Sample Questions from Past Qualifying Exams

This list may give the impression that the exams consist of a series of questions fired at the student one after another. In fact most exams have more the character of a conversation with considerable give and take. Hence this list cannot be expected to indicate accurately the difficulties involved.

The list indicates the professor associated to each question where available. Some have been in the MGSA files for a while, and this information has been lost (if it was ever there).

The listing by section is approximate, since some questions may fit under more than one heading.

Differential Topology

- What is Sard's Theorem?
- Give an application of Sard's Theorem.
- Give a smooth map from S^3 to S^3. Can "most" points have an infinite number of preimages?
- Define the Lie bracket of two vector fields on a $\mathbb{C}^\infty$ manifold. [**Casson**]
- What does it mean to compose two vector fields? i.e. what does $XY - YX$ mean? [**Casson**]
- Define vector field in terms of the ring $\mathcal{F}$ of $\mathbb{C}^\infty$ functions $\mathbb{M} \to \mathbb{R}$. What does it mean to compose two vector fields? [**Casson**]
  - Is XY necessarily a vector field?
  - Why is $[X, Y]$ a vector field?
- When does a vector field determine a flow? [**Casson**]
- What does it mean for a vector field to have compact support? [**Casson**]
- Define flow. [**Casson**]
- In what sense do the diffeomorphisms in a flow vay "in a $\mathbb{C}^\infty$ fashion"? [**Casson**]
- Does a flow determine a vector field? [**Casson**]
- Give conditions on $\mathbb{M}$ so that every vector field on $\mathbb{M}$ determines a flow. [**Casson**]
- Relate "tangent vector to a curve at a point" to "point derivation". [**Casson**]
- Give an example of a vector field on a manifold that does not determine an everywhere-defined flow. [**Casson**]
- A *knot* is a $\mathbb{C}^\infty$ embedding $S^1 \to \mathbb{R}^3$. Consider the following two statements about two knots $f, g\colon S^1 \to \mathbb{R}^3$:

  (i) There is an isotopy between f and g.

  (ii) There is a diffeomorphism of $\mathbb{R}^3$ inducing a diffeomorphism $f(S^1) \to g(S^1)$.

  Relate these conditions. [**Casson**]

    - Can you use any of this information to say something about classifying diffeomorphisms $\mathbb{R}^3 \to \mathbb{R}^3$? [Hint: the trefoil knot cannot be deformed into its mirror image] [**Casson**]
- How many components does $\mathrm{Diff}(\mathbb{R}^3, \mathbb{R}^3)$ have, and what is meant by this? [**Casson**]
- State a Lemma about a diffeomorphism $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ if $f(0) = 0$; in particular, how may f be rewritten? [**Casson**]
- Write down a path from f to $\mathrm{Dif}|_0$, where $f(0) = 0$ and $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism. [**Casson**]
- Define a Morse function. Define index. [**Casson**]
- What is $h^{-1}(a, b)$ if (a, b) does not contain any critical values? What if it contains exactly one critical value? [**Casson**]
- If a morse function on a manifold M has exactly two critical points, what can you say about M? [**Casson**]
- What is the Frobenius Integrability Theorem? [**Serganova**]
- What is an integral submanifold? [**Serganova**]
- What is $[x, y]$? [**Serganova**]
- Can you give an example of a distribution which is not integrable? [**Serganova**]
- Explain how the characteristic classes of a vector bundle arise. What are all the characteristic classes of a vector bundle? [**Wodzicki**]
- What is the Thom isomorphism? [**Wodzicki**]
- What is the Thom class? What is it an obstruction to? [**Wodzicki**]
- Construct the Thom class explicitly for a trivial bundle. [**Frenkel**]
- Prove that the cohomology of a compact Lie group is that of its Lie algebra. [**Wodzicki**]
- How does one use Sard's theorem to prove the Whitney embedding theorem? [**Harrision**]