Punctual Hilbert Schemes of the Plane

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Contents

1 Introduction

This paper discusses punctual Hilbert schemes, Hilbert schemes that parametrize collections of points. When the points lie on a two dimensional surface, these schemes are well understood. Section 2 of this paper briefly describes such schemes, and motivates what will be this paper's central topic: Understanding schemes of degree d supported at a point. Section 3 and 4 deal with some examples. Section 3 describes the variety parametrizing schemes of degree 3 supported at a point, and section 4 examines the spaces parametrizing schemes supported at a point and contained in a curve. Sections 5 and 6 generalizes a result from section 4 and describes an affine cover for schemes supported at a point when degree and hilbert function are fixed. Section 7, using earlier results, computes the dimension of the varieties from sections 5 and 6, and, by extension, the dimension of $\mathrm{Hilb}^n \mathbb{P}^2$, showing it to be $2n$.

2 Punctual Hilbert Schemes of the Plane

Definition 2.1. For a fixed scheme X, and a polynomial f such that $f(n) \in \mathbb{Z}$ for any integer n the **Hilbert Scheme** $\mathrm{Hilb}^f(X)$ is the scheme that parametrizes all subschemes of X with Hilbert polynomial f.

When X is projective, this scheme always exists, see [2] for a construction.

The purpose of this thesis is to study a specific type of Hilbert scheme. For our purposes, the fixed scheme X will be $\mathbb{P}^2$ (or really any smooth surface) over an algebraicly closed field of characteristic 0, to be called k. We will parametrize schemes with constant Hilbert polynomial, i.e those supported at a point $\mathrm{Hilb}^n(\mathbb{P}^2)$ parametrizes schemes of dimension 0, and degree n. If such a scheme is also reduced, then it is a collection of n distinct points. If such a scheme is not reduced it is supported at m points in $\mathbb{P}^2$ for $m < n$. Since the schemes parametrized are supported on finite sets of points, the space is called the Punctual Hilbert Scheme.

There is a map from $\mathrm{Hilb}^n(\mathbb{P}^2) \to (\mathbb{P}^2)^n/S^n$. The target space is an n-fold product of copies of $\mathbb{P}^2$ modulo the action of the symmetric group on n letters permuting the coordinates. The map sends a punctual scheme to the closed points of its support, counted with multiplicity. The map is surjective, and on the open set excluding the diagonal (tuples of points in $\mathbb{P}^2$ where at least two of the points are nondistinct) the map is one-to-one, because there is exactly one scheme of degree n supported at n distinct points, the collection of points themselves.

This map from $\mathrm{Hilb}^n(X) \to (X)^n/S^n$ for smooth projective varieties X is known as the *Hilbert-Chow Morphism*. In the case of $X = \mathbb{P}^2$ this map is a desingularization, but that will not be shown here.

Understanding the fibers away from the open set of distinct points will allow us to understand $\mathrm{Hilb}^n \mathbb{P}^2$. This paper will spend most of its time studying fibers over the small diagonal, tuples of points in $\mathbb{P}^2/S^n$ where every point is the same. The main theorem, that a fiber over such a point has dimension $n-1$ allows us to conclude that $\dim \operatorname{Hilb}^n \mathbb{P}^2 = 2n$. This does not seem surprising, but the analogous result for $\mathbb{P}^3$ or higher is false!

If a point is in the small diagonal of $(\mathbb{P}^2)^n$, its pre-image under the Hilbert-Chow morphism is the set of degree n schemes supported at a single point.

This set can be understood as a variety and as a Hilbert scheme by replacing $\mathbb{P}^2$ with $\operatorname{Spec} k[[x, y]]$. This is easier to work with because $\operatorname{Spec} k[[x, y]]$ is affine, schemes of degree n are in bijection with ideals I in $k[[x, y]]$ with $\dim_k k[[x, y]]/I = n$. Since the rest of the paper will study this construction it will be helpful to abbreviate and set $R = k[[x, y]]$, and in an abuse of notation write $\operatorname{Hilb}^n \operatorname{Spec} k[[x, y]]$ as $\operatorname{Hilb}^n R$.

2.1 Ideals of $k[[x, y]]$

First, some preliminary notions and definitions that will appear throughout the paper. If I is an ideal in R then denote by I_j the elements in R_j that are initial forms of elements in I, i.e. the $u \in R_j$ such that there is some $v \in \mathfrak{m}^{j+1}$ with $u + v \in I$.

Next, and ideal I is graded if it can be generated by homogenous polynomials. To each ideal I, we define the *associated graded ideal* I^* as the ideal generated by the initial forms of elements in I.

Definition 2.2. The *type* of I is the tuple $(t_0(I), t_1(I), t_2(I), ...)$ where $t_j = \dim_k R_j/I_j$.

There are several important but simple claims to be made about the type of and ideal.

1. If I has colength n, then $\sum_j t_j(I) = n$.
2. $t(I) = t_j(I^*)$
3. Say that $\dim R_j - t_j(I) = j + 1 - t_j(I)$ is greater than 0. This means that $I_j \neq 0$. Since $\dim R_{j+1} = j+2$ and multiplying by x and by y gives that $\dim I_{j+1} \geq \dim I_j + 1$ it is the case that $t_{j+1}(I) \leq t_j(I_j)$. This means that the set of possible types is must be of the form $(1, 2, 3...d, t_d(I), t_{d+1}(I), ...)$. d is the largest integer with $I \subset \mathfrak{m}^{d+1}$, and for $k > d$ we have $t_k(I) \leq t_{k-1}(I)$.
4. It is also true that for any tuple of the form $(1, 2, 3...d, n_d, n_{d+1}, ...)$ with $n_k > n_{k+1}$ for $k \geq d$ there is an ideal I in R with exactly this tuple as its type. This is relatively easy to construct. Generate I by choosing $d + 1 - n_d$ elements in R_d, and in successive graded piece of R pick new generators not already contained in I to produce the proper codimension.

Later, it will be convenient to have defined the *jump index* of a type T as the following sequence of numbers. For fixed T and an integer $j \geq d$, define the jump index at j as $e_j = t_j - t_{j-1}$.

Finally, let Z_T denote the locally closed subscheme of $\mathrm{Hilb^n}(R)$ consisting of ideals of type T. It is clear that

$$\mathrm{Hilb}^n k[[x,] = \bigcup_{|T|=n} Z_T$$

2.2 Variety Structure and Inclusion into the Grassmanian

Theorem 2.3. $\mathrm{Hilb}^n(R)$ *is a variety (i.e. it is reduced)*

This will be done by constructing $\mathrm{Hilb}^n(R)$ as a subvariety of a grassmanian.

