Special Lagrangians in the Landau-Ginzburg Mirror of $\mathbb{C}P^2$

Senior thesis submitted by
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Abstract

The Fukaya category associated to a Lefschetz fibration is a central object of study in mirror symmetry. This paper is primarily an exposition of Fukaya-Seidel category via the example of the Landau-Ginzburg mirror to $\mathbb{C}P^2$, given explicitly by $W = x + y + \frac{1}{xy} : (\mathbb{C}^*)^2 \to \mathbb{C}$. Two different models depending on the position of the reference fiber, introduced in [AKO08] and [Sei12] respectively, are discussed. We focus on the complex geometry of a regular fiber and the way it varies in family. In the second half of the paper, we use these geometric results to explore the existence of special Lagrangian submanifolds fibered over an embedded path in the fibration $W : (\mathbb{C}^*)^2 \to \mathbb{C}$.

Contents

1 Introduction

In his seminal paper [Kon94], Maxim Kontsevich proposed the Homological Mirror Symmetry (HMS) conjecture, which states a categorical duality for certain Calabi-Yau manifolds. More precisely, for a Calabi-Yau manifold X, we can associate its derived category of coherent sheaves, $\mathcal{D}^b\mathrm{Coh}(X)$, called the B-model, as well as (some suitably derived version of) its Fukaya category, $\mathcal{D}\mathrm{Fuk}(X)$, called the A-model. The HMS conjecture states that for a Calabi-Yau manifold X, there should exist a mirror Calabi-Yau $X^\vee$, such that there are equivalences of categories

$$\mathcal{D}^b\mathrm{Coh}(X) \cong \mathcal{D}\mathrm{Fuk}(X^\vee) \qquad \text{and} \qquad \mathcal{D}^b\mathrm{Coh}(X^\vee) \cong \mathcal{D}\mathrm{Fuk}(X).$$

In subsequent works, the HMS conjecture had been extended to more general settings. For a toric Fano variety X, the mirror of X is a Landau-Ginzburg model consisting of a Calabi-Yau manifold Y equipped with a holomorphic function W, called the superpotential. In this paper, we discuss the Fukaya-Seidel category associated to a Lefschetz fibration, introduced by Paul Seidel in [Sei01a],[Sei01b], [Sei08], [Sei12], which has the nice property that it is generated by a finite exceptional collection consisting of vanishing cycles (or Lefschetz thimbles). In particular, this will be our version of Fukaya category associated to a Landau Ginzburg model $W : Y \to \mathbb{C}$. In particular, we discuss the HMS conjecture for the specific case of $\mathbb{C}P^2$, originally proved in [Sei01b] and extended to the case of weighted projective planes and their noncommutative deformations in [AKO08].

Special Lagrangian (sLag) submanifolds are central objects in the study of mirror symmetry. In [Joy14], Dominic Joyce conjectured that a holomorphic volume form Ω on a Calabi-Yau manifold X should give rise to a stability condition on $\mathcal{D}\mathrm{Fuk}(X)$, where the semistable objects are sLags with respect to Ω. In the second half of the paper, we compute explicit examples of sLag thimbles in the Landau-Ginzburg model $W : (\mathbb{C}^*)^2 \to \mathbb{C}$, where $W = x+y+x^{-1}y^{-1}$, mirror to $\mathbb{C}P^2$. We show that a generic fiber $W^{-1}(\lambda)$ is biholomorphic to a complex torus with three punctures, and we choose a holomorphic volume form Ω whose residue along $W^{-1}(\lambda)$ is identified with the standard flat structure. The sLag condition for a thimble can then be reduced to studying the fiberwise tangent direction of its vanishing cycles and its base tangent direction. For a thimble in general position, we will need to modify the symplectic form in order to obtain a genuine sLag.

The organization of this paper is as follows. In Section 2, we introduce the notion of a Lefschetz fibration and its Fukaya-Seidel category following [Sei01a], [Sei08]. In Section 3, we continue the exposition and study the mirror Landau-Ginzburg model of $\mathbb{C}P^2$ and prove the HMS conjecture in this case. Then, we move on to the original part of this paper, starting with showing that the standard Lefschetz thimbles are sLags (after modifying the symplectic form) when the reference fiber is at the origin. In Section 4, we explore a more 'general' situation where the reference fiber is at $+\infty$, and prove the main result regarding the existence of sLag thimbles fibered over an embedded curve. Nonetheless, the correctness of our approach relies on certain smoothness property in the construction that the author has yet to prove, and is stated as a conjecture (Conjecture 4.2.2). Finally, in Remark 4.5.2, we briefly discuss the interpretation of our results in the context of mirror symmetry and Bridgeland stability conditions. In the appendix, we review some basic facts about derived categories and A_∞-categories.

Acknowledgements

First and foremost, I would like to thank my advisor Denis Auroux for introducing me to the subject of symplectic geometry and for his most patient and helpful guidance through this thesis. I am deeply indebted to him for his generosity in sharing ideas and wisdoms, for lengthy and rewarding discussions every week for almost an entire year, and for his continuous support throughout my undergraduate career. I would like to thank Hiro Tanaka, who mentored my first undergraduate research in Bridgeland stability conditions, which ultimately led me to the topic of this thesis. I would also like to thank Maxim Jeffs for helping me a lot in learning mirror symmetry in the past semester. Finally, I want to thank my family for their unconditional love and support, and my friends for always being there for me despite having to endure my ramblings about abstract (and concrete) nonsense. May the Force be with you, through 2020 and through all times of hardship.

2 Lefschetz fibrations and Fukaya-Seidel categories

In this section, we follow the general framework of [Sei01a],[Sei08] and define the notion of an exact Lefschetz fibration. Roughly speaking, these are smooth fibrations $\pi : E \to B$ of exact symplectic manifolds with possible nondegenerate singularities. The key property of a Lefschetz fibration is that each fiber $\pi^{-1}(b)$ is symplectic and that T_xE at each $x \in E$ splits into the horizontal direction and the fiber direction. This allows us to define parallel transport along a path in the base. Next, we define vanishing cycles and Lefschetz thimbles, which are objects of the Fukaya-Seidel category.

2.1 Lefschetz fibrations

For the purpose of this paper, we won't introduce the full generality of Lefschetz fibration as covered in [Sei01a],[Sei08]. In particular, we restrict our attention to those of type $\pi : E \to \mathbb{C}$, where E is an open exact symplectic manifold with a compatible almost complex structure J_E.

Definition 2.1.1. A *Lefschetz fibration* is a smooth map $\pi : E \to \mathbb{C}$ such that
1) each fiber of π is a symplectic submanifold of E;
2) π has finitely many critical points $p_1, \cdots, p_k$ that are *integrable* and *nondegenerate*, i.e. in a neighborhood of each p_i, J_E is integrable and π can be written as $\pi(x_1, \cdots, x_n) = \pi(p_i) + x_1^2 + \cdots + x_n^2$ in local coordinates;
3) there is a unique critical point lying over each critical value.

Note that aside from the critical points, the symplectic orthogonal to each fiber defines a horizontal distribution which allows us to do parallel transport.

2.2 Vanishing cycles and Lefschetz thimbles

Let $\pi : E \to \mathbb{C}$ be a Lefschetz fibration where $\dim(E) = 2n + 2$. A *vanishing path* is an embedded path $\gamma : [0, 1] \to \mathbb{C}$ with $\gamma(1)$ a critical value and $\gamma([0, 1))$ disjoint from $\mathrm{Crit}(\pi)$. For each such path, we can associate its *Lefschetz thimble* Δ_γ, which is the unique embedded Lagrangian $(n + 1)$-ball in E satisfying $\pi(\Delta_\gamma) = \gamma([0, 1])$ and $\pi(\partial\Delta_\gamma) = \gamma(0)$. The boundary $\mathrm{VC}(\gamma) := \partial\Delta_\gamma$ is called the *vanishing cycle* of γ. As an abuse of terminology, the Lagrangian spheres $E_{\gamma(t)} \cap \Delta_\gamma$ for $0 \leq t < 1$ are sometimes also called vanishing cycles.

The following lemma ensures that Δ_γ is in fact well defined.

Lemma 2.2.1. *Let β be an embedded path in $\mathbb{C}\backslash\mathrm{Crit}(\pi)$, and $F \subset E$ a submanifold fibered over β such that each $F_{\beta(t)} \subset E_{\beta(t)}$ is Lagrangian. The $F \subset E$ is Lagrangian if and only if parallel transport maps $F_{\beta(t)}$ into each other.*

Proof. First assume that parallel transport maps $F_{\beta(t)}$ into each other. Let t be arbitrary within the interval of definition of β, and $\dot\beta(t)$ the corresponding tangent vector. For $p \in F$ such that $\pi(p) = \beta(t)$, let $\tilde\beta(p)$ be the unique lift of $\dot\beta(t)$ to the horizontal part T_pE^h. By assumption, we have $\tilde\beta(p) \in T_pF$. In particular, T_pF is spanned by $\tilde\beta(p)$ and $T_pF_{\beta(t)}$ and $\omega(\tilde\beta(p), T_pF_{\beta(t)}) = 0$. Since $F_{\beta(t)} \subset E_{\beta(t)}$ is Lagrangian, we conclude that $\omega|_{T_pF} = 0$, and hence $F \subset E$ is Lagrangian.

Conversely, assume F is Lagrangian. For $p \in F$, choose a lift $v(p) \in T_pF$ of $\dot\beta(t)$. Since F is Lagrangian, $v(p) \in (T_pF)^{\perp\omega} \subset (T_pF_{\beta(t)})^{\perp\omega} = T_pE^h + T_pF_{\beta(t)}$. By adding an element of $T_pF_{\beta(t)}$, we may as well assume that the lift $v(p)$ is in T_pE^h. This shows that F is invariant under parallel transport. $\square$

This implies that the Lefschetz thimble Δ_γ is given uniquely by the formula

$$\Delta_\gamma = \{p \in E_{\gamma(t_0)}, 0 \le t_0 < 1 \,:\, \lim_{t_1 \to 1} h_{\gamma|_{[t_0,t_1]}}(p) = p_1\} \cup \{p_1\},$$

where h denotes parallel transport and p_1 is the unique critical point in $E_{\gamma(1)}$.

This description may face several technical difficulties, e.g. if the fibers are noncompact, then parallel transport might not be well defined. However, we will only work with the case such that no such difficulties are present, for instance, by choosing a Kähler form whose induced metric is complete and such that $|\nabla\pi|$ is bounded below outside of a compact set (see [AKO08, Section 4.1]). Such technical difficulties can also be resolved by considering a suitable Hamiltonian functional and its flow, see [Sei08, Section III.16(b)].

2.3 Dehn twist and the symplectic Picard-Lefschetz theorem

Fix a Lefschetz fibration $\pi : E \to \mathbb{C}$ where $\dim(E) = 2n + 2$ and a vanishing path γ. Let $\mathrm{VC}(\gamma)$ be the vanishing cycle, which is a Lagrangian n-sphere in $E_{\gamma(0)}$. Let λ be a loop based at $\gamma(0)$ that winds around $\gamma(1)$ once counterclockwise.

Parallel transport along λ defines a monodromy action h_λ on $E_{\gamma(0)}$. The purpose of this subsection is to relate this monodromy action to another action on $E_{\gamma(0)}$, called the *Dehn twist* along $\mathrm{VC}(\gamma)$.

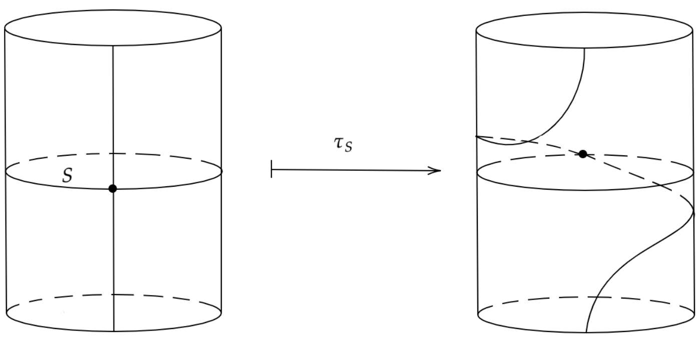

Figure 1: Dehn twist about the zero section

We first define Dehn twist in the standard model T^*S^n with the standard symplectic form and metric. Let p be the fiber coordinate and q be the base coordinate, we can define a Hamiltonian functional $H(p,q) = h(\|p\|)$, where $h : [0,\infty) \to \mathbb{R}$ is a smooth function such that $h'(0) = \pi, h'' \leq 0$ and h is constant outside of a compact neighborhood of zero. The Hamiltonian flow of h defines a diffeomorphism of $T^*S^n \backslash S^n$, which can be extended to the zero section by defining it to be the antipodal map on the zero section. When $n = 1$, we get the familiar picture shown in Figure 1.

