of W_j (i.e. if $x \in N q_j$, $x \mapsto (x, x, \ldots, x)$ acting on $q_j V_i$). The number of copies of W_j in $q_j V_i$ will be denoted by λ_{ij}.

The q_j are minimal central idempotents, so that $q_i q_j = 0$ for $i \neq j$. This means that $q_i V \cap q_j V = \{0\}$ for $i \neq j$ because $q_i x = q_j y$ implies that $q_i x = q_i^2 x = q_i q_j y = 0$. Further, as $\sum q_j = 1$, we have that any $v \in V$ can be expressed as a sum of elements in the $q_j V$. Applying this to the case that we had before, we conclude that

$$V_i = \bigoplus_j q_j V_i = \bigoplus_j W_j^{\lambda_{ij}}.$$

Thus we have a decomposition of each V_i in terms of how the $q_j N$ act upon them. We can now collect the λ_{ij} into an "inclusion matrix" Λ, whose i,j-th component is given by λ_{ij}.

9.2. **A Cleaner way of Analysis.** Let us formulate this in a more abstract fashion. The following interpretation will be especially useful for our discussion of representations.

Recall the construction of K_0 of a ring R: the equivalence classes under isomorphism of the finitely generated projective R modules form a semigroup under the operation of direct sum; turning the semigroup into a group we get $K_0(R)$. For a group algebra $\mathbb{C}[G]$, $K_0(\mathbb{C}[G])$ is the Grothendieck group of its representations. More generally, for a semisimple algebra M with irreducible modules V_i, it is the group

$$K_0(M) = \mathbb{Z}[V_1] \oplus \mathbb{Z}[V_2] \oplus \cdots \oplus \mathbb{Z}[V_m] \simeq \mathbb{Z}^m,$$

where $[V_i]$ represents the equivalence class of the module V_i.

Thinking of this in another way, note that for an idempotent $e \in M$, we can consider the equivalence class of the module Me in $K_0(M)$ Then if e_i is a minimal idempotent (note not minimal *central* idempotent) in $M p_i$, $M e_i$ is a minimal left ideal of M; in particular, it is equivalent to V_i as a M module (for example, the matrix e_1 with a 1 in the top left corner and zeros everywhere else is a minimal idempotent, and $\oplus \mathrm{End}_k(V_i) e_1 = \mathrm{End}_k(V_1) e_1 = V_1$). Further, each V_i is equivalent to an $M e_i$, so that if we take an e_i a minimal idempotent in each $M p_i$, we have a basis $[M e_i]$ of $K_0(M)$.

Now, if we have an inclusion of matrix algebras $N \subset M$ with $N = \oplus q_j N = \oplus \mathrm{End}_k(W_j)$ and $M = \oplus p_i M = \oplus \mathrm{End}_k(V_i)$, we can consider the corresponding map $K_0(N) \to K_0(M)$, where we take the class of Ne to the class of Me for an idempotent e. We claim that:

Theorem 9.5. *This map is given by the matrix Λ with respect to the bases $[Nf_j]$ and $[Me_i]$ of $K_0(N)$ and $K_0(M)$ respectively.*

Proof. To see that this is true, it suffices to show that $[Nf_j]$ maps to $\sum_i \lambda_{ij}[Me_i]$. But since $M = \oplus p_i M = \oplus M p_i$, we have $[Mf_j] = \sum_i [Mf_j p_i] = \sum_i [Mf_j q_j p_i]$. And from the arguments in section 9.1, we have that $f_j q_j p_i$ is the sum of λ_{ij} minimal idempotents in M_{ij}. A minimal idempotent in M_{ij} is still minimal in $p_i M$, so that these minimal idempotents are equivalent to e_i, and we have

$$[Mf_j] = \sum_i \lambda_{ij}[Me_i],$$

as desired. $\square$

To think of this in terms of representation theory, we note that K_0 corresponds to the Grothendieck group, so that the map $K(H) \to K(G)$ from the matrix Λ represents induction, as it takes W_i to $\mathbb{C}[G] \underset{\mathbb{C}[H]}{\otimes} W_i$, i.e. (for the second we use Frobenius reciprocity)

$$\mathrm{Ind}_H^G(W_i) = \oplus_j (V_j)^{\lambda_{ji}} \qquad\qquad \mathrm{Res}_H^G(V_i) = \oplus_j (W_j)^{\lambda_{ij}}.$$