Lemma 2.4. *Every ideal in R of colength n is contains $\mathfrak{m}^n$ where $\mathfrak{m}$ is the maximal ideal (x,y).*

Proof. This is shown by induction. First, as a base case, R itself is the only ideal of colength 0, and it contains $\mathfrak{m}^0 = R$. Now assume the theorem holds true for ideals of colength $n-1$ and less. Let I be an ideal of colength n that does not contain $\mathfrak{m}^n$. Pick some f in $\mathfrak{m}^n$ and not in I. Then the ideal generated by I and f is strictly bigger than I so has colength less than n, so it contains $\mathfrak{m}^{n-1}$. Since $\mathfrak{m}^{n-1} = (I,f)_{n-1} = I_{n-1}$ we conclude that I contains $\mathfrak{m}^{n-1}$. Contradiction. □

This lets us realize the punctual Hilbert scheme as a variety within a grassmanian. We know that

$$\dim_k R]/\mathfrak{m}^n = \frac{m^2+m}{2}$$

Then every ideal of colength n can be realized as a point within the grassmanian $G(R/\mathfrak{m}^n, \frac{m^2+m}{2} - n)$.

Not every n-dimensional subspace of $k[[x,y]]/\mathfrak{m}^n$ is an ideal. The subspaces V that are ideals satisfy the constraint that $xV_{i-1} \cup yV_{i-1} \subset V_i$ for all i. But this can be expressed as an algebraic constraint, so the subspaces that are ideals form a subvariety.

3 The case where $n = 3$

Without too much work, $\mathrm{Hilb}^3(k[[x,y]])$ admits an explicit description, and can be recognized as a cone over a cubic curve.

First, we describe its stratification.

Definition 3.1. A scheme supported at a point is called *curvilinear* if it is contained in a smooth curve passing through its support

Now we give a more algebraic characterization

Proposition 3.2. *The ideals in $k[[x,y]]$ characterizing curvilinear schemes are exactly the nontrivial ideals (of finite colength) that are not contained in $\mathfrak{m}^2$.*

Proof. Let S be a scheme supported at the origin in $\mathbb{A}^2$ and C be a smooth curve containing S. Let S correspond to the ideal $I \subset k[[x,y]]$ and $I(C) = J$. The image of J localized to $k[[x,y]]$ is not contained in $\mathfrak{m}^2$, and since $J \subset I$ neither is I.

In the other direction let I be a nontrivial ideal of $k[[x,y]]$ not contained in $\mathfrak{m}^2$. If $I = \mathfrak{m}$, then we are done. Suppose $I \neq \mathfrak{m}$ we must have $t_1(I) = 1$. This means $t_n(I) \leq 1$ for $n > 1$. Since I has finite colength this means the type of I is $(1,1,1,...1,0,0,0,...)$. This means $I/\mathfrak{m}^n$ is a principle ideal, generated by some polynomial f. f cuts out a smooth curve passing through the origin, containing $\operatorname{Spec} I$. □

Notice now that if an ideal in $k[[x,y]]$ has colength 3 and *is* contained in $\mathfrak{m}^2$ it must be $\mathfrak{m}^2$. So there is only one point in $\operatorname{Hilb}^3(k[[x,y]])$ that is not curvilinear. Every other ideal is contained in a smooth curve. In particular, it is of the form $(f, \mathfrak{m}^3)$ for f a curve that is smooth at the origin, and of degree at most 2.

The subvariety of curvilinear schemes, which will be called C^3, has a nice description: it is the total space of $\mathcal{O}_{\mathbb{P}^1}(3)$, the degree 3 line bundle on $\mathbb{P}^1$. First note that there is a natural inclusion of $i : \mathbb{P}^1 \to C^3$. Interpret $\mathbb{P}^1$ as the space of lines through the origin in $\mathbb{P}^2$, and send every line ℓ to the ideal $(I(\ell), \mathfrak{m}^3)$ where $I(\ell)$ is the linear form cutting out ℓ.

We also have a projection map $p : C^3 \to \mathbb{P}^2$. Send an ideal of the form $(f, \mathfrak{m}^3)$ to the line tangent to f at the origin. This is equivalent to deleting the degree 2 terms in f, so it is clear that $p \circ i = id$.

Proposition 3.3. *C^3 is the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(3)$, with i being the zero section.*

Proof. Let I be a curvilinear ideal written as $(f, \mathfrak{m}^3)$. Say the linear part of f is $ax+by$ and assume without loss of generality that $a \neq 0$. Then it is clear that $x^2 + \frac{b}{a}xy$, and $xy + \frac{b}{a}y^2$ are contained in I, so we can pick a new generator for I and write it as $(x + \frac{a}{b}y + cy^2, \mathfrak{m}^3)$. This form ($x$ has coefficient 1, no x^2 or xy term) is unique. Let ℓ be the point in $\mathbb{P}^1$ that i sends to $(x + \frac{a}{b}y, \mathfrak{m}^3)$. Then the fiber of p over ℓ is identified with $\mathbb{A}^1$ by the coefficient c of y^2.

The fibers are locally trivial over the two open sets of D_x and D_y, by construction. Further, we can write down the explicit transition functions. If I can be written as $(x + c_0 y + c_1 y^2, \mathfrak{m}^3)$, then modulo I we have the equivalences

$$x^2 + c_0 xy \equiv 0 \text{ and } xy + c_0 y^2 \equiv 0$$

so I can be rewritten as $(y + \frac{1}{c_0}x + \frac{c_1}{c_0^3}x^2, \mathfrak{m}^3)$. This is the transition function for $\mathcal{O}(3)$. □

The fact that this bundle is $\mathcal{O}(3)$ can be seen in one other way. Pick 3 general points p_i in $\mathbb{A}^2$. Then if ℓ is a line through the origin there is a unique conic containing the three points tangent to ℓ at the origin. By taking ℓ to the curvilinear scheme contained in that conic at the origin, the three points determine a section of C^3, as seen in figure 1. The

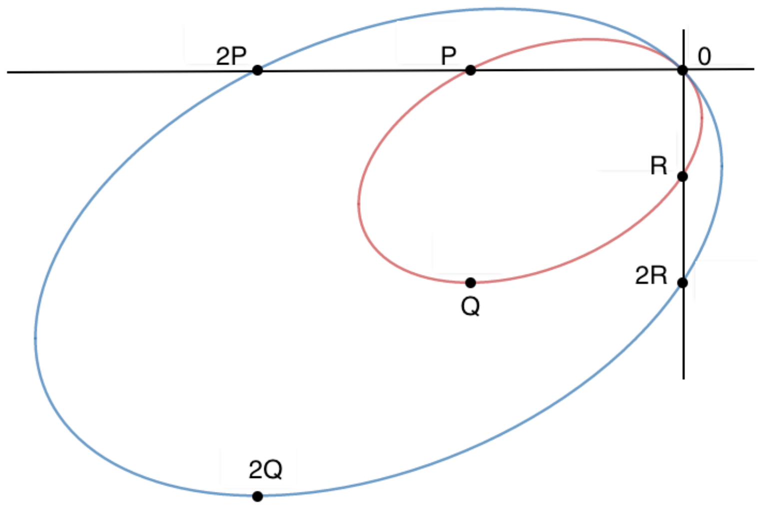

Figure 1: The points P, Q, R, and the points $2P, 2Q, 2R$ determine two conics through the origin with identical slope.

section has a zero whenever the curvilinear scheme produced is contained in a line through the origin, and that happens whenever $\ell = \overline{0p_i}$, so it happens three times.