In the more general setting where we have a Lagrangian sphere S inside a symplectic manifold (M,ω), Weinstein's neighborhood theorem allows us to identify a neighborhood of S in M with (S, T^*S). Performing the above construction inside this neighborhood, we get a symplectomorphism τ_S of M supported inside the neighborhood and restricts to the antipodal map on S, called the *Dehn twist* about S. The isotopy class of τ_S is independent of the choice of h and the choice of neighborhood.

Now, we are ready to state the symplectic Picard-Lefschetz theorem.

Theorem 2.3.1. *Let $\pi : E \to \mathbb{C}, \gamma$ and λ be as in the first paragraph of this subsection. Then, there is an isotopy*

$$h_\lambda \simeq \tau_{\mathrm{VC}(\gamma)}.$$

We sketch a proof of the theorem when $\dim E = 4$, which is the case of interest. For a detailed proof in the general case, see [Sei03].

Since deforming the Lefschetz fibration will not change the isotopy class of the monodromy, we may without loss of generality work in the standard Morse chart $\pi : \mathbb{C}^2 \to \mathbb{C}$ where $\pi(z_1, z_2) = z_1^2 + z_2^2$, with critical value 0. After parallel transporting, we may assume our basepoint to be 1, and consider the monodromy along the loop $\gamma(t) = e^{2\pi i t}$.

In each fiber E_λ, there is a distinguished symplectic vanishing cycle $\sqrt{\lambda}S^1 = \{z = (z_1, z_2) | z_1^2 + z_2^2 = \lambda, z_1, z_2 \in \sqrt{\lambda}\mathbb{R}\}$. Under the standard symplectic form on $\mathbb{C}^2$, the horizontal tangent space at $z = (z_1, z_2)$ is given by $T_z E^h = \mathbb{C}\overline{z}$, and the horizontal lift of a vector $v \in \mathbb{C}$ is given by $\frac{v}{2|z|^2}\overline{z}$. Thus, parallel transport along any curve maps the vanishing cycles $\sqrt{\lambda}S^1$ to each other. Moreover, parallel transport along the unit circle preserves complex norm. To see this, fix λ a point on the unit circle, $v \in \mathbb{C}$ a tangent vector at λ and $z \in E_\lambda$. For small t, parallel transport along tv can be approximated by

$$z \mapsto z + \frac{tv}{2|z|^2}\overline{z} + O(t^2).$$

Then,

$$\left|z + \frac{tv}{2|z|^2}\overline{z}\right|^2 = |z|^2 + \frac{t}{|z|^2}\mathrm{Re}(\overline{v}\lambda) + O(t^2).$$

But since λ is on the unit circle, v is perpendicular to λ and hence $\mathrm{Re}(\overline{v}\lambda) = 0$, which proves the claim. Let $\Phi_t : E_1 \to E_{e^{2\pi i t}}$ be the parallel transport map.

For each t, consider the double cover $\pi_1 : E_{e^{2\pi i t}} \to \mathbb{C}$ given by $(z_1, z_2) \mapsto z_1$, with branch points $\pm(e^{\pi i t}, 0)$. Since the horizontal distribution is $\mathbb{C}\overline{z}$ (in particular, points with the same z_1 coordinate will be sent to points with the same z_1 coordinate), Φ_t descends to a well defined diffeomorphism ϕ_t of $\mathbb{C}$. Note that for each t, ϕ_t is not compactly supported, yet it is close to the identity for $|z_1|$ large. Up to composing Φ_t with an isotopy, we can assume that ϕ_t is the identity outside some large disk D for each t and that Φ_1 is the identity outside the tubular neighborhood $S^1 \times [0,1] \cong \pi_1^{-1}(D)$. Since ϕ_t pushes the branch points counterclockwise, ϕ_1 is the positive half Dehn twist on D with two marked points. Thus, Φ_1 is the unique lift of the half Dehn twist, and by classical results from mapping class groups, it is the positive Dehn twist on $\pi_1^{-1}(D) \cong S^1 \times [0,1]$.

Remark 2.3.2. In fact, in the case of exact Lefschetz fibrations we are considering, the monodromy is Hamiltonian isotopic to the symplectic Dehn twist in $\mathrm{Symp}(E_{\gamma(0)})$. Similarly, vanishing cycles with respect to homotopic paths rel endpoints (where the homotopy does not cross a critical value) are Hamiltonian isotopic to each other in the fiber. We will assume these facts for the rest of the paper (see [Sei01a], [Sei03]).

2.4 The Fukaya-Seidel category

In this subsection, we define the (derived)Fukaya-Seidel category associated to a Lefschetz fibration, following [Sei01a],[Sei08] and [AKO08]. Let $\pi : E \to \mathbb{C}$ be a Lefschetz fibration of dimension $2n + 2$ as above. Fix a base point $x_0 \in \mathbb{C}\backslash\mathrm{Crit}(\pi)$. A *distinguished basis of vanishing paths* is an ordered family $\boldsymbol{\gamma} = (\gamma_1, \cdots, \gamma_m), m = |\mathrm{Crit}(\pi)|$ of vanishing paths such that
1) for each i, $\gamma_i(0) = x_0$ and for $i \neq j$, γ_i and γ_j only intersect at x_0;
2) The tangent directions of γ_i at x_0 are ordered clockwise.

Definition 2.4.1. Given a Lefschetz fibration $\pi : (E, \omega) \to \mathbb{C}$ and a distinguished basis of vanishing paths $\boldsymbol{\gamma}$, the *directed category of vanishing cycles* $\mathrm{Lag}_{\mathrm{vc}}(\pi, \boldsymbol{\gamma})$ is the A_∞-category with m objects $\{L_i = \mathrm{VC}(\gamma_i)\}_{i=1}^m$, and morphisms given by

$$\mathrm{Hom}(L_i, L_j) = \begin{cases} CF^*(L_i, L_j) \cong \mathbb{C}^{|L_i \cap L_j|}, & \text{for } i < j \\ \mathbb{C} \cdot \mathrm{id}_{L_i}, & \text{for } i = j \\ 0, & \text{for } i > j \end{cases}$$

where CF^* is the Floer cochain complex with coefficients in $\mathbb{C}$ (after perturbing such that the L_i intersect transversely in E_{x_0}). When $i_0 < i_1 < \cdots < i_k, k \geq 1$, we define the operations

$$\mu^k : \mathrm{Hom}(L_{i_0}, L_{i_1}) \otimes \cdots \otimes \mathrm{Hom}(L_{i_{k-1}}, L_{i_k}) \to \mathrm{Hom}(L_{i_0}, L_{i_k})[2 - k]$$

in terms of Lagrangian Floer theory inside the fiber E_{x_0}. Specifically, given $p_j \in L_{i_{j-1}} \cap L_{i_j}, 1 \leq j \leq k$ and $q \in L_{i_0} \cap L_{i_k}$, we consider the following. Let D be the closed unit disc minus $\zeta_0, \zeta_1, \cdots, \zeta_k$, where $\{\zeta_i\}$ is a cyclically ordered tuple of marked points on the boundary circle (whose positions are allowed to vary). Fixing a compatible almost complex structure J on E_{x_0}, we consider the space of all J-holomorphic maps $u : D \to E_{x_0}$ which extends continuously to the punctures and maps $[\zeta_j, \zeta_{j+1}]$ to L_{i_j} and $\zeta_1, \cdots, \zeta_k, \zeta_0$ to $p_1, \cdots, p_k, q$, respectively. The moduli space $\mathcal{M}(p_1, \cdots, p_k, q; J)$ is defined as the quotient of the above space by the complex automorphism group of the disk $\mathrm{Aut}(D)$. If $\deg(q) = \sum_{j=1}^k \deg(p_j) + 2 - k$, the expected dimension of $\mathcal{M}(p_1, \cdots, p_k, q; J)$ is zero. Thus, we define

$$\mu^k(p_1, \cdots, p_k) = \sum_{\substack{q \in L_{i_0} \cap L_{i_k} \\ \deg(q) = \sum \deg p_j}} \Big(\sum_{u \in \mathcal{M}(p_1, \cdots, p_k, q; J)} \pm e^{-2\pi\omega([u])} \Big) q.$$

On the other hand, if $i_0 < \cdots < i_k$ fails to hold, we set $\mu^k = 0$. The elements id_{L_j} are defined such that $\mu^1(\mathrm{id}) = 0, \mu^2(\mathrm{id}, a) = \mu^2(a, \mathrm{id}) = a$ and all $\mu^k, k \geq 3$ involving id are zero.

Finally, the *Fukaya-Seidel category* is defined by $\mathrm{FS}(\pi, \boldsymbol{\gamma}) = \mathcal{D}\mathrm{Lag}_{\mathrm{vc}}(\pi, \boldsymbol{\gamma})$, where the derived category is defined as the cohomological category of $\mathrm{Tw}(\mathrm{Lag}_{\mathrm{vc}})$ (see Appendix).

Showing that these μ^k satisfy the A_∞-relations is a standard (but very technically challenging) argument in Lagrangian Floer theory (see [Aur14] for a friendly explanation). Despite the tremendous difficulty in studying Lagrangian Floer theory in general, the setting of a Lefschetz fibration saves us most of the trouble. In particular, the fact that E is an exact symplectic manifold ($\omega = d\theta$) and L_i are exact Lagrangian submanifolds ($\theta|_{L_i} = df_i$) ensure that no sphere or disk bubbling occur and thus the moduli space above is well defined and orientable.

Remark 2.4.2. It is widely understood that the generators of the Fukaya-Seidel category should be the Lefschetz thimbles Δ_{γ_i} instead of the vanishing cycles. Since Lefschetz thimbles are topological disks, they are easier to work with. For instance, they automatically have Maslov index zero, and hence gradable (although in the case we'll consider the vanishing cycles also have this property). It will also be clear later that Lefschetz thimbles are the 'correct' objects when studying special Lagrangian properties.

Nonetheless, this distinction should not bother us. By definition of the vanishing paths γ_i, it is clear that $\mathrm{VC}(\gamma_i) \cap \mathrm{VC}(\gamma_j) = \Delta_{\gamma_i} \cap \Delta_{\gamma_j}$. Moreover, by the open mapping theorem, a pseudo-holomorphic map $u : D \to E$ with boundary on the thimbles must in fact be contained in E_{x_0}. Therefore, up to grading, the Floer cohomology of the Lefschetz thimbles is the same as the Floer cohomology of the vanishing cycles.

It is immediate from the definition that $\mathrm{FS}(\pi, \boldsymbol{\gamma})$ is generated by the *exceptional collection* $(\mathrm{VC}(\gamma_1), \cdots, \mathrm{VC}(\gamma_m))$. However, a priori $\mathrm{FS}(\pi, \boldsymbol{\gamma})$ depends on the choice of a distinguished basis of vanishing paths. We now sketch the idea of proof that $\mathrm{FS}(\pi, \boldsymbol{\gamma})$ is independent of $\boldsymbol{\gamma}$.

Theorem 2.4.3. *([Sei01a],[Sei08]) Given a Lagrangian sphere $S \subset (M, \omega)$ and an object $L \in \mathrm{Fuk}(M)$, there exists an exact triangle (in $\mathrm{Tw}(\mathrm{Fuk}(M))$)*

$$\begin{tikzcd}
HF^*(S,L) \otimes S \arrow[rr, "ev"] & & L \arrow[dl] \\
& \tau_S(L) \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

There is an action of the Braid group Br_m of $m-1$-generators on the set of isotopy classes of distinguished bases, where the standard generator σ_k $(1 \le k \le m-1)$ acts by a 'braid' shown in Figure 2.