9.3. **Bratteli Diagrams.** To simplify even further, we introduce a "Bratteli" Diagram to each inclusion $N \subset M$, which is a bicolored weighted multigraph with points and lines as defined by the following: For each minimal central idempotent p_i in M such that $Mp_i = \mathrm{End}_k(V_i) = \mathrm{Mat}_{\mu_i}(k)$, we have a black point (which for now we call the i-black point) on our Bratteli Diagram with weight μ_i, and for each minimal central idempotent q_j in N such that $Nq_j = \mathrm{End}_k(W_j) = \mathrm{Mat}_{\nu_j}(k)$, we have a white point (which we call the j-th white point) on our Bratteli Diagram with weight ν_j. The i-th black point and the j-th white point are connected by λ_{ij} lines.

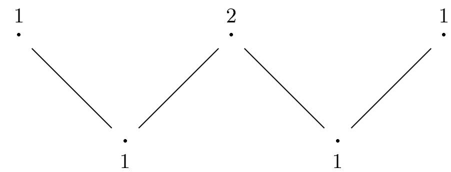
9.4. **An Example.** As an example of what we have done so far, consider the inclusion of matrices $\mathbb{C}[S_2] \subset \mathbb{C}[S_3]$, where S_2 and S_3 are the symmetric groups of permutations on 2 and 3 letters, respectively. Then modules over $\mathbb{C}[S_i]$ are just representations over S_i, irreducible representations correspond to the W_i and V_i considered before, and the $K_0(S_i)$ are the Grothendieck groups of S_i, given by

$$K_0(S_i) = \underset{irreps}{\oplus} \mathbb{Z}\chi_s.$$

From basic representation theory, we have that the 2-dimensional representation of S_3 restricts to the sum of a trivial representation and a sign representation of S_2, the trivial representation of S_3 restricts to the trivial representation of S_2, and that the sign representation of S_3 restricts to the sign representation of S_2. Thus our inclusion matrix is (with S_2 representations ordered by trivial, sign and S_3 reps ordered by trivial, 2d, and sign)

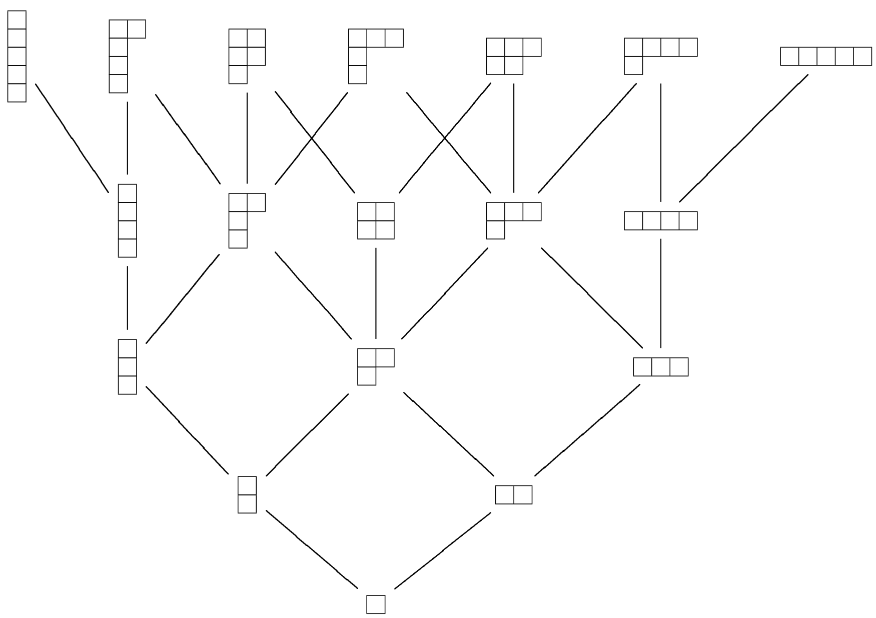
$$\Lambda = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

On the Bratteli Diagram, we put the "black" dots above and the "white" dots below, so that coloring becomes unnecessary, and we obtain



In the future, we will always put the dots for the larger algebra up top, and the dots for the smaller algebra below.