Now, fixing ℓ, we can explore the fiber by looking at tp_1, tp_2, tp_3 and letting t vary. Let S_t be the scheme associated to t in this set up. To compute $\lim_{t\to\infty S_t}$ make the same setup in $\mathbb{P}^2$, define $[1:0:0]$ as the origin and take p_i to the line at infinity. Once the p_i are collinear, the only conics containing them are unions of two lines, one through the p_i and on through the origin. So $\lim_{t\to\infty} S_t$ is a scheme contained in a line.

To compute $\lim_{t\to 0}$ it is easier to take the limit in $G(3, k[[x,y]]/\mathfrak{m}^3)$. The subspace corresponding to the ideal $(x + c_0 y + cy^2, \mathfrak{m}^3)$ has $x + c_0 y + cy^2, xy + c_0 y^2, x^2 + c_0 xy$ as a basis.

If the points p_1, p_2, p_3 and the line ℓ determine the ideal $(x + c_0 y + cy^2, \mathfrak{m}^3)$ then the points tp_1, tp_2, tp_3 determine the ideal $(tx + tc_0 y + cy^2, \mathfrak{m}^3)$. This corresponds to the subspace $\langle tx + tc_0 y + cy^2, xy + c_0 y^2, x^2 + c_0 xy\rangle$. We take $t \to 0$ to see the limit is the subspace $\langle y^2, xy, x^2\rangle$ which of course is the ideal $\mathfrak{m}^2$.

$G(3,6)$ is embedded in $\mathbb{P}(\bigwedge^3 k^6)$. In this embedding the curve parametrized by

$$t \mapsto \langle tx + tc_0 y + cy^2, xy + c_0 y^2, x^2 + c_0 xy\rangle$$

is in fact a line. So the variety $\operatorname{Hilb}^3 K[[x,y]]$ is a cone over the image of $\mathbb{P}^1$ as the curvilinear schemes contained in a line. To find the degree of this image, take an arbitrary hyperplane that does not contain the point corresponding to the ideal $\mathfrak{m}^2$. The intersection of this hyperplane with $\operatorname{Hilb}^3 K[[x,y]]$ defines a section of the bundle $\mathcal{O}(3)$ so the hyperplane meets the graded curvilinear ideals 3 times, so the graded curvilinear ideals are a curve of degree 3. Thus $\mathrm{Hilb}^3 k[[x, y]]$ is a cone over a curve of degree 3.

4 Curvilinear schemes of higher degree

When $n > 3$, curvilinear schemes become more complicated. The ideal corresponding to a curvilinear scheme of degree n can be written as $(f, \mathfrak{m}^n)$ for f a degree $n-1$ polynomial in x and y that is smooth at the origin. Denote by C^n the variety of curvilinear schemes of degree n supported at a point.

Similarly to the first part of the proof of theorem 3.3, if we assume without loss of generality that the linear part of f is $x + c_1 y$, we can write the ideal as

$$(x + c_1 y + c_2 y^2 + \cdots c_{n-1} y^{n-1}, \mathfrak{m}^n)$$

Once again there are the natural maps from $i : \mathbb{P}^1 \to C^n$ taking a line through the origin to the degree n scheme it contains, and $p : C^n \to \mathbb{P}^1$ taking a curvilinear scheme to its tangent line. The fibers of p are $\mathbb{A}^{n-2}$, since for $i > 1$ c_i can be freely determined.

C^n is then an affine bundle over $\mathbb{P}^1$ with a distinguished section. However, for $n > 3$, C^n is not algebraic vector bundle.

Proposition 4.1. *For each n, and any $k > n$, the line bundle over $\mathbb{P}^1$ corresponding to ideals of the form $(ax + by + cy^n, \mathfrak{m}^k)$ is $\mathcal{O}(n)$.*

Proof. Reduce to the case $k = n+1$ by noting that for f and g two polynomials in x and y of degree n and $k > n$ it's the case that $(f, \mathfrak{m}^k) = (g, \mathfrak{m}^k)$ if and only if $(f, \mathfrak{m}^{n+1}) = (g, \mathfrak{m}^{n+1})$

It suffices to describe the transition functions on the open set in $\mathbb{P}^1$ where neither x or y is 0. The ideal $(x - c_0 y + cy^n, \mathfrak{m}^{n+1})$ contains $x^i y^{n-i} + c_0 x^{i-1} y^{n-i+1}$ for all i. Therefore, $cy^n \equiv \frac{c}{c_0} xy^{n-1}$. We can continue this process, trading powers of y for powers of x and picking up additional $\frac{1}{c_0}$ terms to see that the ideal can be rewritten as can be rewritten as $(y - \frac{1}{c_0} x - \frac{c}{c_0^n} x^n, \mathfrak{m}^{n+1})$. This is the transition function for $\mathcal{O}(n)$ □

For each n there is a map $p_n : C^n \to C^{n-1}$ coming from the map $I \mapsto I + \mathfrak{m}^n$. There is also a map $i_n : \mathbb{P}^1 \to C^n$ as usual coming from the inclusion of linear forms among polynomials of degree at most $n-1$. Then $p_n^{-1}(i_{n-1}(\mathbb{P}^1))$ is the $\mathcal{O}^n$ just described. Hence it is sufficient to show that C^4 is not a vector bundle. The rest of the proof follows from induction. If C^n were a vector bundle, it would contain $\mathcal{O}(n)$, and, as a vector bundle over $\mathbb{P}^1$, it would split. Then C^{n-1} would be isomorphic to $C^n/\mathcal{O}(n)$ and thus a vector bundle as well.

Theorem 4.2. *C^4 is not a vector bundle*

Proof. Assume for the sake of contradiction that C^4 is a vector bundle. We first show it must be isomorphic to the bundle $\mathcal{O}(4) \oplus \mathcal{O}(3)$. Note that it sits in the middle of

$$0 \longrightarrow \mathcal{O}(4) \longrightarrow C^4 \xrightarrow{p} C^3 \longrightarrow 0$$

We will show that,assuming C^4 is a vector bundle, then p is a map of vector bundles, so this will be a bonafide short exact sequence of vector bundles, and hence must split.