The symplectic Picard-Lefschetz theorem implies that

$$\mathrm{VC}(\sigma_k(\boldsymbol{\gamma})_j) = \begin{cases} \tau_{\mathrm{VC}(\gamma_k)}(\mathrm{VC}(\gamma_{k+1})), & \text{for } j = k \\ \mathrm{VC}(\gamma_k), & \text{for } j = k+1 \\ \mathrm{VC}(\gamma_j), & \text{otherwise} \end{cases}$$

and Theorem 2.4.3. implies that the new exceptional collection of vanishing cycles associated to $\sigma_k(\boldsymbol{\gamma})$ is related to the original collection by a mutation (which induces a mutation on the category). Using these insights, Seidel showed that

Theorem 2.4.4. *For any two distinguished bases $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$, $\mathrm{Lag}_{\mathrm{vc}}(\pi, \boldsymbol{\gamma})$ and $\mathrm{Lag}_{\mathrm{vc}}(\pi, \boldsymbol{\gamma}')$ are related by a sequence of mutations, and there is an exact equivalence between $\mathrm{FS}(\pi, \boldsymbol{\gamma})$ and $\mathrm{FS}(\pi, \boldsymbol{\gamma})$ as ordinary triangulated categories.*

Hence from now on, we may write $\mathrm{FS}(\pi)$ instead of $\mathrm{FS}(\pi, \boldsymbol{\gamma})$.

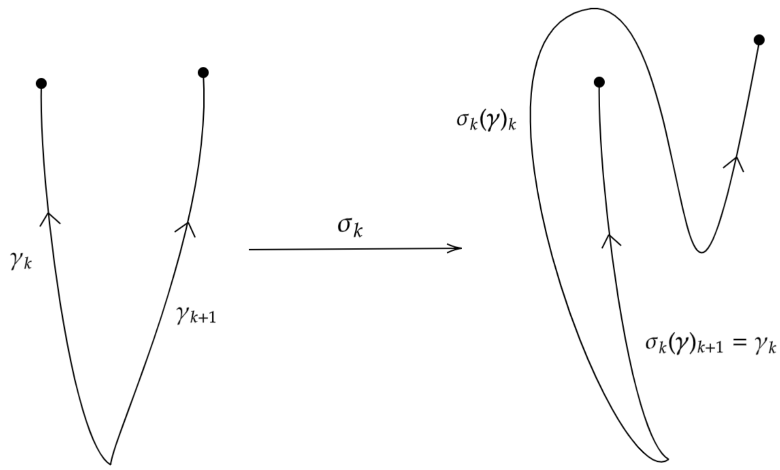

Figure 2: Braid group action on a basis of vanishing paths

Remark 2.4.5. In [Sei12], Seidel presented an alternative geometric description that gives rise to the same Fukaya category. Here we briefly discuss a simplified version of that picture. We take our basepoint to be $+\infty$, i.e. take a basis of disjoint vanishing paths γ_i such that for each i, $\lim_{t\to 0} \gamma_i(t) = +\infty$ and outside of a compact neighborhood of $\gamma_i(1)$, γ_i have constant imaginary part c_i. Moreover, assume $c_i > c_j$ if $i > j$ (i.e. in 'clockwise' order at $+\infty$). Then, $\mathrm{Hom}(\Delta_{\gamma_i}, \Delta_{\gamma_j}), i < j$ is a perturbed Floer cochain complex. More specifically, for each $i < j$, we fix a Hamiltonian functional $h_{ij} : \mathbb{C} \to \mathbb{R}$, depending only on $\mathrm{Re}(z)$, such that $h_{ij} = 0$ in a compact neighborhood of $\gamma_i(1)$ and $h'_{ij} = \text{constant} > c_j - c_i$ close to infinity. Hence, $X_{h_{ij}}$ pushes γ_i in the positive imaginary direction and produces a transverse intersection with γ_j.

Let H_{ij} be the pullback of h_{ij} to the total space. Then, we define $\mathrm{Hom}(\Delta_i, \Delta_j) = CF^*(\Delta_i, \Delta_j, H_{ij})$ (see [Sei08] for perturbed Floer theory) assuming h_{ij} is chosen such that $\phi^1_{H_{ij}}(\Delta_i)$ intersects transversely with Δ_j. In this version of the Fukaya category, the directedness of the Lagrangians Δ_i is naturally explained by the choice of our Hamiltonian functionals.

3 The Landau-Ginzburg model, version 1

In this section, we study a specific example of Fukaya-Seidel category associated to a complex dimension 2 Lefschetz fibration, and prove homological mirror symmetry for $\mathbb{C}P^2$.

3.1 Geometry of the Landau-Ginzburg model

The mirror to $\mathbb{C}P^2$ is a Landau-Ginzburg model $W : (\mathbb{C}^*)^2 \to \mathbb{C}$, where $W = x+y+x^{-1}y^{-1}$ is the superpotential coming from counting holomorphic discs bounded by the product tori in $\mathbb{C}P^2$ that has intersection number 1 with the anticanonical divisor (see [Aur07]). Following [AKO08], we embed the total space into $(\mathbb{C}^*)^3$, i.e. as the subspace $X = \{xyz =$

$1\} \subset (\mathbb{C}^*)^3$ equipped with the superpotential $W = x + y + z$. This equivalent description gives us additional symmetries that will be useful later.

Let the symplectic form ω on X be the restriction of the standard symplectic form on $(\mathbb{C}^*)^3$

$$\frac{i}{2}\Big(\frac{dx}{x} \wedge \frac{d\overline{x}}{\overline{x}} + \frac{dy}{y} \wedge \frac{d\overline{y}}{\overline{y}} + \frac{dz}{z} \wedge \frac{d\overline{z}}{\overline{z}}\Big).$$

Then, ω is exact, invariant under the diagonal $\mathbb{Z}_3$ action by the cubic root of unity and anti-invariant under complex conjugation. Together with the complex structure induced from the standard one on $(\mathbb{C}^*)^3$, ω defines a Kähler structure on X whose Kähler metric is complete and $|\nabla W|$ is bounded below outside of a compact set. Thus, parallel transport is well defined.

The Lefschetz fibration $W : X \to \mathbb{C}$ has three critical values $3, 3\zeta, 3\zeta^2$, where $\zeta = e^{2i\pi/3}$, corresponding to three critical points $(1,1,1), (\zeta,\zeta,\zeta), (\zeta^2,\zeta^2,\zeta^2)$. Away from the critical values, the fiber $\Sigma_\lambda = W^{-1}(\lambda) \subset X$ is a smooth curve defined by $xy(\lambda - x - y) = 1$, which degenerates to a singular curve as λ approaches a critical value. Our first goal is to understand the geometry of the fiber, particularly as a complex manifold, as well as its (nodal)degeneration at the critical values. Although this is not completely necessary for just proving HMS, it will be crucial to the study of special Lagrangians in X.

Recall that the *Weierstrass's elliptic function* with periods ω_1, ω_2 is defined as

$$\wp(z;\omega_1,\omega_2) = \frac{1}{z^2} + \sum_{(n,m)\neq(0,0)} \Big(\frac{1}{(z + n\omega_1 + m\omega_2)^2} - \frac{1}{(n\omega_1 + m\omega_2)^2}\Big).$$

It is a meromorphic function that establishes the elliptic curve $\mathbb{C}/\langle\omega_1,\omega_2\rangle$ as a double cover of the Riemann sphere with four branch points corresponding to $0, \omega_1/2, \omega_2/2, (\omega_1 + \omega_2)/2$. Moreover, $(\wp(z), \wp'(z))$ gives a parametrization of the elliptic curve via the famous functional equation

$$[\wp'(z)]^2 = 4[\wp(z)]^3 - g_2\wp - g_3,$$

where

$$g_2 = 60 \sum_{(n,m)\neq(0,0)} (n\omega_1 + m\omega_2)^{-4}, \qquad g_3 = 140 \sum_{(n,m)\neq(0,0)} (n\omega_1 + m\omega_2)^{-6}.$$

With this understanding, we sometimes write $\wp(z; g_2, g_3)$ instead of $\wp(z;\omega_1,\omega_2)$. The inverse of the Weierstrass elliptic function is given by

$$u = \int_y^\infty \frac{ds}{\sqrt{4s^3 - g_2 s - g_3}},$$

i.e. $y = \wp(u)$. We don't specify a path of integration, since u modulo the lattice does not depend on the choice of path (but as a complex number it does because of the branching behavior of the integrand). Finally, at the branch points of $\wp$, we have $\wp' = 0$. Hence,

$$e_1 = \wp(\frac{\omega_1}{2}) \qquad e_2 = \wp(\frac{\omega_2}{2}) \qquad e_3 = \wp(\frac{\omega_3}{2}),$$

where $\omega_3 = \omega_1 + \omega_2$, are the three complex roots of $4s^3 - g_2 s - g_3$. In particular, for suitable paths of integration

$$\frac{\omega_i}{2} = \int_{e_i}^\infty \frac{ds}{\sqrt{4s^2 - g_2 s - g_3}}.$$

Now we are ready to prove the following proposition.

Proposition 3.1.1. *For a regular value λ, the fiber Σ_λ is biholomorphic to a complex elliptic curve with three punctures. Moreover, there exists a holomoprhic volume form Ω on X whose residue along each regular fiber Σ_λ is identified with the standard flat structure under the above biholomorphism. When λ is a critical value, Σ_λ has a unique nodal singularity.*

Proof. The fiber Σ_λ can be identified with the curve $\{xy(\lambda - x - y) = 1\} \subset (\mathbb{C}^*)^2$, with a projection $\pi_x : \Sigma_\lambda \to \mathbb{C}^*$ onto the x-coordinate. Since the polynomial is quadratic in y, π_x is a double cover with three branch points given by the roots of $x(x-\lambda)^2 - 4 = 0$, corresponding to the vanishing of the discriminant of the quadratic polynomial. Under the change of coordinates

$$\begin{cases} t = \frac{1}{\sqrt[3]{4}}\left(\frac{1}{x} - \frac{\lambda^2}{12}\right) \\ s = \frac{i}{x}\left(y + \frac{x-\lambda}{2}\right) \end{cases}$$

the equation becomes $s^2 = 4t^3 - g_2(\lambda)t - g_3(\lambda)$, where

$$g_2(\lambda) = \sqrt[3]{4}\left(\frac{\lambda^4}{48} - \frac{\lambda}{2}\right) \qquad g_3(\lambda) = \frac{\lambda^6}{864} - \frac{\lambda^3}{24} + \frac{1}{4}.$$

The discriminant is given by

$$\Delta = g_2^3 - 27g_3^2 = \frac{\lambda^3 - 27}{16}.$$

Hence, away from the critical values, $\Delta \neq 0$ and we get an smooth elliptic curve. Let $\wp(z;\lambda) = \wp(z; g_2(\lambda), g_3(\lambda))$, and let $\Lambda(\lambda)$ be the associated lattice. Then, there exist biholomorphisms $\psi_\lambda : \mathbb{C}\backslash\{-\frac{\lambda^2}{12\sqrt[3]{4}}\} \to \mathbb{C}^*$ given by

$$\psi_\lambda(t) = \frac{1}{\sqrt[3]{4}t + \frac{\lambda^2}{12}}$$

and $\Psi_\lambda : \mathbb{C}/\Lambda(\lambda) - \{0\} - \{\wp^{-1}(-\frac{\lambda^2}{12\sqrt[3]{4}};\lambda)\} \to \Sigma_\lambda$ given by the composition of

$$z \mapsto (\wp(z;\lambda), \wp'(z;\lambda))$$

and

$$(t,s) \mapsto \left(\frac{1}{\sqrt[3]{4}t + \frac{\lambda^2}{12}}, \frac{-is - \frac{1}{2}}{\sqrt[3]{4}t + \frac{\lambda^2}{12}} + \frac{\lambda}{2}\right),$$

fitting into a commutative diagram

$$\begin{tikzcd}
\mathbb{C}/\Lambda(\lambda) - \{0\} - \{\wp^{-1}(-\frac{\lambda^2}{12\sqrt[3]{4}};\lambda)\} \arrow[r, "\Psi_\lambda"] \arrow[d, "\wp(-;\lambda)"] & \Sigma_\lambda \arrow[d, "\pi_x"] \\
\mathbb{C}\backslash\{-\frac{\lambda^2}{12\sqrt[3]{4}}\} \arrow[r, "\psi_\lambda"] & \mathbb{C}^*
\end{tikzcd}$$

On the other hand, $\Delta = 0$ if and only if $\lambda \in \mathrm{Crit}(\pi)$, and in that case it can be easily checked that $g_2 \neq 0$. By standard results regarding elliptic curves, Σ_λ has a nodal singularity when $\lambda \in \mathrm{Crit}(\pi)$.