Now recalling the classical theory of representations of S_n, we can actually associate these representations with Young Diagrams, and noting the restriction rule for representations, we have that the Bratteli diagram for the inclusion $S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5$ is given by



10. Back to the Hecke Algebras

Let us return to the Hecke algebras $\mathcal{H}_{S_n}(q)$ and show another construction of irreducible representations which like before will be associated with partitions $\lambda \dashv n$. This discussion will also give us a hint on what set of $q \in \mathbb{C}$ make the $\mathcal{H}_{S_n}(q)$ be non-semi-simple.

Consider a symmetric group S_n and the associated Hecke Algebra $\mathcal{H}_{S_n}(q)$. Define

$$Q_i := \frac{T_{s_i} + 1}{q+1},$$

for $i \in \{1, \ldots, n-1\}$ and $q \neq -1$. It follows through computation [6] that the Q_i form a presentation of $\mathcal{H}_{S_n}(q)$ with relations

$$Q_iQ_i = Q_i \qquad i \leq n-1 \tag{7}$$

$$Q_iQ_{i+1}Q_i - Q_{i+1}Q_iQ_{i+1} = \frac{q}{(q+1)^2}(Q_i - Q_{i+1}) \qquad i \leq n-2 \tag{8}$$

$$Q_iQ_j = Q_jQ_i \qquad |i-j| \geq 2 \text{ and } i, j \leq n-1. \tag{9}$$

Now let $\Omega_n \subset \mathbb{C}$ be the set of roots of the polynomials $x^i - 1$, $i \leq n-1$, without 1 and with 0. In other words,

$$\Omega_n = \{0, -1\} \cup \left\{ a \in \mathbb{C} | \exists i \in \{1, \ldots, n-1\} \text{ s.t. } \sum_0^i a^i = 0 \right\}.$$

For $q \notin \Omega_n$, consider the Bratelli diagram above for the inclusions $S_1 \subset S_2 \subset \ldots S_n$ with a "-1" floor corresponding to an empty diagram $\cdot$ and a single line to the Young Diagram with one box, and take a partition $\lambda \dashv n$. We define V_λ to be a vector space generated over $\mathbb{C}$ by the set of paths p from $\cdot \to \lambda$ in n steps. We claim that this vector space can be seen as an irreducible $\mathcal{H}_{S_n}(q)$ representation J_λ, and that for $q = 1$ this representation J_λ is in fact the representation associated with λ under the classical theory and via the previous part, the representation associated with the cell representation $\mathcal{K}_\lambda$. In other words, all three of these constructions give the same irreducible representations of S_n, with the same assocation to the partitions of n.

10.1. **Construction of the Tower Representation.** To define a representation, we merely have to give an action of the Q_i on V_λ, and make sure that it satisfies rules (7)-(9). For $n = 1$, we define $V_\lambda = \mathbb{C}$ where $\lambda = \square$, and give it the trivial action by $\mathcal{H}_{S_1}(1) = \mathbb{C}$. Now let $p = (p_0, p_1, \ldots, p_n)$ be a path from $\cdot \to \lambda$, with p_i being where the path ends after i steps. In other words, p_i corresponds to a partition of i. Let e_p be the basis vector of V_λ corresponding to this path. We define J_λ as follows

$$Q_i(e_p) = \begin{cases} e_p & p_{i+1} \text{ comes from } p_{i-1} \text{ by adding two boxes to the same row,} \\ 0 & p_{i+1} \text{ comes from } p_{i-1} \text{ by adding two boxes to the same column,} \\ d_p^i(q)e_p + (1 - d_p^i(q))e_{p'} & p_{i+1} \text{ comes from } p_{i-1} \text{ by adding boxes to different columnss.} \end{cases}$$