Restricted to the inclusion of $\mathcal{O}(4) \to C^4$ the map p is the zero map. Then if u is a section of $\mathcal{O}(3)$, and w a section of $\mathcal{O}(4)$, viewed as a section of C^4 we have $p(u+w) = p(u) + p(w) = p(u)$ since a map sending $w \mapsto p(u+w) - p(u)$ is a fiber bundle map from $\mathcal{O}(4) \to \mathcal{O}(3)$ and must be zero. Similarly, since the only fiber bundle maps from $\mathcal{O}(3)$ to itself are vector bundle homomorphisms $p|\mathcal{O}(3)$ must be a vector bundle isomorphism.

This derives a contradiction because there is no section from $C^3 \to C^4$. This becomes apparent when the transition functions for C^4 are written out. If $(x+c_1y+c_2y^2+c_3y^3, \mathfrak{m}^4)$ and $(y+b_1x+b_2x^2+b_3x^3, \mathfrak{m}^4)$ define the same ideal I then substituting $y \equiv -(b_1x+b_2x^2+b_3x^3)(\bmod I)$ shows that

$$x - c_1(b_1x + b_2x^2 + b_3x^3) + c_2(b_1x + b_2x^2 + b_3x^3)^2 - c_3(b_1x + b_2x^2 + b_3x^3)^3 \in I$$

This polynomial is also in (x), but $I \cap x \subset x^4$, or else I contains x^3 and $y+b_1x+b_2x^2+b_3x^3$, contradicting the fact that it is codimension 4. By setting the linear through cubic terms of this polynomial equal to 0, we derive the transition functions

1. $1 - c_1b_1 = 0$
2. $c_2b_1^2 - c_1b_2 = 0$
3. $2c_2b_1b_3 - c_1b_3 - c_3b_1^3 = 0$

Combining these equations we can write $b_3 = b_1^{-1}b_2^2 - c_3b_1^4$. Now say c_3 can be written as a polynomial function of c_1 and c_2, or as a polynomial function g of b_1^{-1} and $b_1^{-3}b_2$. That means $b_3 = b_1^{-1}b_2^2 - g(b_1^{-1}, b_1^{-3}b_2)b_1^4$. But it is impossible for a term of $g(b_1^{-1}, b_1^{-3}b_2)b_1^4$ to cancel $b_1^{-1}b_2^2$, so this section g cannot be extended over $b_1 = 0$.

This completes the proof □

This concludes the bulk of the examples for the paper. The rest will be devoted to computing the dimension of $\mathrm{Hilb}^n\, k[[x,y]]$ and, in order to do this, proving that for any type T, the ideals of that type form an affine bundle over the graded ideals of this type.

5 Affine Bundle Structure for Ideals with Fixed Pattern

The plan of attack, drawn from [1], is as follows. First, instead of looking at all the ideals of type T, we will first look at the ideals of type T that are disjoint from a fixed, large-as-possible subspace of $R/\mathfrak{m}^{\mathfrak{n}}$. When this is assumed, it is possible to straightforwardly produce generators for an ideal. Upon further examination, some terms in the generators can be freely chosen, and the other parts are then uniquely determined. This shows that the subscheme of ideals disjoint from the fixed vector space is actually an affine space and lets us determine its dimension.

After showing this for a fixed vector space. It will be shown that after selecting only finitely many such vector spaces, *every* ideal will be disjoint from at least one of the vector spaces chosen, i.e. Z_T can be covered by a finite union of affine spaces.

5.1 The structure of ideals in $k[[x, y]]$

Definition 5.1. For T a type, A *pattern of type* T is a set P of polynomials in x and y with $|P_j| = t_j$. The *normal pattern of type* T is the pattern of type T where terms are chosen to have highest possible x degree.

Definition 5.2. An ideal $I \subset R$ has pattern P if one of the following equivalent formulations holds:

1. For all $j \in \mathbb{N}$ it is the case that $\langle P \cap \mathfrak{m}^j \rangle \oplus (I \cap \mathfrak{m}^j) = \mathfrak{m}^j$
2. For all $j \in \mathbb{N}$ it is the case that $\langle P_j \rangle \oplus \langle I_j \rangle = R_j$
3. $\langle P \rangle \cap I = 0$ and $|P_j| = t_j(I)$

Of course, equivalence needs to be proven.

Proof. $1 \to 2$) Given that $(I \cap \mathfrak{m}^j) \oplus \langle P \cap \mathfrak{m}^j \rangle = \mathfrak{m}^j$ and that $(I \cap \mathfrak{m}^{j+1}) \oplus \langle P \cap \mathfrak{m}^{j+1} \rangle = \mathfrak{m}^{j+1}$ it follows that

$$\begin{aligned} R_j = \mathfrak{m}^j/\mathfrak{m}^{j+1} &= \big((I \cap \mathfrak{m}^j) \oplus \langle P \cap \mathfrak{m}^j \rangle\big) \,/\, \big((I \cap \mathfrak{m}^{j+1}) \oplus \langle P \cap \mathfrak{m}^{j+1} \rangle\big) \\ &= \big((I \cap \mathfrak{m}^j)/(I \cap \mathfrak{m}^{j+1})\big) \oplus \big(\langle P \cap \mathfrak{m}^j \rangle / \langle P \cap \mathfrak{m}^{j+1} \rangle\big) \\ &= I_j \oplus P_j \end{aligned}$$

$2 \to 3$) Since $t_j(I)$ is defined as $\dim R_j - \dim I_j$ it is clear that $\dim P_j = t_j(I)$, and that $\langle P_j \rangle \cap I_j = 0$. Say the intersection $\langle P \rangle \cap I$ has some nonzero element. Then that element has an initial form of degree j, but this contradicts $\langle P_j \rangle \cap I_j = 0$.

$3 \to 1$) If $\langle P \rangle \cap I = 0$ and $|P_j| = t_j(I)$ then $\langle P \cap \mathfrak{m}^j \rangle \cap (I \cap \mathfrak{m}^j) = 0$. Quotienting out by $\mathfrak{m}^n$, where $n = \operatorname{codim} I$ we have that the two subsapces are disjoint and of complementary dimension, so for all $j < n$ $\langle P \cap \mathfrak{m}^j \rangle / \mathfrak{m}^{\mathfrak{n}} \oplus (I \cap \mathfrak{m}^j)/\mathfrak{m}^n = \mathfrak{m}^j/\mathfrak{m}^n$. We know that for $k \geq n$ $P_k = 0$ and $I_k = R_k$, so this result lifts to what is desired □

It will frequently be helpful to think of R, and ideals and patterns in R as arranged in a sort of Pascal's Triangle, as in figure 2.