Finally, let $\Omega_0 = d\log x \wedge d\log y \wedge d\log z$ be the standard holomorphic volume form on $(\mathbb{C}^*)^3$, let Ω' be the holomorphic 2-form on X obtained by taking residue of Ω_0 along X, and let Θ' be the holomorphic one form obtained by further taking residue of Ω' along a level set $W = \lambda$, i.e.

$$\Theta' \wedge d(x+y+z) \wedge d(xyz) = \Omega_0.$$

Note that Θ' is not well defined as a one form on X, but $\Theta'|_{\Sigma_\lambda}$ is well defined. By an abuse of notation, we denote Θ'_{Σ_λ} as Θ'. When $\lambda \notin \mathrm{Crit}(\pi)$ and away from the branch points of π_x, Θ' has an explicit expression $\frac{dx}{x(y-z)}$ (but keep in mind that Θ' is defined on all of Σ_λ). Let ζ denote the flat coordinate on the complex torus with punctures $\mathbb{C}/\Lambda(\lambda) - \{0\} - \{\wp^{-1}(-\frac{\lambda^2}{12\sqrt[3]{4}};\lambda)\}$. Then, via the map Ψ_λ, we have

$$d\Big(\frac{1}{x}\Big) = d(\sqrt[3]{4}t + \frac{\lambda^2}{12}) = \sqrt[3]{4}\wp'(\zeta;\lambda)d\zeta.$$

Hence,

$$\begin{aligned} d\zeta &= \frac{d\Big(\frac{1}{x}\Big)}{\sqrt[3]{4}\wp'(\zeta;\lambda)} \\ &= \frac{-\frac{1}{x^2}dx}{\sqrt[3]{4}\frac{i}{x}(y+\frac{x-\lambda}{2})} \\ &= \frac{2i}{\sqrt[3]{4}}\frac{dx}{x(y-z)}. \end{aligned}$$

Thus, the one form $\Theta = \frac{2i}{\sqrt[3]{4}}\Theta'$, which is the residue of $\Omega = \frac{2i}{\sqrt[3]{4}}\Omega'$ along Σ_λ, is identified with $d\zeta$. $\square$

3.2 Vanishing cycles, Floer complex and mirror symmetry for $\mathbb{C}P^2$

In this subsection, we fix the basepoint $\lambda_0 = 0$, and consider the distinguished basis of vanishing paths $(\gamma_0, \gamma_1, \gamma_2)$, where γ_i is the straight line path connecting 0 to the critical value $3\zeta^{-i}$, where $\zeta = e^{2i\pi/3}$. For simplicity of notation, let $L_i = \mathrm{VC}(\gamma_i)$ be the vanishing cycles in Σ_0. There is a $\mathbb{Z}_3$ action on X by diagonal multiplication by third roots of unity. Since this action preserves Σ_0 and both ω and W are equivariant with respect to it, the three Lefschetz thimbles (or vanishing cycles) are related to each other by this action. Hence, it suffices to describe L_0.

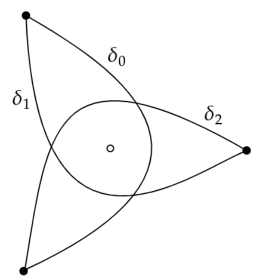

Figure 3: The arcs $\delta_i \subset \mathbb{C}^*$

At $\lambda = 0$, the three branch points of π_x are $\sqrt[3]{4}\zeta^i, i = 0, 1, 2$. The two branch points $\sqrt[3]{4}\zeta$, $\sqrt[3]{4}\zeta^2$ become closer and eventually merge as λ approaches the critical value 3 from the positive real axis. Hence the projection of L_0 under π_x is an arc δ_0 connecting the two branch points $\sqrt[3]{4}\zeta$, $\sqrt[3]{4}\zeta^2$ and intersecting the positive real axis once. $\mathbb{Z}_3$-equivariance implies that the the other two arcs are obtained by multiplying δ_0 with an appropriate cubic root of unity, which gives us the configuration as in Figure 3.

This gives us a topological description of the vanishing cycle L_0 as the union of the two lifts of δ_0 under π_x. However, since ω is exact and complex conjugate anti-invariant, the topological vanishing cycle is not only homotopic, but in fact Hamiltonian isotopic to the symplectic vanishing cycle([AKO08]), and hence is not distinguished in the Fukaya category.

In fact, we can give a more precise description of the vanishing cycles. By Proposition 3.1.1 the fiber Σ_0 is a smooth complex elliptic curve with $g_2 = 0, g_3 = 1/4$. Hence, Σ_0 is *equianharmonic*, i.e. there exists a positive real number a such that the two periods are given by $\omega_1 = a, \omega_2 = ae^{2i\pi/3}$. From here, it is not hard to see that up to Hamiltonian isotopy, the three vanishing cycles have the following configuration.

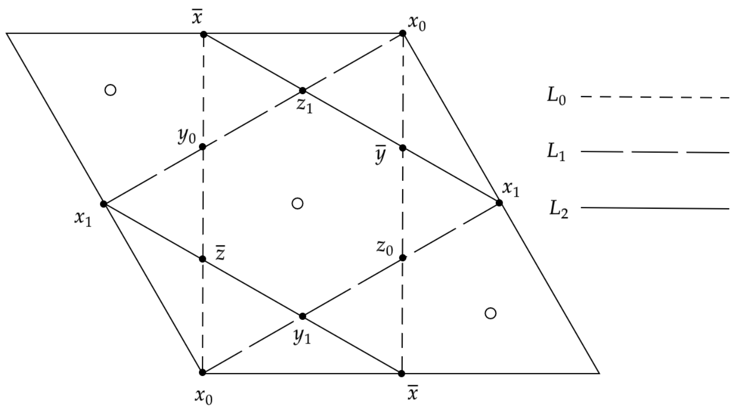

Figure 4: The vanishing cycles in Σ_0

Hence, the category $\mathrm{Lag}_{\mathrm{vc}}(\pi)$ has three objects L_0, L_1, L_2, with three pairwise intersection points. From the above figure, we observe that $\mu^1 = 0$ since there are no holomorphic strips and the only nontrivial products are

$$\mu^2(x_0, y_1) = \pm e^{-2\pi\alpha_{x_0 y_1 \bar{z}}}\bar{z} \qquad \mu^2(x_0, z_1) = \pm e^{-2\pi\alpha_{x_0 z_1 \bar{y}}}\bar{y}$$

$$\mu^2(y_0, x_1) = \pm e^{-2\pi\alpha_{y_0 x_1 \bar{z}}}\bar{z} \qquad \mu^2(y_0, z_1) = \pm e^{-2\pi\alpha_{y_0 z_1 \bar{x}}}\bar{x}$$

$$\mu^2(z_0, x_1) = \pm e^{-2\pi\alpha_{z_0 x_1 \bar{y}}}\bar{y} \qquad \mu^2(z_0, y_1) = \pm e^{-2\pi\alpha_{z_0 y_1 \bar{x}}}\bar{x},$$

where α_{abc} denotes the sympletic area of the triangle T_{abc}. By symmetry, all these triangles have identical area, and thus we get the relations

$$\mu^2(x_0, y_1) = \pm\mu^2(y_0, x_1) \quad \mu^2(x_0, z_1) = \pm\mu^2(z_0, x_1) \quad \mu^2(y_0, z_1) = \pm\mu^2(z_0, y_1).$$

Since there are only three objects, from the definition of $\mathrm{Lag}_{\mathrm{vc}}(W)$ it is automatic that $\mu^k, k \geq 3$ all vanishes.

Finally, we want to give appropriate grading to the L_i such that the morphisms are of the correct degree. Recall from Proposition 3.1.1 that there exists a holomorphic one form Θ on Σ_0 which is identified with $d\zeta$ via uniformization. Let $\phi_i = \arg(\Theta) : L_i \to \mathbb{R}/2\pi\mathbb{Z}$

be the its functions. Since each L_i has vanishing Maslov class, we can choose a real lift $\tilde{\phi}_i$ for each ϕ_i, and the degree of $p \in CF^*(L_i, L_j)$ can be defined as the ceiling function of $\frac{1}{\pi}(\tilde{\phi}_j(p) - \tilde{\phi}_i(p))$. In our case, the L_i are in fact sLag with respect to Θ, and we can easily deduce from Figure 4 that

$$\deg(x_i) = \deg(y_i) = \deg(z_i) = 1 \qquad \deg(\overline{x}) = \deg(\overline{y}) = \deg(\overline{z}) = 2.$$

Hence, as an exceptional collection in $\mathrm{FS}(W)$, $\{L_0, L_1, L_2\}$ has exactly the same relations, up to shifts, as the exceptional collection $\{\mathcal{O}(-1), \Omega^1(1), \mathcal{O}\}$ in $\mathcal{D}^b\mathrm{Coh}(\mathbb{P}^2)$. Since both collections generate, this proves a version of HMS for $\mathbb{C}P^2$.

Remark 3.2.1. Note that there is an S_3 action on X generated by $(x \mapsto x, y \mapsto z, z \mapsto y)$ and $(x \mapsto y, y \mapsto z, z \mapsto x)$. Both the superpotential $W = x + y + z$ and the symplectic form ω are invariant under this action. This implies that the fibers Σ_λ and the parallel transport vector field (associated to any curve in the base) are both invariant as well. Hence, for any vanishing path γ, its associated vanishing cycle L_γ has an induced S_3 symmetry. Moreover, if γ is a line segment on the real axis, there is an additional $\mathbb{Z}_2$ symmetry of L_λ induced by complex conjugation.

Using Proposition 3.1.1 to identify Σ_λ with a complex torus with three punctures, the S_3 action can be extended to its compactification. Hence, for instance, the action $(x \mapsto x, y \mapsto z, z \mapsto y)$ corresponds to rotation by π around the puncture $x = 0$, and $(x \mapsto y, y \mapsto z, z \mapsto x)$ corresponds to a $\mathbb{Z}_3$ translation cyclically permuting the punctures. When $\lambda \in \mathbb{R}$, the action $(x \mapsto \overline{x}, y \mapsto \overline{y}, z \mapsto \overline{z})$ corresponds to reflection along $\{\mathrm{Re}(\zeta) = 0\}$ (where ζ is the flat coordinate on the torus).

3.3 Δ_{γ_i} are (not quite) special Lagrangians

First we recall the definition of a special Lagrangian submanifold. Let (X, ω, J) be a Kähler manifold and Ω a nowhere vanishing holomorphic top form. For a Lagrangian submanifold $L \subset X$, let $\theta_L : L \to \mathbb{R}/2\pi\mathbb{Z}$ be the smooth function defined by $\theta_L = \arg(\Omega|_L)$.

Definition 3.3.1. In the above situation, we call L a *special Lagrangian* with phase ϕ if θ_L is constant with value ϕ.

In this subsection, we take Ω to be the holomorphic 2-form from Proposition 3.1.1 and consider the following construction.

For $0 \leq \lambda < 3$, let $a_1(\lambda), a_2(\lambda)$ be the two branch points of π_x that merge at λ approaches 3 from the positive real axis. Let L_λ denote the vanishing cycle $\mathrm{VC}(\gamma_0|_{[\lambda,1]}) \subset \Sigma_\lambda, 0 \leq \lambda < 3$. By the S_3 symmetry of L_λ (see Remark 3.2.1), there exists a sLag $\tilde{L}_\lambda \subset \Sigma_\lambda$ with respect to Θ in the Hamiltonian isotopy class of $L_\lambda \subset \Sigma_\lambda$, i.e. by 'straightening it out' inside Σ_λ, viewed as a complex torus with 3-punctures. The idea is to define $\tilde{\Delta}_{\gamma_0}$ as the submanifold fibered over γ_0 whose fiber over λ is $\tilde{L}_\lambda$ (the same construction can be done for the other γ_i's). However, there are several caveats towards this construction. First of all, we haven't showed that $\tilde{\Delta}_{\gamma_0}$ defined this way is smooth. Secondly, since parallel transport induced by ω doesn't necessarily map $\tilde{L}_\lambda$ into each other, even if $\tilde{\Delta}_{\gamma_0}$ is a smooth submanifold, it need not be Lagrangian with respect to ω. We will postpone the discussion of these technical details to Section 4 in a more general context. With these issues aside, I claim that $\arg\Omega$ is constant over $\tilde{\Delta}_{\gamma_0}$.