In the third case, p' is defined by $p'_j = p_j$ for all $j \neq i$, and $p'_i \neq p_i$. In other words, p' is the path that gets p_{i+1} from p_{i-1} by adding the boxes in the opposite order as p. Now suppose that the path p adds the i-th block in column r, and the $i+1$-th block in column s. We define, by abuse of notation,

$$d_p^i = (s - r) + (p_{i+1}^r - p_{i+1}^s),$$

where p_{i+1}^j is the number of boxes in the j-th column of p_{i+1}. Finally, we define $d_p^i(q)$ by (writing d for d_p^i)

$$d_p^i(q) = \begin{cases} \frac{1-q^{d+1}}{(1+q)(1-q^d)} & q \neq 1 \\ \frac{d+1}{2d} & q = 1. \end{cases}$$

Note that since there is no Q_n, $|d|$ is always less than or equal to $n-1$, and $q \notin \Omega_n$. Thus $Q_i(e_p)$ is well defined for all $i \in \{1, \dots, n-1\}$ and paths p to λ.

Now since $d^i_{p'} = -d^i_p$ by definition, we have $d^i_p(q) + d^i_{p'}(q) = 1$, so that on the subspace of V_λ generated by $\{e_p, e_{p'}\}$, we have

$$Q_i = \begin{pmatrix} d^i_p(q) & 1 - d^i_{p'}(q) \\ 1 - d^i_p(q) & d^i_{p'}(q) \end{pmatrix} = \begin{pmatrix} d^i_p(q) & d^i_p(1) \\ 1 - d^i_p(q) & 1 - d^i_p(q) \end{pmatrix}.$$

Theorem 10.1. *This is a true represntation of $\mathcal{H}_{S_n}(q)$*

Proof. It is pretty easy to check through computation that the Q_i satisfy equations (7) and (9), and for (8), we refer to [16]. □

We have thus created representations J_λ of $\mathcal{H}_{S_n}(q)$ for all partitions $\lambda \dashv n$.

Theorem 10.2. *These representations J_λ are irreducible and mutually inequivalent*

Proof. We will prove this by induction on S_n. The cases of S_1 and S_2 are easily calculated. So we assume the theorem up to $n-1$, and prove for n. Let $\mu \dashv (n-1)$ and $\lambda \dashv n$; we write $\mu \rightharpoonup \lambda$ if μ is obtained from λ by removing a box. Also note that we have the natural injection $\mathcal{H}_{S_{n-1}}(q) \hookrightarrow \mathcal{H}_{S_n}(q)$ given by $Q_i \mapsto Q_i$. By construction then, we have

$$V_\lambda = \bigoplus_{\mu \rightharpoonup \lambda} V_\mu,$$

as $\mathcal{H}_{S_{n-1}}(q)$ modules. By induction, this shows that the V_λ are mutually inequivalent, because they reduce to different things for $n \geq 3$. Also by induction, the V_μ are irreducible $\mathcal{H}_{S_{n-1}}(q)$ modules. So if $V \subset V_\lambda$ is an irreducible $\mathcal{H}_{S_n}(q)$ module, we must have $V = \bigoplus_I V_i$ for $I \subset \{\mu | \mu \rightharpoonup \lambda\}$. By the third case of the definition of $Q_{n-1}(e_p)$, we have that if $V_\mu \subset V$, then we must have $V_{\mu'} \subset V$ if μ' can be obtained from μ through removing a box and adding a box. But we can get any $\mu' \rightharpoonup \lambda$ from any $\mu \rightharpoonup \lambda$ through successive iterations of removing a box and adding a box. Thus if $V \neq 0$, we must have $V = V_\lambda$, as desired. □

Theorem 10.3. *For $q \notin \Omega_n$, we have $\mathcal{H}_{S_n}(q)$ is semisimple and*

$$\mathcal{H}_{S_n}(q) = \bigoplus_{\lambda \dashv n} End(V_\lambda).$$

Proof. From the construction of irreducibles above, we have that $\mathcal{H}_{S_n}(q)$ contains a quotient isomorphic to $\bigoplus_{\lambda \dashv n} \mathrm{End}(V_\lambda)$. But since the set of paths from $\cdot \to \lambda$ is isomorphic to the set of set of standard Young tableaux SYT_λ (we can see the standard Young tableaux as recording the path), we have that

$$\dim\left(\bigoplus_{\lambda \dashv n} \mathrm{End}(V_\lambda)\right) = \sum_{\lambda \dashv n} |SYT_\lambda|^2 = |S_n| = \dim(\mathcal{H}_{S_n}(q)).$$

where the second to last equality follows from the classical theory of representations of S_n. So we indeed have the result □