Now, pick a pattern P that is a valid pattern for an ideal, it can be used to determine a generating set for I. Let v_j be the monomial in R_j with the highest x-degree that is *not*

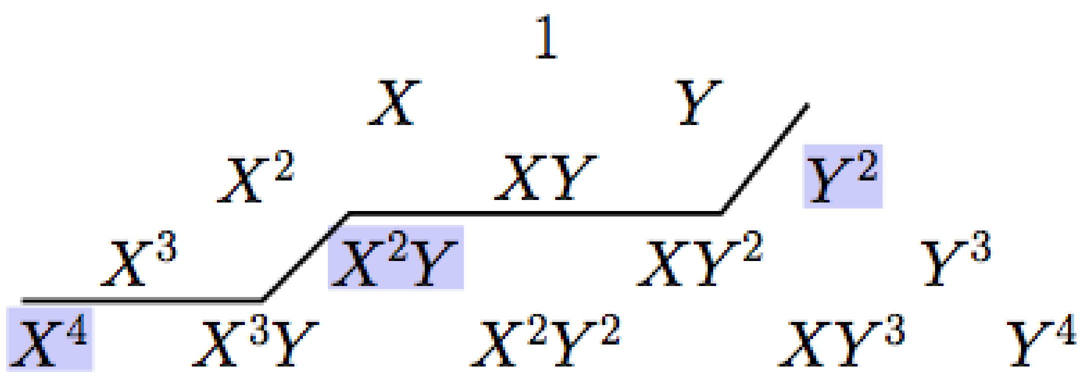

Figure 2: In a pattern of type $(1, 2, 2, 1, 0...)$. By our notation $u_0 = X^4$, $u_1 = X^2Y$, and $u_2 = Y^2$

in P_j. For convenience when $P_j = R_j$ say $v_j = 0$. It is clear that, the x-degree of xv_j is less than or equal to the x-degree of v_{j+1}.

Let U be the set of monomials such that $U_j = \{x^{j-i}y_i | \deg_x v_j \leq j - i < \deg_x xv_{j-1}\}$. The set U is highlighted in figure 2.

Proposition 5.3. *Some simple facts about U:*

1. *If d is as above, the largest integer for which $P_{d-1} = R_{d-1}$, then $U_k = \emptyset$ for $k \leq d$.*
2. *U has exactly $d + 1$ elements*
3. *Distinct elements have distinct y degrees going from 0 to d. For the future, u_i will refer to the monomial $x^{k_i}y^i$ contained in U. Note that u_0 is a power of x, and $u_d = y^d$*
4. *Define U_k as the subset of U consisting of elements of degree k. Letting $T = (1, 2, 3, ...d, t_d, t_{d+1}, ..)$ then U_d has size equal to $e_d + 1$, and for $j > d$ U_j has size equal to e_j.*

As a vector space $\langle P \rangle$ spans R/I, since it is disjoint from I and dimensions check out. Thus for every $u_i \in U$ there is an $h_i \in \langle P \rangle$ such that $f_i = u_i - h_i \in I$. These f_i will be called the *standard generators* for I, and now we prove this name has merit.

Theorem 5.4. *Defined as above, $I = (f_0, ...f_d)$*

Proof. This is first shown in the case that I is graded. Let $J = (f_0, ...f_d)$ Clearly $J \subset I$. The fact that $J = I$ will follow from the claim that for all k $J_k + \langle P_k \rangle = R_k$.

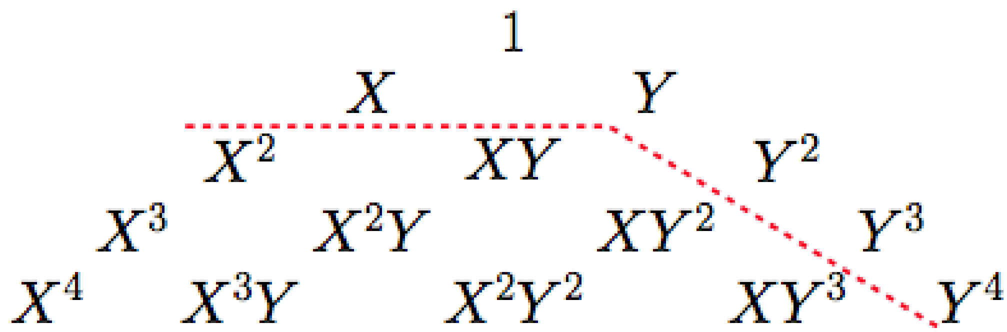

Figure 3: The set $S(xy)$

The proof of this claim goes by induction. Clearly for $k < d$ $J_k + \langle P_k \rangle = 0 + R_k$ and for $k = d$ $J_d + \langle P_d \rangle = \langle U_d \rangle + \langle P_d \rangle = R_d$.

Now assume $J_k + \langle P_k \rangle = R_k$. Then

$$\begin{aligned} R_{k+1} &= \langle y^{k+1}, xR_k \rangle \\ &= \langle y^{k+1}, xJ_k, xP_k \rangle \\ &= \langle y^{k+1-d} f_d, xJ_k, P_{k+1}, U_{k+1} \rangle \end{aligned}$$

By definition $\langle P_{k+1}, U_{k+1} \rangle \subset \langle P_{k+1}, J_{k+1} \rangle$, so $R_{k+1} = \langle P_{k+1}, J_{k+1} \rangle$

This shows the result if I is graded. Suppose that I is not graded, and let $f_1, ... f_d$ be the standard generators for I. Denote by F_i the initial form of f_i. It is not hard to see that $F_1, ... F_d$ will be the standard generators for I^*. Since I^* is equal to $I/\mathfrak{m}I$ we can apply Nakayama's lemma to see that $I = (f_1, ... f_d)$ □

Definition 5.5. If u is a monomial, then $S(u)$, called the shadow of u, is the set of all monomials of degree greater than or equal to the degree of u, and x degree greater than or equal to the x degree of u.

In the Pascal's triangle arrangement, $S(u)$ is all monomials in or to the left of the triangle extending below u. Figure 3 shows $S(xy)$ below the dotted line.

Now we show an alternate way to generate $S(u)$

Lemma 5.6. *For a fixed pattern P, let U be as above, and let f_i be equal to $u_i - h_i$ for $h_i \in \langle P \cap \mathfrak{m}^{\deg u_i} \rangle$.*

Then $(f_0, ... f_s) + \langle P \cap \mathfrak{m}^{\deg u_s} \rangle = \langle S(u_s) \rangle$

Proof. The proof proceeds by induction. In the case $s = 0$ $f_0 = u_0 = x^{k_0}$, $P \cap \mathfrak{m}^{k_0} = \emptyset$, and $S(x_0) = (x_0) = (f_0)$.