To see this, let $F \subset X$ be any submanifold fibered over some embedded path γ. For each $p \in F$, we can write the volume element of F at p as (v_1, v_2), where $v_1 \in T_pF_p \subset T_p\Sigma_{W(p)}$ and v_2 is a lift of $\dot{\gamma}|_{W(p)}$. Since

$$\Omega(v_1, v_2) = \det \begin{pmatrix} \Theta(v_1) & 0 \\ \Theta(v_2) & dW(v_2) \end{pmatrix} = \Theta(v_1) \cdot \dot{\gamma}|_{W(p)}$$

(up to a real scalar), F has constant argument with respect to Ω if and only if each fiber F_λ is sLag under $\Theta|_\lambda$ and $\arg(\Theta(F_\lambda)) + \arg(\dot{\gamma}|_\lambda)$ is constant along $\lambda \in \gamma$ (here we view γ as a submanifold of $\mathbb{C}$).

In our case, $\tilde{\Delta}_{\gamma_0}$ is fibered over the straight line segment γ_0 and its fibers $\tilde{L}_\lambda \subset \Sigma_\lambda$ are constructed to be special Lagrangians. Hence, it suffices to show that $\arg(\Theta(\tilde{L}_\lambda))$ is constant for $0 \leq \lambda < 3$. By the description of Σ_0 from the previous subsection, $\arg(\Theta(\tilde{L}_0)) = \frac{\pi}{2}$, so we need to show that all straight line vanishing cycles $\tilde{L}_\lambda, 0 \leq \lambda < 3$ are 'vertical' with respect to the fiberwise flat structure. Since $\tilde{L}_\lambda$ is the vanishing cycle connecting $\frac{\omega_1+\omega_2}{2}$ and $\frac{3}{2}\omega_1 + \omega_2$, for an appropriate choice of periods ω_1, ω_2, this is equivalent to showing that $\arg(\omega_1 + \frac{1}{2}\omega_2) = \frac{\pi}{2}$.

Recall that

$$g_2(\lambda) = \sqrt[3]{4}\Big(\frac{\lambda^4}{48} - \frac{\lambda}{2}\Big) \qquad g_3(\lambda) = \frac{\lambda^6}{864} - \frac{\lambda^3}{24} + \frac{1}{4}$$

and

$$\Delta = g_2^3 - 27g_3^2 = \frac{\lambda^3 - 27}{16}.$$

Hence, g_2, g_3 are real and $\Delta < 0$ for $0 \leq \lambda < 3$. Let e_2 be the unique real root of $4t^3 - g_2t - g_3 = 0$ when $0 \leq \lambda < 3$, then we have

$$\frac{\omega_2(\lambda)}{2} = \int_{e_2}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2t - g_3}}$$

,modulo the lattice Λ. If we take the path of integration to be from e_2 to $+\infty$ along the real axis, then the integral is real since $4t^3 - g_2t - g_3 \geq 0$ for $t \in [e_2, \infty)$. On the other hand, if the take the path of integration to be from e_2 to $-\infty$ along the real axis, then the integral is purely imaginary since $4t^3 - g_2t - g_3 \leq 0$ for $t \in (-\infty, e_2]$. Hence, we conclude that $\frac{\omega_2(\lambda)}{2}$ is congruent to both a real number and a purely imaginary number modulo $\Lambda(\lambda)$. However, at $\lambda = 0$, we have $\frac{\omega_2}{2} \in \mathbb{R}_+$ and $\frac{\omega_2}{2} + \omega_1 \in i\mathbb{R}_+$. Thus by continuity, these relations must be true for all $\lambda \in [0, 3)$ (after a consistent choice of ω_1, ω_2, of course).

4 The Landau-Ginzburg model, version 2

4.1 A full exceptional collection in $\mathrm{FS}(W)$ mirror to $\{\mathcal{O}(-1), \mathcal{O}, \mathcal{O}(1)\}$

In this subsection, we consider the Landau-Ginzburg model mirror to $\mathbb{C}P^2$, but with reference fiber $\Sigma_{+\infty}$ (see Remark 2.4.5). Here we only sketch the idea of most arguments, since their proofs are the same as in Section 3.

Let $\{\beta_{-1}, \beta_0, \beta_1\}$ be the basis of vanishing paths in the figure below and $L_{\beta_i} \subset \Sigma_{+\infty}$ be the vanishing cycles(to be inserted). Without changing the Floer cohomology, we may assume that β_i intersect at some large $c \in \mathbb{R}_{>0}$ (see the dashed curves in Figure 5) and the vanishing cycles lie in Σ_c.

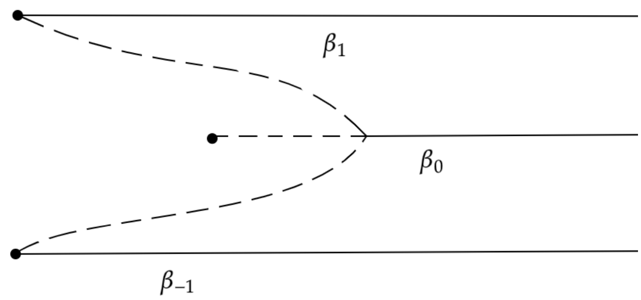

Figure 5: The new vanishing paths

The new basis of vanishing paths can be obtained from the original one by moving the basepoint from 0 to c, via a path that goes above the critical value 3, and then performing a braid (see Figure 6).

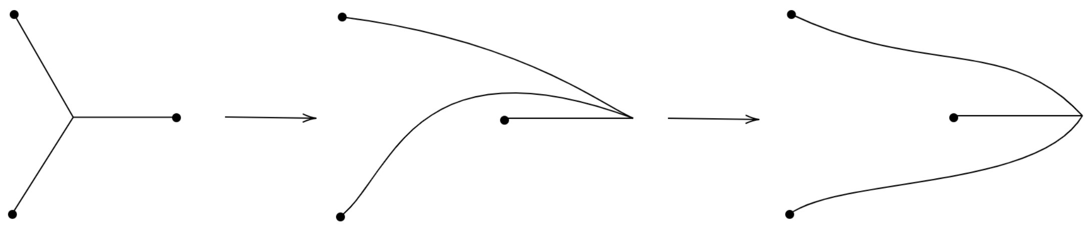

Figure 6: Three stages of vanishing paths

Therefore, the new configuration of vanishing cycles in Σ_c can be obtained from Figure 4. by first shearing the parallelogram (corresponding to stage 0→stage 1) and then performing a (left)Dehn twist about a vanishing cycle (corresponding to stage 1→stage 2) by the symplectic Picard-Lefschetz theorem. This process can be visualized as in Figure 7 and 8.

Remark 4.1.1. Since we've chosen $c \gg 3$, in particular $\Delta(c) > 0$ and thus the cubic equation $4t^3 - g_2t - g_3 = 0$ defining Σ_c has three real roots. Therefore, the inverse formula for $\wp$ implies that there is an appropriate choice of half periods such that $\frac{\omega_1}{2} \in \mathbb{R}_+, \frac{\omega_3}{2} \in i\mathbb{R}_+$. Therefore, the flat structure on Σ_c is indeed a rectangle.

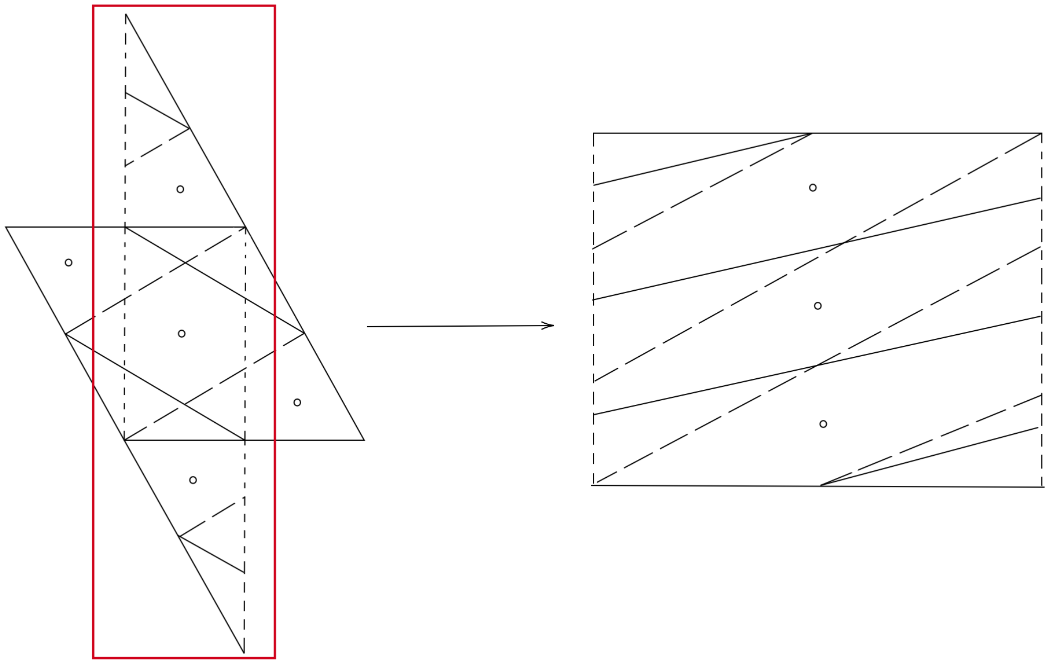

Figure 7: The corresponding vanishing cycles from stage 0 to stage 1

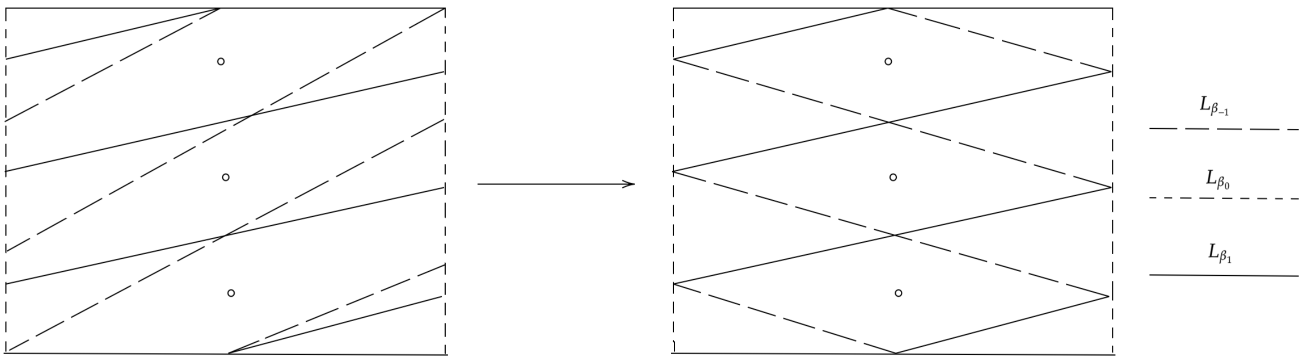

Figure 8: The corresponding vanishing cycles from stage 1 to stage 2

Now it is not hard to observe from Figure 8 that

$$\dim \mathrm{Hom}(L_{\beta_{-1}}, L_{\beta_0}) = 3 \quad \dim \mathrm{Hom}(L_{\beta_0}, L_{\beta_1}) = 3 \quad \dim \mathrm{Hom}(L_{\beta_{-1}}, L_{\beta_1}) = 6$$

and they have the same relations as the exceptional collection $\{\mathcal{O}(-1), \mathcal{O}, \mathcal{O}(1)\}$ in $\mathcal{D}^b\mathrm{Coh}(\mathbb{P}^2)$.