By the previous theorems, we have that the Bratteli diagram for the Hecke Algebra inclusions $\mathcal{H}_{S_1}(q) \subset \mathcal{H}_{S_2}(q) \subset \dots$, for $q \notin \Omega = \cup_1^\infty \Omega_n$, is the same as the Bratteli diagram for $\mathbb{C}[S_1] \subset \mathbb{C}[S_2] \subset \dots$ constructed before. We now come to the final result of this section

Theorem 10.4.

$$\rho_\lambda \simeq J_\lambda$$

Proof. For S_1, both generate the trivial representation.

For S_2, there are two representations, the trivial representation and the sign representation. Under the classical theory [8], the trivial representation is associated with $\lambda_1 = \square\square$, while the sign representation is associated with $\lambda_2 = \begin{smallmatrix}\square\\\square\end{smallmatrix}$. From the constructions above, we have that there is only one path p to λ_1, and that

$$Q_1(e_p) = e_p,$$

so that we have

$$T_{s_1}(e_p) = 2Q_1(e_p) - T_1(e_p) = e_p.$$

So that we conclude that J_{λ_1} is indeeed the trivial representation ρ_{λ_1}.

Similarly, there is only one path p to λ_2 so we have

$$Q_1(e_p) = 0 \implies T_{s_1}(e_p) = -e_p,$$

so that again we have that $J_{\lambda_2} \simeq \rho_{\lambda_2}$

We prove the rest of the theorem by induction; so we assume the case up to $n-1$, and prove for n, for $n \geq 3$. From the comment above, both of these sets of irreducibles induce the same Bratteli diagram for $\mathcal{H}_{S_1}(1) \subset \mathcal{H}_{S_2}(1) \subset \dots$. In particular, if we let $\lambda \dashv n$, we have on characters (we are abusing notation now to have J_λ, ρ_λ also refer to characters)

$$\operatorname{Res}^{S_n}_{S_{n-1}}(J_\lambda) = \sum_{\mu \rightharpoonup \lambda} J_\mu.$$

Similarly, the classical theory gives us

$$\operatorname{Res}^{S_n}_{S_{n-1}}(\rho_\lambda) = \sum_{\mu \rightharpoonup \lambda} \rho_\mu.$$

By induction we have

$$\operatorname{Res}^{S_n}_{S_{n-1}}(\rho_\lambda) = \sum_{\mu \rightharpoonup \lambda} \rho_\mu = \sum_{\mu \rightharpoonup \lambda} J_\mu = \operatorname{Res}^{S_n}_{S_{n-1}}(J_\lambda).$$

To finish off the proof, we note that Theorem 10.3 shows that J_λ is irreducible. Thus, we must have

$$\rho_{\lambda'} = J_\lambda$$

for some $\lambda' \dashv n$. But we claim that for $n \geq 3$, $\rho_{\lambda'}$ is uniquely determined by $\operatorname{Res}^{S_n}_{S_{n-1}} \rho_{\lambda'}$, so that the equation above implies that we must have $\lambda = \lambda'$.

To see this, it is enough to show that the set $\{\mu \rightharpoonup \lambda\}$ uniquely determines $\lambda \dashv n$, for $n \geq 3$. To see this, note that if $|\{\mu \rightharpoonup \lambda\}| \geq 2$, then taking any two unequal elements x and y of this set, we must have λ be the Young Diagram with boxes the union of the boxes of x and y. For example,

$$\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}, \begin{smallmatrix}\square\square\square\\\square\end{smallmatrix} \in |\{\mu \rightharpoonup \lambda\}| \implies \lambda = \begin{smallmatrix}\square\square\square\\\square\square\end{smallmatrix}.$$

On the other hand, if $|\{\mu \rightharpoonup \lambda\}| = 1$, then we must have that μ and λ are diagrams that are either both 1 column or both 1 row (because if λ was not either one column or one row, $|\{\mu \rightharpoonup \lambda\}| \geq 2$). So μ determines λ, as desired. [14] $\square$

11. Example for $n = 3$

To illustrate the 3 constructions that we have created, let us use the methods above to create 3 different constructions of the same irreducible representations of the symmetric S_3.

