Sample Questions from Past Qualifying Exams

This list may give the impression that the exams consist of a series of questions fired at the student one after another. In fact most exams have more the character of a conversation with considerable give and take. Hence this list cannot be expected to indicate accurately the difficulties involved.

The list indicates the professor associated to each question where available. Some have been in the MGSA files for a while, and this information has been lost (if it was ever there).

The listing by section is approximate, since some questions may fit under more than one heading.

Also available from the MGSA is an extended syllabus and sample Qual questions with answers that Danny Calegari wrote. The questions on algebraic topology are included here, but his write-up contains answers as well.

Algebraic Topology

- Compute the homotopy group $\pi_3(S^2)$.
- Which homotopy classes $\alpha\colon \mathbb{P}^2 \to \mathbb{P}^2$ are there which is the identity on $\subset_1$ $(\mathbb{P}^2)$, the fundamental group?
- What can you say about the homotopy type of the "dunce cap"?
- Tell us about the Van Kampen theorem. [**Stallings**]
- Can you use the Van Kampen theorem to compute the fundamental group of the Hawaiian earring? [**Kirby**]
- Is the fundamental group of the Hawaiian earring finite? Free? Countable or Uncountable? [**Kirby**]
- Why do you have the Fundamental Theorem of Algebra under algebraic topology in your syllabus? [**Stallings**]
- How do you know the fundamental group of S^1 is $\mathbb{Z}$? [**Stallings**]
- Find all 2-fold coverings of the figure 8.
- Find an example of:
  (1) $H_p(X) = H_p(Y)$ for all p, but X and Y not homeomorphic.
  (2) $\pi_p(X) = \pi_p(Y)$ for all p, but $H_*(X) \neq H_*(Y)$.
- The example to (2) above seems to contradict Whitehead's Theorem. Do you know why it doesn't contradict it?
- Compute $H_*(\mathbb{C}\mathbf{P}^n)$ and $H^*(\mathbb{C}\mathbf{P}^n)$.
- Define Chern classes and compute them for some examples like $\mathbb{C}\mathbf{P}^n$.
- Does "Euler Class" classify all disk bundles over S^2?
- Does C_1, the first Chern class, classify all complex line bundles over T^2?
- Compute the intersection form from the framed link which represents the 4-manifold.
- What is $\pi_2(S^2 \vee S^2)$?

- Give an example of two spaces which are not homotopy equivalent, but have the same homology.
- What is π_2 of $S^2 \vee S^1$?
- Calculate $\pi_1(X)$ where X is the three manifold obtained from $T^2 \times I$ by identifying the opposite faces by the glueing map $(1,0) \mapsto (2,1)$, $(0,1) \mapsto (1,1)$.
- Show that the free group on two generators contains the free group on n generators with finite index.
- Show that every subgroup of a free group is free.
- Given an example of a pair (X, A) such taht $\pi_i(X, A) \neq \pi_i(X/A)$ for some i.
- Give an example of two spaces with the same cohomology groups but with a different ring structure.
- Show that a compact surface with sectional curvature positive everywhere is homeomorphic to S^2.
- Calculate the homology with coefficients in $\mathbb{Z}$ of the Lens space $L(a, b)$.
- Prove that for any orientable compact 3-manifold M with boundary ∂M that half the first rational homology of ∂M is killed by inclusion into M.
- Show that an element of H_{n-1} for an orientable n-manifold is represented by a smoothly embedded $n-1$-manifold.
- If a simply-connected CW complex Σ satisfies $H_2(\Sigma) = \mathbb{Z} \oplus \mathbb{Z}$ and $H_i(\Sigma) = 0$ for all $i \neq 2$, then show that Σ is homotopy equivalent to $S^2 \vee S^2$.
- Show that if G is a finitely generated finitely presented group, then G is the fundamental group of some compact 4-manifold.
- Show that a simply connected differentiable manifold is orientable.
- Classify S^3 bundles over S^5.
- Show that any two embeddings of a connected closed set X in S^2 has homeomorphic complements C_1, C_2.
- Show that $\mathbb{CP}^2$ does not cover any manifold other than itself.
- Compute the homology of $\mathbb{P}^n$. [**Stallings**]
- Compute the homology of $\mathbb{P}^n$ with $\mathbb{Z}/2\mathbb{Z}$. [**Stallings**]
- Compute the cohomology of $\mathbb{P}^n$. [**Stallings**]
- Use intersection theory to compute the cup structure of the cohomology of $\mathbb{P}^n$ with $\mathbb{Z}/2\mathbb{Z}$ coefficients where n is odd. [**Stallings**]
- What are all of the n-fold covers of the genus 2 surface. [**Stallings**]
- What is an H-space? What special property does π_1 of an H-space have? Prove it. [**Casson**]
- Why can't S^2 be an H-space? [**Stallings**]
- What is the homology of $S^2 \times S^2$? Cohomology? How is the cohomology related to the homology? What is the cup product structure? [**Casson**]

- Suppose that X and Y are simply-connected cell spaces which have the same homology groups. Do they necessarily have the same homotopy groups? Are they necessarily homotopy equivalent? [**Givental**]
- Why does Hurwitz's theorem fail for non-simply connected spaces? Give an example of a space X where the action of $\pi_1(X)$ on the higher homotopy groups is not trivial. [**Weinstein**]
- What is the Thom class? Let E be the universal bundle over $BU(n)$ and consider $F = \mathbb{P}(E \oplus \mathbb{C})$. What is the Thom class of the normal bundle to the zero section in F? [**Givental**]