Sample Questions from Past Qualifying Exams

This list may give the impression that the exams consist of a series of questions fired at the student one after another. In fact most exams have more the character of a conversation with considerable give and take. Hence this list cannot be expected to indicate accurately the difficulties involved.

The list indicates the professor associated to each question where available. Some have been in the MGSA files for a while, and this information has been lost (if it was ever there).

The listing by section is approximate, since some questions may fit under more than one heading.

Geometry, Riemannian and Symplectic

- Let M be a Riemannian manifold, $exp_p : T_pM \to M$ the exponential map at a point p. Is it true that for each $q \in T_pM$ the exponential map is locally a diffeomorphism around q?
- List all the Jacobi fields along a great circle joining the north and south poles on S^n. [**Weinstein**]
- Find two equal-volume flat tori that are not isometric.
- Compute the curvature of the unit sphere in $\mathbb{R}^3$.
- What is parallel translation? How is it related to the notion of a connection on a principal bundle?
- Given a principal $O(n)$ bundle, give some examples of associated vector bundles.
- Give S^2 the usual induced metric from $\mathbb{R}^3$ and the (local) parameterization

  $$(\theta, \phi) \to (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi)$$

  Construct an orthonormal basis for T^*S^2. Calculate the Levi-Civita connection with respect to this basis. Calculate the Christoffel symbols. Calculate the curvature and the first Chern class.
- What are the geodesics on $\mathbb{R}^n$? On S^2? On the hyperbolic plane?
- Derive the Euler-Lagrange equations.
- Prove that a closed surface in $\mathbb{R}^3$ cannot have everywhere negative gaussian curvature.
- Let M contain a totally geodesic surface S. What can you say about the curvature of S?
- Give an example of a vector field in $\mathbb{R}^2$ which is not uniquely integrable.
- A Riemannian metric naturally identifies TX and T^*X. What kind of form does a volume-preserving flow correspond to?
- What is the area of a right-angled hyperbolic heptagon? What is the smallest genus closed hyperbolic surface that can be decomposed into right-angled hyperbolic heptagons?

- What is the condition on a 1-form α on a three manifold that $\ker(\alpha)$ be integrable?
- Is the space of Riemannian metrics on M path connected, for a fixed smooth structure? Is it contractible for $M = S^1$? What about $M = S^2$?
- What is the exponential map for a Riemannian manifold? [**Pugh**]
- What is a geodesic? Do they always exist? Are they unique? What are the Christoffel symbols? [**Pugh**]
- Let $\gamma_v(t)$ denote the geodesic through $p \in M$ with initial tangent vector $v \in T_pM$. Why is $\gamma_{tv}(1) = \gamma_v(t)$? [**Pugh**]
- What does it mean for two points $p, q \in M$ to be conjugate? [**Weinstein**]
- If any two points in a Riemannian manifold can be joined by a minimizing geodesic, does that mean the manifold is geodesically complete? Give some examples of incomplete manifolds. [**Weinstein**]
- Let S be a surface homeomorphic to S^2. Let p, q be two points on S and γ a minimizing geodesic connecting them. Prove that there must exist at least one more geodesic connecting p and q. [**Weinstein**]
- What is a symplectic manifold? Why must the dimension be even? List all the symplectic manifolds you know. (!) [**Weinstein**]
- Why is $S^1 \times S^3$ not symplectic? What about $\mathbb{C}P^2 \# \mathbb{C}P^2$?
- Give an example of a symplectic group action which is not hamiltonian.
- Give an example of a hamiltonian group action which is not Poisson.
- Given an example of a non-integrable almost-complex structure.
- Calculate the moment map for the standard action of $SO(3, \mathbb{R})$ on $\mathbb{R}^3$.
- Calculate the symplectic volume of $S^{2n+1}_{\sqrt{2E}}$ as a function of E, where this denotes the inverse image of $2E$ under the moment map

  $$\mu : (z_0, \ldots, z_n)) \to 1/2(|z_0|^2 + \ldots + |z_n|^2)$$

  for the standard S^1 action on $\mathbb{C}^{n+1}$. [**Weinstein**]
- Give a counterexample to Moser's theorem if M is not compact.
- Explain geodesics to a symplectic geometer.
- Let (M, ω) be symplectic where ω is actually an integral class. Can you find a principal S^1 bundle P over M and a connection form θ on P such that the curvature of θ is precisely ω?
- Let P and θ be constructed as above. Show that the horizontal distribution on P defined by θ is a contact structure on P.
- In the setup above, what is P if $M = \mathbb{C}P^n$ with the canonical symplectic structure? Can you calculate θ in this case? [**Weinstein**]
- Again, let P, θ be as above. Suppose we have another S^1-action on M which lifts to an action of P on P such that the action of the two S^1's commute and for which the second S^1 leaves θ invariant. Is this action hamiltonian (on M)?

- Characterize the Lagrangian submanifolds of T^*X with the canonical symplectic structure which are "close" to the 0-section.
- Let (M, ω) be symplectic, X a submanifold defined as the intersection of the 0-levels of functions $f_1, \ldots, f_k : M \to \mathbb{R}$. (Suppose 0 is a regular value for each f_i). Suppose each $T_x X$ is coisotropic. What can you say about X?
- Let (M, ω) and X be as above, but now suppose that $T_x X$ is a symplectic subspace of $T_x M$ for each $x \in X$. What can you say about X?
- Given (M, ω) symplectic, why is the space of compatible almost-complex structures contractible, and what is this fact good for?
- What is a contact manifold? Give some examples of contact manifolds. [**Weinstein**]
- What is a moment map? What is symplectic reduction? Give an example. [**Weinstein**]
- What is the Duistermaat-Heckman theorem? Give an example of its use. [**Borcherds**]
- What does a Hamiltonian vector field look like in local coordinates? [**Halpern**]