4.2 Manifolds fibered over paths

In contrast to the situation in Section 3.3, one can show that the Lefschetz thimble Δ_{β_0} is already special Lagrangian with respect to Ω by itself. The important observation is that Δ_{β_0} lies in the real locus of X. To see this, let $\lambda \in \beta_0$ be a regular value and consider the projection $\pi_x : \Sigma_\lambda \to \mathbb{C}^*$. The image of the vanishing cycle L_λ passes through two branch points, both of which lie on the positive real axis. However, since L_λ is symmetric under complex conjugation, we must have $\pi_x(L_\lambda) \subset \mathbb{R}^* \subset \mathbb{C}^*$, i.e., it is in fact pointwise fixed by complex conjugation. Similarly, $\pi_y(L_\lambda)$ and $\pi_z(L_\lambda)$ are also contained in the real part of $\mathbb{C}^*$. As a result, Δ_{β_0} has constant argument, either $\frac{\pi}{2}$ or $-\frac{\pi}{2}$, with respect to Ω. This argument obviously does not apply to the other two thimbles $\Delta_{\beta_1}, \Delta_{\beta_{-1}}$.

Finding sLag representatives of a given Lagrangian submanifold in its Hamiltonian isotopy class is a very hard problem in general. A relatively successful approach is Lagrangian mean curvature flow (see [Joy14]), which has the nice property that it preserves the Hamiltonian isotopy class of a Lagrangian. Even then, finite time singularities are often unavoidable and one has to perform surgeries in order to continue the flow; only a few special cases are known that the above procedure will indeed eventually converge to a sLag.

We approach this problem by constructing a submanifold $\tilde{\Delta}_\beta \subset X, \dim(\tilde{\Delta}_\beta) = 2$ with the following heuristics:
1) $\tilde{\Delta}_\beta$ is fibered over a vanishing path β that is asymptotically horizontal.
2) For some large $a \in \mathbb{R}$, $\arg e^{-aW}\Omega|_{\tilde{\Delta}_\beta}$ is constant.
3) $\tilde{\Delta}_\beta$ is 'close' to the thimble Δ_{β_1}.

Roughly speaking, preserving the fiberedness over an embedded path reduces the problem of finding sLag in the total space to calculations regarding certain vector fields in the base $\mathbb{C}$, but the tradeoff is that $\tilde{\Delta}_\beta$ will not be Lagrangian under the original symplectic form ω, but only after deforming ω to a new symplectic form $\tilde{\omega}$, we can ensure that $\tilde{\omega}|_{\tilde{\Delta}_\beta} = 0$. Unfortunately, the author has yet to rigorously prove the smoothness of $\tilde{\Delta}_\beta$ and the existence of $\tilde{\omega}$, and those issues will be addressed in the form of a conjecture.

First let's fix some notations. Recall that we have a Lefschetz fibration $W : X \to \mathbb{C}, X = \{xyz = 1\} \subset (\mathbb{C}^*)^3$ where $W = x + y + z$. Let ω be the standard symplectic form from Section 3.1 and Ω be defined as in Proposition 3.1.1. Since the Lagrangians L we consider are noncompact in the $+\infty$ direction in the base, we will use the twisted holomorphic volume form $\Omega_a = e^{-aW}\Omega$ for some positive real number a so that the integral $\int_L e^{-aW}\Omega$ converges. Integrals of this kind, sometimes called oscillatory integrals, also arise in the context of the Gamma conjecture (see [Iri19]).

Let $p_0 = 3e^{2\pi i/3}$ be one of the critical values. Let $U_0 \subset \mathbb{C}$ be an open subset containing p_0 whose boundary is the graph of a strictly decreasing convex function lying above the other two critical values. We also assume that U_0 is not contained in $\{\lambda \in \mathbb{C} | \mathrm{Im}(\lambda) > c\}$ for any $c > -\infty$ (see Figure 9).

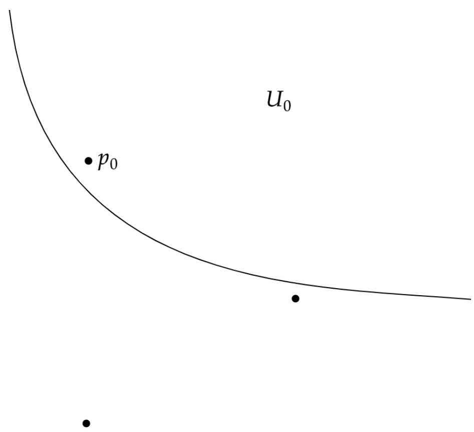

Figure 9: The region U_0

For $\lambda \in U_0\backslash\{p_0\}$, let $l_{\lambda p_0}$ be the straightline vanishing path joining λ to p_0, and consider the associated vanishing cycle $L(\lambda)$. Let $\alpha : S^1 \to U_0$ be any loop enclosing p_0 such that $\alpha(0) = \alpha(1) = \lambda$. By the symplectic Picard-Lefschetz theorem,

$$h_\alpha(L(\lambda)) \simeq \tau_{L(\lambda)}(L(\lambda)) \simeq L(\lambda).$$

In fact, the Dehn twist restricted to the zero section is just the antipodal map, which is an orientation preserving automorphism in dimension 1. Hence, it is possible to choose consistent orientation of $L(\lambda)$ as λ ranges over $U_0\backslash\{p_0\}$. After picking such a family of orientation, we obtain a well defined smooth function $P : U_0\backslash\{p_0\} \to \mathbb{C}$ given by

$$P(\lambda) = \int_{[L(\lambda)]} \Theta,$$

where $[-]$ denotes the homology class. Finally, by the S_3 symmetry it is an easy observation that the Hamiltonian isotopy class of $L(\lambda)$ has a unique sLag (with respect to Θ) representative $\tilde{L}(\lambda) \subset \Sigma_\lambda$, with an induced orientation. For a smoothly embedded path β in U_0, let $\tilde{\Delta}_\beta$ denote the topological submanifold of X fibered over β such that the fiber over $\beta(t)$ is $\tilde{L}_{\beta(t)}$.

Lemma 4.2.1. *If β is contained in the regular part of W, then $\tilde{\Delta}_\beta$ is smooth.*

Proof. This is an immediate consequence of Proposition 3.1.1. More explicitly, for each $\lambda \in \beta$, a point on $\tilde{L}_\lambda$ can be identified with $\zeta_1(\lambda) + s(\zeta_2(\lambda) - \zeta_1(\lambda)), s \in [0, 2)$ under Ψ_λ, where ζ_1, ζ_2 are suitable half periods on the punctured complex torus $\mathbb{C}/\Lambda(\lambda) - \{0\} - \{\wp^{-1}(-\frac{\lambda^2}{12\sqrt[3]{4}}; \lambda)\}$. Since, $\zeta_1, \zeta_2, \Psi_\lambda$ all depend complex analytically on λ, the coordinate chart on $\tilde{\Delta}_\beta$ given by

$$(t, s) \mapsto \Psi_{\beta(t)}(\zeta_1(\lambda) + s(\zeta_2(\lambda) - \zeta_1(\lambda)))$$

is smooth. ☐

Despite not being able to give a rigorous proof, we conjecture that if $\beta \subset U_0$ is a vanishing path with $\beta(1) = p_0$, then $\tilde{\Delta}_\beta$ is smooth at the critical point $(1, 1, 1)$.

Conjecture 4.2.2. *Let $\beta : (-\infty, 1] \to U_0$ be a smoothly embedded path with $\beta(1) = p_0$ and $\beta'_-(1) \neq 0$. Then $\tilde{\Delta}_\beta$ is a smooth submanifold of X. Moreover, there exists a neighborhood $\tilde{U} \supset \tilde{\Delta}_\beta$ in X, a smooth function $f : \tilde{U} \to \mathbb{R}$ with $f(1, 1, 1) = 0$ and a smooth vector field ξ on $\tilde{U}$ tangent to the fibers of W (besides the critical point) and with $\xi_{(1,1,1)} = 0$ satisfying the following conditions:*
For $\lambda \in \beta\backslash\{p_0\}$, let $f_\lambda = f|_{\Sigma_\lambda \cap \tilde{U}}$ and $\xi_\lambda = \xi|_{\Sigma_\lambda \cap \tilde{U}}$, then
i) ξ_λ is the Hamiltonian vector field associated to f_λ, i.e.

$$\omega|_{\Sigma_\lambda}(\xi_\lambda, -) = df_\lambda,$$

ii) for all $\sigma \in S_3$ (see Remark 3.2.1), $f_\lambda \circ \sigma = f_\lambda$. Consequently, $\sigma_* \xi_\lambda = \xi_\lambda$,
iii) let ϕ^t_λ be the time t flow of ξ_λ, then $\phi^1_\lambda(\tilde{L}_\lambda) = L_\lambda$.

In particular, if Φ^t is the time t flow of ξ, then $\Phi^1(\tilde{\Delta}_\beta) = \Delta_\beta$. But note that ξ is not necessarily the Hamiltonian vector field associated to f. Now, choose a smooth bump function $g : X \to \mathbb{R}$ satisfying $g|_V = 1, g|_{X\backslash\tilde{U}} = 0$, where V is an open set such that

$\tilde{\Delta}_\beta \subset V \subset \overline{V} \subset \tilde{U}$. This gives us a smooth vector field $g\xi$ on X, and let Φ_g^t be its time t flow. Then, $\tilde{\omega} = (\Phi_g^1)^*\omega$ is a symplectic form on X satisfying

$$\tilde{\omega}|_{\tilde{\Delta}_\beta} = 0 \quad \text{and} \quad \tilde{\omega}|_{X\backslash\tilde{U}} = \omega|_{X\backslash\tilde{U}}.$$

From now on, we will assume Conjecture 4.2.2. and view $\tilde{\Delta}_\beta$ as a Lagrangian with respect to the new symplectic form $\tilde{\omega}$.

4.3 Some preliminary calculations

The rest of this paper will be concerned with finding vanishing paths β with certain constraints such that $\arg \Omega_a|_{\tilde{\Delta}_\beta}$ is constant. We start with studying some properties of the period function $P(\lambda)$.

Proposition 4.3.1. *P defines a complex analytic function on U_0.*

Proof. Let's first show that P is complex analytic in a neighborhood of p_0. Recall that when λ is a regular value, the projection $\pi_x : \Sigma_\lambda \to \mathbb{C}^*$ defines a double cover with three branch points. As $\lambda \to p_0$, two of the branch points $a_1(\lambda), a_2(\lambda) \to e^{2\pi i/3}$ and $a_3(\lambda) \to 4e^{2\pi i/3}$. Let $U \subset U_0$ be a small neighborhood of p_0 so that for $\lambda \in U\backslash\{p_0\}$, the lift of the line segment $l_{a_1a_2}$ to Σ_λ is homologous to $\tilde{L}(\lambda)$. Then

$$\begin{aligned}
P(\lambda) &= \int_{\tilde{L}(\lambda)} \Theta \\
&= \pm 2 \int_{l_{a_1a_2}} \frac{2i}{\sqrt[3]{4}} \frac{dx}{x(y-z)} \\
&= \pm \frac{4i}{\sqrt[3]{4}} \int_{l_{a_1a_2}} \frac{dx}{\sqrt{x(x-a_1)(x-a_2)(x-a_3)}} \\
&= \pm \frac{4i}{\sqrt[3]{4}} \int_0^1 \frac{dx}{\sqrt{x(x-1)((a_2-a_1)x+a_1)((a_2-a_1)x+a_1-a_3)}},
\end{aligned}$$

where the sign depends on the chosen orientation (from now on we fix it to be $+$). Since a_1, a_2, a_3 are the three roots of $x(x-\lambda)^2 = 4$, it is immediate that $P(\lambda)$ is a complex analytic in $U\backslash\{p_0\}$. Note that although the expression we obtained above might a priori have branching behaviour near p_0, the monodromy has to be trivial: recall that $P(\lambda)$ is well defined as a smooth function in a punctured disk near p_0, which is a result of Picard-Lefschetz.