Now assume the theorem holds for $s-1$. It is pictorially clear that $\langle S(u_s)\rangle = (u_s) + \langle S(u_{s-1})\rangle + \langle P \cap \mathfrak{m}^{\deg u_s}\rangle$. Take $g \in \langle S(u_s)\rangle$, and write g as $\alpha_0 u_s + k$ for $k \in \langle S(u_{s-1})\rangle + \langle P \cap \mathfrak{m}^{\deg u_s}\rangle$. Then

$$g - \alpha_0 f_s = \alpha_0 u_s - \alpha_0 f_s + k = \alpha_0 h_s + k$$

Examining αh_2, it is possible to express α_0 as $\beta + \gamma$ so that $\gamma h_s \in \langle S(u_{s-1})\rangle + \langle P \cap \mathfrak{m}^{\deg u_s}\rangle$, and $\beta h_s \in (u_s)$. This means $\beta h_s = \alpha_1 u_s$ and it is neccessary that the highest power of x dividing α_1 is greater than the highest power dividing β, which is in turn greater than or equal to the highest power of x dividing α_0 We then can rewrite $g - \alpha_0 f = \alpha_1 x + k'$ for $k' \in \langle S(u_{s-1})\rangle + \langle P \cap \mathfrak{m}^{\deg u_s}\rangle$

This can be repeated until $x^{k_{s-1}-k_s}$ divides the resultant α_n. This means

$$g - (\alpha_0 + \alpha_1 + \cdots \alpha_{n-1}) f_s \in x^{k_{s-1}-k_s}(u_s) + \langle S(u_{s-1})\rangle + \langle P \cap \mathfrak{m}^{\deg u_s}$$

From the definition $x^{k_{s-1}-k_s}(u_s) \subset \langle S(u_{s-1})\rangle$ so that means

$$g \in (f_s) + \langle S(u_{s-1})\rangle + \langle P \cap \mathfrak{m}^{\deg u_s}$$

Apply the induction hypothesis to see $g \in (f_0, ... f_s) + \langle P \cap \mathfrak{m}^{\deg u_s}\rangle$. □

As a corrollary of this, if $(f_0, ... f_s) \cap \langle P\rangle = 0$ then $\langle S(u_s)\rangle = (f_0, ... f_s) \oplus \langle P\rangle$.

This concludes the helpful lemmas and allows us to state and prove the main result.

5.2 Relations on Generators

The goal is to prove that the subvariety of ideals with pattern P is affine. From earlier results, we know that to specify and ideal with pattern P, it is enough to specify all the coefficients of the monomials in P for each f_i. Now we show that the relations on such coefficients are particularly nice.

Theorem 5.7. *Let P be a normal pattern, and $f_0, ... f_s$ be polynomials of the form $f_i = u_i + h_i$ for $h_i \in \langle P \cap \mathfrak{m}^{\deg u_i}\rangle$ so that $(f_0, ... f_s) \cap \langle P\rangle = 0$. We want to choose f_{s+1} so that $(f_0, ... f_s, f_{s+1}) \cap \langle P\rangle = 0$. Write*

$$f_{s+1} = u_{s+1} + \sum a_i \alpha_i + \sum b_j \beta_j$$

so that $\alpha_i, \beta_j \in P \cap \mathfrak{m}^{\deg u_i}$ and $x^{k_s - k_{s+1}} \alpha_i \notin P$ while $x^{k_s - k_{s+1}} \beta_i \in P$. Then the set of a_i uniquely determine the set of b_j and moreover, the set of a_i can be chosen arbitrarily.

Proof. Say

$$g = x^{k_s - k_{s+1}} \sum a_i \alpha_i \text{ and } h = x^{k_s - k_{s+1}} \sum b_j \beta_j$$

Then, $g + h = x^{k_s - k_{s+1}} f_{s+1} \in \langle S(u_s)\rangle$. So modulo $(f_0, ... f_s)$, g is equivalent to some $h' \in \langle P \cap \mathfrak{m}^{\deg u_s}\rangle$.

This means that $x^{k_s-k_{s+1}}f_{s+1} \equiv h'+h \bmod (f_0,...f_s)$. To enforce that $(f_0,...f_{s+1})\cap P = 0$, it must be the case that $x^{k_s-k_{s+1}}f_{s+1} \equiv 0 \bmod (f_0,...f_s)$. Since h' and h are contained in $\langle P\cap \mathfrak{m}^{\deg u_s}\rangle$, that means $h'=-h$. Since h' is clearly determined by g, so is h.

Moreover, suppose the a_i are determined. Then we can solve for h, and dividing by $x^{k_s-k_{s+1}}$, uncover the b_j. This gives us a candidate f_{s+1}. We need to show that $(f_0,...f_{s+1})\cap\langle P\rangle = 0$. First note that $(f_{s+1})\cap\langle P\rangle = 0$, since $f_{s+1} = u_{s+1}+h_{h+1}$ and $(u_{s+1})\cap\langle P\rangle = 0$, while $h_{s+1}\in\langle P\rangle$.

Now observe that $(f_0,...f_{s+1})\cap\langle P\rangle \subset \langle P\cap\mathfrak{m}^{\deg u_s}\rangle$: Let $\sum_{i=0}^{s+1} g_i f_i = h_0+h_1$ be in $(f_0,...f_{s+1})\cap\langle P\rangle$ with $\deg h_0 < \deg u_s$. Then modding out by $\mathfrak{m}^{\deg u_s}$ we have $g_{s+1}f_{s+1}\equiv h_0(\bmod\,\mathfrak{m}^{\deg u_s})$. But this is impossible unless $h_0=0$, since f_{s+1} contains u_{s+1} and no terms of lower degree.

Now suppose $\sum_{i=0}^{s+1} g_i f_i \in (f_0,...f_{s+1})\cap\langle P\rangle$. Since $\langle S(u_s)\rangle = (f_0,...f_s)\oplus\langle P\cap\mathfrak{m}^{\deg u_s}\rangle$ then $g_{s+1}f_{s+1}\in\langle S(u_s)\rangle$. But $(f_{s+1})\cap\langle S(u_s)\rangle = (x^{k_s-k_{s+1}}f_{s+1})\subset(f_0,...f_s)$. This means

$$(f_0,...f_{s+1})\cap\langle P\rangle \subset (f_0,...f_s,x^{k_s-k_{s+1}}f_{s+1})\cap\langle P\rangle = (f_0,...f_s)\cap\langle P\rangle = 0$$

This completes the proof. ☐

This theorem is the most important result of this section and the rest follows straightforwardly.

For one thing, consider graded ideals. The following propositions show that the generators described above work well for graded ideals

Proposition 5.8. *Say I has pattern P. I is a graded ideal if and only if the standard generators f_i are homogenous*

Proof. Since $(f_0,...f_d)=I$ the $\Leftarrow$ direction is clear.