11.1. **Classical Theory.** We will construct these representations from scratch by characters, but retain the association with partitions from the general theory of representations of symmetric groups [8].

The group S_3 has 3 conjugacy classes, $\{1\}, T = \{s_1, s_2, s_1s_2s_1\}, C = \{s_1s_2, s_2s_1\}$: the identity, the tranpositions, and the cyclic permutations respectively. Thus there are three irreducible representations. Two of these are common to every symmetric group: the trivial representation 1 and the sign representation σ, both of dimension 1. Thus, as the sum of the dimensions squared must be the order of the group, the last representation θ must have dimension 2. And as the sum of all the characters must be the character of the regular representation, we have the following character table:

	1	T	C
1	1	1	1
σ	1	-1	1
θ	2	0	-1

The more general theory of representations ([8], section 3) gives us the associations

$$1 \leftrightarrow \square\square\square \qquad \sigma \leftrightarrow \begin{array}{c}\square\\\square\\\square\end{array} \qquad \theta \leftrightarrow \begin{array}{l}\square\square\\\square\end{array}.$$

11.2. **Left Cell Representations.** From our examples in the sections before, we have that $\mathcal{H}_{S_n}(q)$ has four left cells, $\{1\}, \{s_1, s_2s_1\}, \{s_2, s_1s_2\}, \{s_1s_2s_1\}$. From the sizes of these cells, we can already begin to guess that the two singleton cells correspond to χ_1 and χ_2, while the larger cells correspond to χ_3. Let us find out.

Let us take the cell $\mathcal{C}_0 = \{1\}$. From Theorem 7.1, we have

$$\begin{aligned} T_1C_1 &= C_1, \\ T_{s_1}C_1 &= qC_1 + q^{1/2}C_{s_1}, \\ T_{s_2}C_1 &= qC_1 + q^{1/2}C_{s_2}. \end{aligned}$$

So we have $T_{s_i}C_1 = qC_1$ in $\mathcal{K}_{\mathcal{C}}$, and for $q = 1$, this indeed corresponds to the trivial representation 1.[15]

Let us take the cell $\mathcal{C}_1 = \{s_1s_2s_1\}$. From the multiplication rule again, we have

$$T_{s_i}C_{s_1s_2s_1} = -C_{s_1s_2s_1},$$

so that this cell corresponds to the sign representation σ.

[14] Note that we had to assume that $n \geq 3$, because for $n = 2$, the diagram $\mu = \square$ is simultaneously only one column and one row

[15] Recall that we quotient out by stuff $\leq_L$ but not in $\mathcal{C}$

Let us take the cell $\mathcal{C}_2 = \{s_1, s_2s_1\}$. From the multiplication rule, we have that the actions of T_{s_1} and T_{s_2} on the vector space $\mathcal{K}_\mathcal{C}$ is given by the matrices (indexed with $C_{s_1} < C_{s_2s_1}$) is given by

$$T_{s_1} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \qquad\qquad T_{s_2} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

And since this is a representation, we have

$$T_{s_1s_2} = T_{s_1}T_{s_2} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \qquad\qquad T_{s_2s_1} = T_{s_2}T_{s_1} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix},$$

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, we have that the character of this cell has $\chi_\mathcal{C}(T) = 0$, $\chi_\mathcal{C}(C) = -1$, where T and C are the conjugacy classes of transpositions and cycles, respectively, as before. From the character table of the classical theory, we conclude that this representations corresponds to θ.

Note that the last cell $\{s_2, s_1s_2\}$ has the same representation as $\mathcal{C}_2$, but with 1s and 2s reversed. So it also corresponds to θ.

