Sample Questions from Past Qualifying Exams

This list may give the impression that the exams consist of a series of questions fired at the student one after another. In fact most exams have more the character of a conversation with considerable give and take. Hence this list cannot be expected to indicate accurately the difficulties involved.

The list indicates the professor associated to each question where available. Some have been in the MGSA files for a while, and this information has been lost (if it was ever there).

The listing by section is approximate, since some questions may fit under more than one heading.

Field Theory and Galois Theory

- Give a polynomial not solvable by radicals.
- If α and β are algebraic over $\mathbb{Q}$, show that $\alpha+\beta$ is also algebraic over $\mathbb{Q}$.
- If α is algebraic over $\mathbb{Q}$, show that $\mathbb{Q}[\alpha]$ is a field.
- Given a non-normal separable extension $[E\colon F]=4$, bound the degree $[K\colon F]$ of the normal closure of E. [**Bergman**]
- What if there exists an intermediate field $E \subseteq LsubseteqF$ of degree 2? Then what can ou say about $\mathrm{Gal}(F/E)$? [**Bergman**]
- Are there any Galois extensions whose group is $\{\pm 1, \pm i, \pm j, \pm k\}$? [**Bergman**]
- What is an algebraic closure? [**Lenstra**]
- What is an algebraically closed field? What does it mean when a polynomial splist? [**Lenstra**]
- Show that "every polynomial over a field has a root in the field" implies that the field is algebraically close. [**Lenstra**]
- What is the index of a field in its algebraic closure? [**Lenstra**]
- What is $[\overline{\mathbb{Q}}\colon Q]$? [**Lenstra**]
- Give an example of an inseparable field extension.
- What is a splitting field of a polynomial?
- Let F be a field, and $p(x) \in F[x]$ irreducible over F. Suppose that $p(x)$ has roots α and β isn some extension field, with $\alpha \neq \beta$. Is it true that $F(\alpha) \cong F(\beta)$?
- Is it true that a polynomial of degree n over a division ring has at most n roots in any extending division ring?
- Let F be a field, θ an element (possibly transcendental) in an extension field E. The ring $F[\theta] \cong F[x]/A$ for some ideal A. What can be said about A?
- Give an example of a field F and a polynomial $p(x)$ such that the splitting field E of p over F has no normal subextensions.
- Show that a polynomial of odd degree always has at least one real root.

- Let $p(x)$ be a polynomial over $\mathbb{Q}$ with Galois group $\mathbb{Z}_4 \times \mathbb{Z}_4$. What can be said about the solvability of $p(x)$ by radicals?
- In a finite extension of fields, when can you conclude that the separable degree is equal to the degree?
- How are normal extensions and splitting fields related?
- Are all algebraic extensions finite?
- Can the cardinality of an infinite field be increased by algebraic closure?
- What is the Galois group of $x^8 - 1$ over $\mathbb{Q}$?
- What is the Galois group of $x^8 + 1$ over $\mathbb{Q}$?
- Define the concept of prime field.
- Show that any two finite fields of the same order are isomorphic.
- Let F be a Galois extension of degree n over k. What is the order of the Galois group of F over k?
- Show that if $F(a)$ is a finite extension of the field F, then the extension if algebraic.
- Show that if $F(a)$ is algebraic over F then $[F(a) \colon F]$ is finite.
- Are extensions of degree 2 over $\mathbb{Q}$ always normal?
- Let t be transcendental over k, and let $x = \frac{t^3+2}{t^2+3}$. Is x algebraic over k?
- What is a cyclotomic extension?
- Construct a field of 27 elements. [**Vojta**]
- Can you construct a field of 27 elements by starting with $\mathbb{F}_3$ and picking an irreducible polynomial of some degree? Which degree? [**Vojta**]
- Prove that any two fields of 27 elements are isomorphic. [**Vojta/Bergman**]