Now suppose I is graded. Then $f_i = h_j+h_{j+1}+\cdots$ with h_k in the kth graded piece of I, and $j=\deg(u_i)$. But this means $h_{j+1}+h_{j+2}+\cdots\in I\cap\langle P\rangle$, so it is 0, so every term is 0 so $f_s=h_j$. ☐

This lets us show a similar result for graded ideals

Theorem 5.9. *Let $f_0,...f_s$ be chosen to be homogenous, of the form $f_i=u_i+h_i$ for $h_i\in\langle P\cap\mathfrak{m}^{\deg u_i}\rangle$ so that $(f_0,...f_s)\cap\langle P\rangle=0$. Also, let a_i, α_i, b_j, and β_j be as above.*

Then say that the a_i of f_{s+1} are determined so that $a_k=0$ whenever $\deg\alpha_k\neq\deg u_{s+1}$. The b_i are uniquely determined so that $(f_0,...f_{s+1})\cap\langle P\rangle=0$, but in addition to this, f_{s+1} will be homogenous of degree $\deg u_s$

Proof. It suffices to show, in the language of the earlier result, that h is homogenous of the same degree as g, since we can then divide by $x^{k_s-k_{s+1}}$. But $g-h\in(f_0,...f_s)$ which is graded, so it can be written as $g-h_j-h_{j+1}-\cdots$ with h_i in the ith graded piece of $(f_0,...f_s)$. But then $h_{j+1},h_{j+2},...$ are contained in $(f_0,...f_s)$ and also $\langle P\rangle$ so they are 0. ☐

This shows that for a fixed pattern P, the ideals of pattern P form a trivial affine bundle over the graded ideals of pattern P. Once we write Z_T as a union of finitely many such open sets, this will show that Z_T is a locally trivial affine bundle over G_T. This is because if an ideal I has pattern P, its associated graded ideal has pattern P as well., so we can check local triviality over the open sets corresponding to a fixed pattern.

6 Finite covering

This will only proves the result for Z_T one we show that Z_T can be written as a union of finitely many Z_P with different patterns P.

A pattern corresponds to a particular parametrization of R. Any polynomial in x and y can be rewritten as a polynomial of the same degree in $x' = x - ay$ and y. When the generators of an ideal are rewritten, we can check whether or not they satisfy the normal pattern of the same type in x' and y.

We will show that for a fixed type T there are finitely many $a_1, ..a_N$ such that every ideal of type T satisfies the normal pattern in the parametrization $x - a_i y, y$.

A stronger claim, which will prove the above is that there is some M such that for any ideal I there are at most M choices of a so that I does not satisfy the normal pattern in $x - ay, y$. Then *any* choice of $a_1, ...a_{M+1}$ will be acceptable.

First, we restate what it means to have a normal pattern.

Proposition 6.1. *I has normal pattern in the parametrization x, y if and only if for all j $(x^{j+1-t_j}) \cap \langle I_j \rangle = 0$.*

Proof. From Definition 5.2, I has normal pattern P if and only if $\langle P_j \rangle \cap \langle I_J \rangle = 0$.

The elements of P_j are $x^j, x^{j-1}y, ...x^{j+1-t_j}y^{t_j}$. Thus $\langle P_j \rangle = (x^{j+1-t_j}) \cap R_j$, so

$$\langle P \rangle \cap I_j = (x^{j+1-t_j}) \cap R_j \cap I_j = (x^{j+1-t_j}) \cap I_j$$

□

Now we prove the main result of this section

Theorem 6.2. *Z_T is a union of finitely many Z_P*

Proof. When $t_j = 0$ it is immediate that $(x^{j+1-t_j}) \cap I_j = 0$, because the two things we are intersecting live in different degrees. Also if I has codimension n, then $t_j = 0$ for $j > n$. So it is enough to find an M_j such that for any ideal I and any j $((x - ay)^{j+1-t_j}) \cap I_j \neq 0$ for at most M_j values of a. For $j > n$ M_j will be 0 so we can then take $M = \sum_{i=0}^{\infty} M_j$.

If $I_j = 0$ we can take $M_j = 0$, so assume otherwise. Since $\dim R_j = j+1$ and $\dim P_j = t_j$ we have $\dim I_j = j + 1 - t_j$. Say that $I = \langle f_1(x, y), ...f_{j+1-t_j}(x, y) \rangle$. Let V be a k vector space that is a subspace of $k[x]$ generated as $\langle f_1(x, 1), ...f_{j+1-t_j}(x, 1) \rangle$. It is clear that $((x - ay)^{j+1-t_j}) \cap I_j \neq 0$ if and only if there is some $f \in V$ with $(x - a)^{j+1-t_j}$ dividing f.

In this case there is a neat way of determining if there is such an f. This is equivalent to $x-a$ dividing the *Wronskian* of $f_1, ...f_{j+1-t_j}$. The Wronskian is equal to the determinant of

$$\begin{bmatrix} f_1 & \cdots & f_{j+1-t_j} \\ Df_1 & \cdots & Df_{j+1-t_j} \\ \vdots & \ddots & \vdots \\ D^{j-t_j}f_1 & \cdots & D^{j-t_j}f_{j+1-t_j} \end{bmatrix}$$

where D^i is the ith derivative.

If $(x-a)^{j+1-t_j}$ divides some element in $f \in \langle f_1, ...f_{j+1-t_j}\rangle$ then we can change basis for V and rewrite the above matrix so one column is $f, \frac{d}{dx}f, ...$ Then $(x-a)$ will clearly divide the wronskian.

Since f_i has degree no greater than j, the wronskian is a polynomial of bounded degree. Furthermore, since the f_i are linearly independent the Wronskian is not uniformly 0, and thus has a bounded number of roots. This will be proven as a separate lemma.

This completes the proof. □

Lemma 6.3. *If $f_1, ...f_n$ are linearly independent polynomials, their wronskian is not uniformly 0.*

Proof. Since differentiation is linear we can replace f_i with linear combinations as long as we preserve linear independence. Then we assume, without loss of generality, that f_1 has the highest degree and the degree of $f_2, ...f_n$ is strictly smaller. Then we make it so f_2 has the highest degree of $f_2, ...f_n$ and the degrees of $f_3, ...f_n$ are strictly smaller, and so on. This means that when we compute $|D^j f_i|$, there is a unique monomial of highest degree found in the product of the terms along the diagonal, so the polynomial cannot be uniformly 0. □

Returning to Hilbert Schemes, this finite union is connected. In fact, the combined intersection of all the open sets in the cover is nonempty. Let P be the normal pattern in x and y, and let I be the ideal generated by $(u_0, ...u_d)$ as before. Then I has every normal pattern in every parametrisation.