Now let us take a representative of each cell, and find its tableau shape under the Robinson-Schensted Correspondence. In the lines below, we have $Q(w) = Q(x_1x_2 \dots x_n)$ where $w = x_1x_2 \dots x_n$ as before:

$$Q(1) = Q(123) = \square\square\square,$$

$$Q(s_1) = Q(213) = \begin{smallmatrix}\square\square\\ \square\end{smallmatrix},$$

$$Q(s_1s_2s_1) = Q(321) = \begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}.$$

so that we indeed have the same associations of representations to Young diagrams, as desired.

11.3. **Tower Construction.** The tower construction of the representation for $\square\square\square$ has

$$Q_i(e_p) = e_p \implies T_{s_i}(e_p) = e_p$$

for e_p the unique path to $\square\square\square$, so that this corresponds to the trivial representation 1.

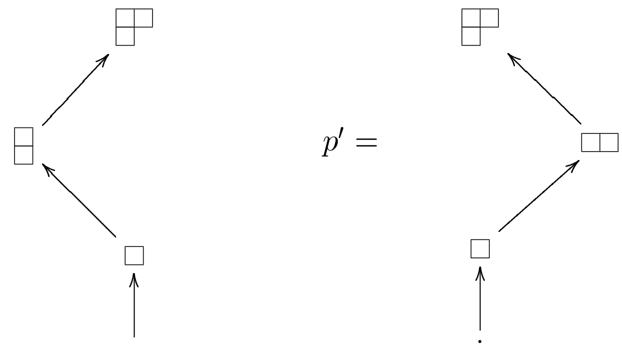
The tower construction of the representation for $\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}$ has

$$Q_i(e_p) = 0 \implies T_{s_i}(e_p) = -e_p$$

for e_p the unique path to $\square\square\square$, so that this corresponds to the sign representation σ.

The tower construction of the representation for is a vector space with two generators, e_p and $e_{p'}$

$p =$ $\qquad$ $p' =$



Then we have $d_p^2 = 2$, so that $d_p^2(q) = 3/4$ and we have, on the basis $\{e_p, e_{p'}\}$

$$Q_2 = \begin{pmatrix} 3/4 & 3/4 \\ 1/4 & 1/4 \end{pmatrix} \implies T_{s_2} = \begin{pmatrix} 1/2 & 3/2 \\ 1/2 & -1/2 \end{pmatrix}.$$

Also, p_2 is obtained from p_0 by adding two boxes in the same column, and p_2' is obtained from p_0' by adding two boxes in the same row, so that we have

$$Q_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \implies T_{s_1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So we have

$$T_{s_2 s_1} = \begin{pmatrix} -1/2 & 3/2 \\ -1/2 & -1/2 \end{pmatrix}.$$

So that by looking at the traces, we again have that this is the representation θ.

12. Conclusions

So where are we now? We have showed three different constructions of the irreducible representations of S_n, and showed that the different constructions can be naturally identified with each other through the language or partitions. But what about for the Hecke Algebras $\mathcal{H}_{S_n}(q)$ for arbitrary q outside of Ω[16]? We finish off by asserting that our identification of J_λ and $\mathcal{K}_\lambda$ indeed extends. Adding arguments to J_λ and $\mathcal{K}_\lambda$ of q, we have

Theorem 12.1. *For all q such that $\mathcal{H}_{S_n}(q)$ is semisimple, we have $\mathcal{K}_\lambda(q) \simeq J_\lambda(q)$ for all $\lambda \dashv n$*

Proof. First note that as the number of paths to λ and left cells do not change for these changes of $q \notin \Omega$ and the $J_\lambda(q)$ at least are still irreducible, we have that the $J_\lambda(q)$ still constitute a complete set of representations. So for arbitray $q \notin \Omega$, $\mathcal{K}_\lambda(q)$ must correspond to some $J_{\lambda'}(q)$. Then as the characters of $J_\lambda(q)$ and $\mathcal{K}_\lambda(q)$ must deform continuously as we move q around, and $\mathcal{K}_\lambda(q)$ must always conform to some $J_{\lambda'}(q)$, we must have $\lambda = \lambda'$, or else at some point in the path $J_\lambda(q)$ corresponds to two partitions λ and λ', a contradiction. $\square$

Thus, the results in this thesis may provide useful insights into understanding the structure of more general Hecke Algebras.

[16]Outside of Ω to ensure semisimplicity

13. Bibliography

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