Hence, by Riemann's removable singularity theorem, it suffices to show that P has a limit as $\lambda \to p_0$. However, note that as $\lambda \to p_0$, $|a_1 - a_2| \to 0$ while $|a_1 - a_3|, |a_2 - a_3|$ are bounded below. Thus,

$$\begin{aligned}
\lim_{\lambda\to p_0} \int_{l_{a_1a_2}} \frac{dx}{\sqrt{x(x-a_1)(x-a_2)(x-a_3)}} &= \lim_{\lambda\to p_0} \frac{1}{\sqrt{a_1(a_1-a_3)}} \int_{l_{a_1a_2}} \frac{dx}{\sqrt{(x-a_1)(x-a_2)}} \\
&= \lim_{\lambda\to p_0} \frac{1}{\sqrt{a_1(a_1-a_3)}} \int_0^1 \frac{dx}{\sqrt{x(x-1)}}.
\end{aligned}$$

This limit exists and is in fact nonzero, which implies that $P|_U$ is complex analytic. Finally, it is clear from the definition that P is the analytic continuation of $P|_U$ to U_0. $\square$

Corollary 4.3.2. *$\frac{\partial \arg P(\lambda)}{\partial \mathrm{Im}(\lambda)}, \frac{\partial \arg P(\lambda)}{\partial \mathrm{Re}(\lambda)}$ are bounded in any bounded region $p_0 \in U \subset U_0$.*

Proof. This is an immediate consequence of Proposition 4.3.1 and the fact that $P(p_0) \neq 0$. □

It turns out that bounding $\frac{\partial \arg P(\lambda)}{\partial \mathrm{Re}(\lambda)}$ and $\frac{\partial \arg P(\lambda)}{\partial \mathrm{Im}(\lambda)}$ is crucial for certain constructions in later sections. The next lemma describes the behavior of these quantities as λ approaches infinity along a horizontal line.

Lemma 4.3.3. *Fix $y \in \mathbb{R}$, then*

$$\lim_{t\to+\infty} P(t+iy) = 0 \ , \quad \lim_{t\to+\infty} \frac{\mathrm{Re(P(t+iy))}}{\mathrm{Im}(P(t+iy))} = 0.$$

Moreover, $\frac{\partial \arg P(\lambda)}{\partial \mathrm{Im}(\lambda)}, \frac{\partial \arg P(\lambda)}{\partial \mathrm{Re}(\lambda)}$ are bounded over each $\{\lambda \in U_0 | \mathrm{Im}(\lambda) = y\}$.

Proof. Let a_1, a_2, a_3 be the branch points of $\pi_x : \Sigma_\lambda \to \mathbb{C}^*$; equivalently, they are the roots of the equation $x(x-\lambda)^2 = 4$. Fix $y \in \mathbb{R}$, as $\mathrm{Re}(\lambda) \gg 0$ with $\mathrm{Im}(\lambda) = y$, the three roots of the above equation can be approximated by

$$a_1 \sim \frac{4}{\lambda^2}, \ \ a_2 \sim \lambda - \frac{2}{\sqrt{\lambda}}, \ \ a_3 \sim \lambda + \frac{2}{\sqrt{\lambda}}.$$

Moreover, $\tilde{L}(\lambda)$ is homologous to the lift $\tilde{\gamma}_{a_1a_3}$ of the matching path $\gamma_{a_1a_3}$ shown in Figure 10.

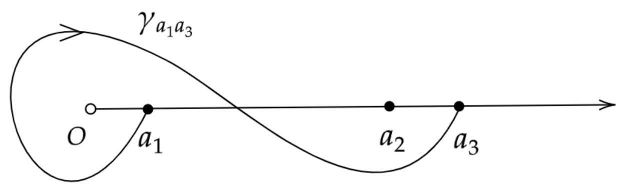

Figure 10: The matching path $\gamma_{a_1a_3}$

In the compactification $\overline{\Sigma}_\lambda$ of Σ_λ, one easily see that $\tilde{\gamma}_{a_1a_3}$ is homologous to $\tilde{l}_{a_1a_2} + 3\tilde{l}_{a_2a_3}$, where $l_{a_1a_2}$ (resp. $l_{a_2a_3}$) is the line segment joining a_1, a_2 (resp. a_2, a_3). Since $\Theta|_{\Sigma_\lambda}$ extends to a holomorphic volume form on $\overline{\Sigma}_\lambda$ (i.e. dz),

$$\begin{aligned} P(\lambda) &= \int_{\tilde{l}_{a_1a_2}} \Theta + 3\int_{\tilde{l}_{a_2a_3}} \Theta \\ &= \frac{4i}{\sqrt[3]{4}}\Big(\int_{a_1}^{a_2} \frac{dx}{\sqrt{x(x-a_1)(x-a_2)(x-a_3)}} + 3\int_{a_2}^{a_3} \frac{dx}{\sqrt{x(x-a_1)(x-a_2)(x-a_3)}}\Big). \end{aligned}$$

When $\lambda \to +\infty$ with fixed $\mathrm{Im}(\lambda)$, the argument of the first integral goes to 0 (mod 2π) and the argument of the second integral goes to $\frac{\pi}{2}$ (mod 2π). Hence, to prove the first half of the lemma, it suffices to show that both integrals tend to 0 as $\lambda \to +\infty$ with fixed

$\mathrm{Im}(\lambda)$, while the (absolute value of the) ratio of the first over the second tends to ∞. Fix $y \in \mathbb{R}$ and a horizontal strip $H_y = \{\lambda \in U_0 | y - 1.1 \le \mathrm{Im}(\lambda) \le y + 1.1\}$. Denote $f \sim g$ if there exist $0 < c_1 < c_2$ and $N > 0$ such that $c_1|f(\lambda)| < |g(\lambda)| < c_2|f(\lambda)|$ whenever $\lambda \in H_y, \mathrm{Re}(\lambda) > N$ (similarly $f = O(g)$ should be interpreted within the strip). Then,

$$\int_{a_1}^{a_2} \frac{dx}{\sqrt{x(x-a_1)(x-a_2)(x-a_3)}} = \Big(\int_{a_1}^{a_3-a_2} + \int_{a_3-a_2}^{\frac{a_3-a_1}{2}} + \int_{\frac{a_3-a_1}{2}}^{a_2-a_1} + \int_{a_2-a_1}^{a_2}\Big) \frac{dx}{\sqrt{x(x-a_1)(x-a_2)(x-a_3)}}.$$

We'll evaluate the four parts separately.

$$\begin{aligned}\int_{a_1}^{a_3-a_2} \frac{dx}{\sqrt{x(x-a_1)(x-a_2)(x-a_3)}} &\sim \frac{1}{a_2}\int_{a_1}^{a_3-a_2} \frac{dx}{\sqrt{x(x-a_1)}} \\ &= \frac{1}{a_2}\int_{1}^{\frac{2a_3-2a_2-a_1}{a_1}} \frac{dx}{\sqrt{x^2-1}} \\ &\sim \frac{1}{\lambda}\operatorname{arccosh}(2\lambda^{\frac{3}{2}}-1),\end{aligned}$$

$$\begin{aligned}\int_{a_3-a_2}^{\frac{a_3-a_1}{2}} \frac{dx}{\sqrt{x(x-a_1)(x-a_2)(x-a_3)}} &\sim \frac{1}{a_2}\int_{a_3-a_2}^{\frac{a_3-a_1}{2}} \frac{dx}{\sqrt{x(x-a_1)}} \\ &= \frac{1}{a_2}\int_{\frac{2a_3-2a_2-a_1}{a_1}}^{\frac{a_3-2a_1}{a_1}} \frac{dx}{\sqrt{x^2-1}} \\ &\sim \frac{1}{\lambda}\Big(\operatorname{arccosh}(\frac{1}{4}\lambda^3) - \operatorname{arccosh}(2\lambda^{\frac{3}{2}})\Big),\end{aligned}$$

$$\begin{aligned}\int_{\frac{a_3-a_1}{2}}^{a_2-a_1} \frac{dx}{\sqrt{x(x-a_1)(x-a_2)(x-a_3)}} &\sim \frac{1}{a_2}\int_{\frac{a_3-a_1}{2}}^{a_2-a_1} \frac{dx}{\sqrt{(x-a_2)(x-a_3)}} \\ &= \frac{1}{a_2}\int_{1+\frac{2a_1}{a_3-a_2}}^{\frac{a_2+a_1}{a_3-a_2}} \frac{dx}{\sqrt{x^2-1}} \\ &\sim \frac{1}{\lambda}\Big(\operatorname{arccosh}(\frac{1}{4}\lambda^{\frac{3}{2}}) - \operatorname{arccosh}(2\lambda^{-\frac{3}{2}}+1)\Big),\end{aligned}$$

and similarly

$$\begin{aligned}\int_{a_2-a_1}^{a_2} \frac{dx}{\sqrt{x(x-a_1)(x-a_2)(x-a_3)}} &\sim \frac{1}{a_2}\int_{a_2-a_1}^{a_2} \frac{dx}{\sqrt{(x-a_2)(x-a_3)}} \\ &= \frac{1}{a_2}\int_{1}^{1+\frac{2a_1}{a_3-a_2}} \frac{dx}{\sqrt{x^2-1}} \\ &\sim \frac{1}{\lambda}\operatorname{arccosh}(2\lambda^{-\frac{3}{2}}+1).\end{aligned}$$

We thus conclude that the original integral is $\sim \frac{1}{\lambda}\operatorname{arccosh}(\lambda^3)$. On the other hand,

$$\begin{aligned}\int_{a_2}^{a_3} \frac{dx}{\sqrt{x(x-a_1)(x-a_2)(x-a_3)}} &= \int_0^1 \frac{dx}{\sqrt{x(x-1)((a_3-a_2)x+a_2)((a_3-a_2)x+a_2-a_1)}} \\ &= O(|\frac{1}{\sqrt{a_2(a_2-a_1)}}|) \\ &= O(|\lambda|^{-1}).\end{aligned}$$

Fixing $\mathrm{Im}(\lambda) = y$, we have

$$\lim_{\lambda \to +\infty} \frac{1}{\operatorname{arccosh}(\lambda^3)} = 0$$

and

$$\lim_{\lambda \to +\infty} \frac{1}{\lambda} \operatorname{arccosh}(\lambda^3) = \lim_{\lambda \to +\infty} 3\lambda^2 \frac{1}{\sqrt{\lambda^6 - 1}} = 0.$$

This proves the first half of the lemma. To prove the second half, we first note that

$$\frac{\partial \arg P(\lambda)}{\partial \mathrm{Im}(\lambda)}, \frac{\partial \arg P(\lambda)}{\partial \mathrm{Re}(\lambda)} = O(|\frac{P'(\lambda)}{P(\lambda)}|).$$

Let $g(\lambda) = \frac{1}{\lambda} \operatorname{arccosh}(\lambda^3)$, $C(\lambda)$ the unit circle centered at λ and define $\|f\|_{C(\lambda)} = \sup\{|f(z)| z \in C(\lambda)\}$. Then for $\lambda \in U_0, \mathrm{Im}(\lambda) = y$,

$$|\frac{P'(\lambda)}{P(\lambda)}| \leq \frac{\|P\|_{C(\lambda)}}{|P(\lambda)|} \leq c \frac{\|g\|_{C(\lambda)}}{|g(\lambda)|}$$

for some constant c, where the first inequality follows from Cauchy integral formula. However, it is clear that

$$\lim_{\lambda \to +\infty, \mathrm{Im}(\lambda) = y} \frac{\|g\|_{C(\lambda)}}{|g(\lambda)|} = 1.$$

This completes the proof of the lemma. □

For real numbers $c_1 < c_2$, let $H_{[c_1,c_2]} = \{\lambda \in U_0 | c_1 \leq \mathrm{Im}(\lambda) \leq c_2\}$. The following is an immediate consequence of Corollary 4.3.2 and Lemma 4.3.3:

Corollary 4.3.4. *For fixed real numbers* $c_1 < c_2$, $\frac{\partial \arg P(\lambda)}{\partial \mathrm{Im}(\lambda)}, \frac{\partial \arg P(\lambda)}{\partial \mathrm{Re}(\lambda)}$ *are bounded in* $H_{[c_1,c_2]}$.