This is clear because $I_j = \langle y^j, xy^{j-1}, ...x^{j-t_j}y^{t_j}\rangle$ so $V_j = \langle 1, x, ...x^{j-t_j}\rangle$ which has an upper triangular wronskian with diagonal 1. Since $x-ay$ never divides 1, I has every normal pattern.

7 Dimension

Recall the properties listed about the set U in Proposition 5.3. In particular, $|U_d| = e_d + 1$ and for $j > d$ $|U_j| = e_j$

We can now compute dimensions. Since Z_T can be covered by finitely many Z_P, it suffices to compute dimension for ideals having a fixed pattern.

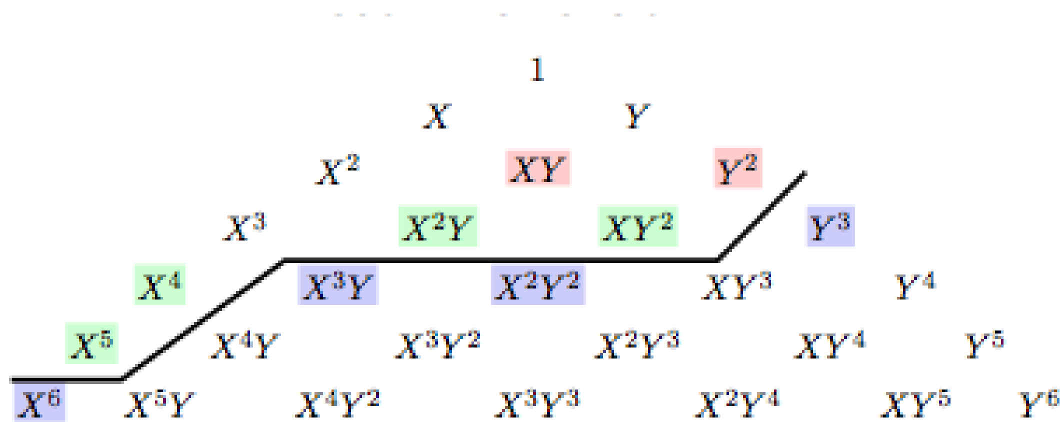

Figure 4: $s = 3$ and with the given pattern $k_2 - k_3 = 2$. The highlight terms are the elements of P that land outside when multiplied by x^2. Those in red are of degree less than $\deg u_3$

7.1 Dimension of the space of all ideals of type T

Let P be a pattern with type T. For each f_s there are $s(k_{s-1} - k_s)$ monomials in P with x-degree no less than u_s such that multiplying by $x^{k_{s-1}-k_s}$ brings them out of P. (For $s = 0$ k_{s-1} is incoherent, but this term is 0 anyway so it doesn't matter). See figure 4 for an example where $u_s = u_3 = y^3$ and $u_2 = x^2y^2$.

Also note that

$$\sum_{s=0}^{d} s(k_{s-1} - k_s) = \sum_{s=0}^{d} k_s = n$$

This however, overcounts the total number of terms. For each f_s we have counted terms of degree less than the degree of u_s. These should not contribute to f_s. One such term in the excess is produced for every u_a with $a < s$ and $\deg u_a = \deg u_{s-1}$, and all such terms are produced this way.

Say $j > d$. We will compute the number of overcounted terms when $\deg u_{s-1} = j$. Let $U_j = \{u_a, u_{a+1}, ...u_{a+e_j-1}\}$. f_{a+e_j} contributes $|U_j|$ overcounted terms, f_{a+e_j-1} contributes $|U_j| - 1$ overcounted terms, and so on. Remembering that $|U_j| = e_j$ this gives a total of

$$\sum_{i=1}^{e_j} i = \frac{e_j(e_j+1)}{2}$$

Now say that $j = d$ This is the same as before except there is no u_s of degree less than d with $\deg u_{s-1} = d$, so the sum is instead $\sum_{i=0}^{|U_d|-1} i$. However, since $|U_d| - 1 = e_d$, this is exactly $\frac{e_d(e_d+1)}{2}$.

This gives the total dimension of Z_P, and thus Z_T as $n - \sum_{j=d}^{n} \frac{e_j(e_j+1)}{2}$.

This proves that the dimension of $\mathrm{Hilb}^n R$ is equal to $n-1$, since curvilinears have dimension $n-1$ and are the highest-dimensional component. In fact, $\mathrm{Hilb}^n R$ is irreducible of dimension $n-1$, but that takes more work to show.

7.2 Dimension of graded ideals of type T

For graded ideals, it is easier to count exactly the number of terms. For each s, we want to count elements of P that have degree equal to $\deg u_s$ such that multiplying the element by $x^{k_{s-1}-k_s}$ sends it outside of P. This is equivalent to the number of a such that

$$\deg u_a > \deg u_s, \text{ and } \deg u_{s-1} \leq \deg u_a \leq \deg u_{s-1} + 1$$

Fix u_a with $\deg u_a. = j$. How many allowable options are there for u_{s-1} so that the above relation holds? u_{s-1} could be any element of U_{j-1}, unless $j-1=d$ in which case u_{s-1} could be all elements except u_d. If $j \neq d$ u_{s-1} could also be the element of U_j with highest possible y-degree, guaranteeing that $\deg u_s < \deg u_a$. Thus the total number of terms coming from u_a is

$$\begin{aligned} 0 &\text{ if } \deg u_a = d \\ |U_d| &\text{ if } \deg u_a = d-1 \\ |U_{j-1}| + 1 &\text{ otherwise} \end{aligned}$$

This simplifies because $|U_d| = e_d + 1$ and $|U_j| = e_j$ for $j > d$. The total is then

$$\sum_{j=d+1} \sum_{|U_j|} e_{j-1} + 1 = \sum_{j=d+1} e_j(e_{j-1}+1) = \sum_{j=d} (e_j+1)e_{j+1}$$

As a final note, this shows that $\dim \mathrm{Hilb}^n[[x,y]] = n-1$, since this is the dimension of the subscheme parametrizing curvilinear schemes. This shows that in $\mathrm{Hilb}^n \mathbb{P}^2$, nothing has higher dimension than the open set parametrizing n distinct points, so $\dim \mathrm{Hilb}^n \mathbb{P}^2 = 2n$

8 Conclusion

That concludes the results for this paper. There are many other related areas of interest. When $\mathbb{P}^2$ is replaced with $\mathbb{P}^3$, the above methods fail to generalize easily. For large enough n, $\mathrm{Hilb}^n \mathbb{P}^3$ is not irreducible and has unknown dimension. In a different direction, affine bundle that are not vector bundles are not well understood. The ones constructed in this paper can be distinguished by looking at the tangent bundle of the 'zero section'. For example, the tangent bundle of $\mathbb{P}^1$ included in C^4 is $\mathcal{O}(4) \oplus \mathcal{O}(3)$.

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