4.4 The vector field $\overline{V}_{a,y}(\lambda)$

Let γ be a smoothly embedded path in U_0, $\tilde{\Delta}_\gamma$ the submanifold fibered over γ defined in Subsection 4.2, and consider the twisted holomorphic volume form $\Omega_a = e^{-aW}\Omega (a > 0)$. Then, $\arg \Omega_a|_{\tilde{\Delta}_\gamma}$ is constant if and only if

$$\arg P(\gamma(t)) + \arg \dot{\gamma}(t) - a\mathrm{Im}(\gamma(t))$$

is constant as t ranges over the domain of γ. If we think of γ as a submanifold of $\mathbb{C}$ (i.e. forgetting about the parametrization), the above condition is equivalent to γ being contained in a leaf (i.e. a maximally connected integral submanifold) of (the distribution generated by) the nowhere vanishing vector field

$$\overline{V}_{a,y}(\lambda) = e^{-\frac{\pi i}{2}} e^{-a(\overline{\lambda} + iy)} \overline{P(\lambda)} \in \mathrm{Vect}(U_0)$$

for some $y \in \mathbb{R}$. By Lemma 4.3.3,

$$\lim_{\lambda \to +\infty, \mathrm{Im}(\lambda) = y} \arg \overline{V}_{a,y}(\lambda) = \pi \mod 2\pi.$$

Since the standard generators of the Fukaya-Seidel categories are (Lagrangian) thimbles fibered over vanishing paths which are asymptotically horizontal at ∞, a natural inquiry would be whether there exists $a > 0, y \in \mathbb{R}$ and some arc γ contained in a leaf of $\overline{V}_{a,y}$ such that

i) $\partial\gamma = \{p_0\} \cup \{iy' + \infty\}$ for some $y' \in \mathbb{R}$ and ii) γ is asymptotically horizontal with height y'.

This subsection will be devoted to studying part ii) of the above question.

Proposition 4.4.1. *Let $c_1 < y < c_2, a > 0$ be real numbers such that $|\frac{\partial \arg P(\lambda)}{\partial \mathrm{Im}(\lambda)}| < a$ for all $\lambda \in H_{[c_1,c_2]}$. Then, there exists a unique leaf $\hat{\gamma}_{a,y}$ of $\overline{V}_{a,y}$ such that an arc of $\hat{\gamma}_{a,y} \cap H_{[c_1,c_2]}$ is asymptotically horizontal at $+\infty$ with height y.*

Note that by Corollary 4.3.4, for any fixed $c_1 < c_2$, we can always choose $a > 0$ such that the above assumption holds.

Proof. We first prove the existence of γ. Since $|\frac{\partial \arg P(\lambda)}{\partial \mathrm{Im}(\lambda)}| < a$ in $H_{[c_1,c_2]}$ and

$$\forall r \in \mathbb{R}, \lim_{\lambda \to +\infty, \mathrm{Im}(\lambda)=r} \arg P(\lambda) = \frac{\pi}{2} \mod 2\pi,$$

we can find some small $\epsilon < \frac{\pi}{2a}$ and large N_1 such that
1) $\tan(\arg \overline{V}_{a,y})$ is uniformly bounded inside $H_{y,\epsilon} = \{\lambda \in U_0 | \, |\mathrm{Im}(\lambda) - y| \le \epsilon\}$;
2) for all $\lambda \in H_{y,\epsilon,N_1} = \{\lambda \in U_0 | \mathrm{Re}(\lambda) \ge N_1, |\mathrm{Im}(\lambda) - y| \le \epsilon\} \subset H_{y,\epsilon}$, $|\arg P(\lambda) - \frac{\pi}{2}| < \frac{a\epsilon}{2}$ (mod 2π).
For $\lambda \in H_{y,\epsilon,N_1}$, let $\hat{\gamma}_{\lambda,a,y}$ be the leaf of $\overline{V}_{a,y}$ containing λ, and $\gamma_{\lambda,a,y}$ the path component of $\hat{\gamma}_{\lambda,a,y} \cap H_{y,\epsilon,N_1} \cap \{z | \mathrm{Re}(z) \le \mathrm{Re}(\lambda)\}$ containing λ. Since $\tan(\arg \overline{V}_{a,y})$ is uniformly bounded inside H_{y,ϵ,N_1}, $\gamma_{\lambda,a,y}$ can be viewed as the graph of a function f_λ (suppressing a, y in the notation) supported on the interval $[N_1, \mathrm{Re}(\lambda)]$.

Note that if $\lambda \in H_{y,\epsilon,N_1}$ and $\mathrm{Im}(\lambda) = y \pm \epsilon$, then

$$\arg \overline{V}_{a,y}(\lambda) = -\frac{\pi}{2} + a\epsilon - \arg P(\lambda) \in (\pi \pm \frac{a\epsilon}{2}, \pi \pm \frac{3a\epsilon}{2}).$$

Since $a\epsilon < \frac{\pi}{2}$, we conclude on the boundary of H_{y,ϵ,N_1}, the vector field $\overline{V}_{a,y}$ points inward of H_{y,ϵ,N_1}. As a result, if $\lambda \in H_{y,\epsilon,N_1}$, then $\gamma_{\lambda,a,y}$ does not intersect $\{z | \mathrm{Im}(z) = y \pm \epsilon\} \cap H_{y,\epsilon,N_1}$.

By the above argument, we construct a sequence of functions $f_n = f_{iy+N_1+n}, n \ge 0$, where f_n is defined over $[N_1, N_1 + n]$. After passing to a subsequence, we may assume that the sequence is monotonic. Without loss of generality, say $f_n > f_{n+1}$, and since $|f_n - y| < \epsilon$ for all n, there exists a pointwise limit f of f_n defined over $[N_1, +\infty)$.

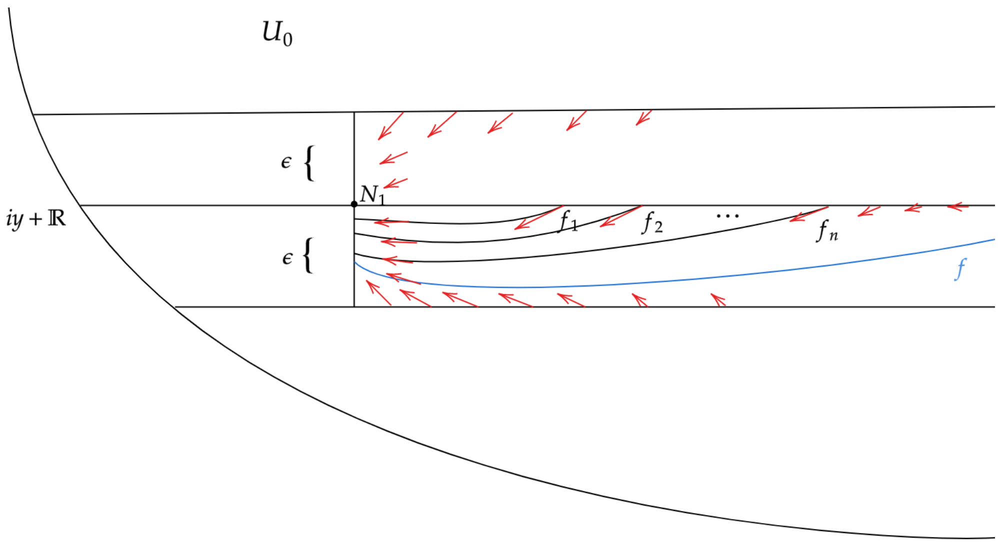

Figure 11: f as a pointwise limit of f_n

This f is clearly asymptotically horizontal with height y. It remains to show that f is a differentiable function such that $f'(t) = \tan(\arg \overline{V}_{a,y}(f(t)))$.

Consider any compact interval $[N_1, N_1 + k], k > 0$. Since $\tan(\arg \overline{V}_{a,y})$ is uniformly bounded inside H_{y,ϵ,N_1}, (in particular, the derivatives of f_n are uniformly bounded), the convergence $f_n \to f$ is uniform on $[N_1, N_1 + k]$. Hence, f is continuous on $[N_1, N_1 + k]$. Next, we show that the derivatives f'_n converges uniformly. Pointwise convergence is a direct consequence of $\overline{V}_{a,y}$ being a smooth vector field. Moreover, since f is continuous, the pointwise limit of f'_n is a continuous function given by $\tan(\arg \overline{V}_{a,y}(f(t)))$. Finally, since $f_n(t) > f_{n+1}(t)$ for all $t \in [N_1, N_1 + k]$,

$$\Big|\frac{\partial \arg P(\lambda)}{\partial \mathrm{Im}(\lambda)}\Big| < a \Rightarrow \arg \overline{V}_{a,y}(f_n(t)) > \arg \overline{V}_{a,y}(f_{n+1}(t)).$$

Equivalently, $f'_n > f'_{n+1}$ on $[N_1, N_1 + k]$. This implies that f'_n converges uniformly on $[N_1, N_1 + k]$. As a result, f is a differentiable function with

$$f'(t) = \lim_{n\to\infty} f'_n(t) = \lim_{n\to\infty} \tan(\arg \overline{V}_{a,y}(f_n(t))) = \tan(\arg \overline{V}_{a,y}(f(t)))$$

for all $t \in [N_1, N_1 + k]$. Since k is arbitrary, this completes the proof of the existence part.

To prove uniqueness, suppose γ_1, γ_2 are two distinct (hence disjoint) leaves of $\overline{V}_{a,y}$ that are asymptotically horizontal at $+\infty$ with height y. Then, restricting to a small neighborhood of $\{\lambda | \mathrm{Im}(\lambda) = y, \mathrm{Re}(\lambda) > N'\}$ for some large N', we may assume that γ_1, γ_2 are graphs of differentiable functions f_1, f_2 with $f_1 > f_2$. Since

$$\lim_{t\to\infty} f_1(t) = \lim_{t\to\infty} f_2(t) = y,$$

we can find arbitrarily large t with $f'_1(t) < f'_2(t)$. This is a contradiction since

$$\Big|\frac{\partial \arg P(\lambda)}{\partial \mathrm{Im}(\lambda)}\Big| < a \Rightarrow \arg \overline{V}_{a,y}(f_1(t)) > \arg \overline{V}_{a,y}(f_2(t))$$

for all t such that $f_1(t), f_2(t) \in H_{[c_1,c_2]}$. □

Corollary 4.4.2. *With notations and assumptions as above, there exists a unique path component $\gamma_{a,y}$ of $\hat{\gamma}_{a,y} \cap H_{[c_1,c_2]}$ such that $\gamma_{a,y}$ is the graph of a smooth function $f_{a,y}$: $[t_0, +\infty)$ for some $t_0 \in \mathbb{R}$ satisfying*

$$\lim_{t\to+\infty} f_{a,y}(t) = y.$$

Proof. Take $\gamma_{a,y}$ to be the unique path component of $\hat{\gamma}_{a,y} \cap H_{[c_1,c_2]}$ that is asymptotically horizontal at $+\infty$ with height y. To see that $\gamma_{a,y}$ is a graph, note that

$$\Big|\frac{\partial \arg P(\lambda)}{\partial \mathrm{Im}(\lambda)}\Big| < a \Rightarrow \frac{\partial \arg \overline{V}_{a,y}(\lambda)}{\partial \mathrm{Im}(\lambda)} > 0$$

for all $\lambda \in H_{[c_1,c_2]}$. This prevents $\gamma_{a,y}$ from 'backtracking' in the positive real direction and hence is a graph. □

4.5 Main result

In this subsection, we prove the main result of this paper regarding the existence of a submanifold of X fibered over an asymptotically horizontal vanishing path that has constant argument with respect to $\Omega_a = e^{-aW}\Omega$.

Theorem 4.5.1. *With notations as in previous subsections, there exists a constant $A > 0$ that satisfies the following. For all $a > A$, there exists a unique real number y_0 and vanishing path γ_{a,y_0} such that*
i) (the image of) γ_{a,y_0} is the graph of a smooth function $f_{a,y_0} : [\mathrm{Re}(p_0), +\infty) \to \mathbb{R}$ with

$$f_{a,y}(\mathrm{Re}(p_0)) = \mathrm{Im}(p_0)\,, \quad \lim_{t\to+\infty} f_{a,y_0}(t) = y_0;$$

ii) the submanifold $\tilde{\Delta}_{\gamma_{a,y_0}}$ has constant argument with respect to Ω_a.

Proof. Fix $c_1 < \frac{3\sqrt{3}}{2} < c_2$, or equivalently $p_0 \in H_{[c_1,c_2]}$, and $0 < \epsilon < \min\{\frac{\sqrt{3}}{2} - \frac{c_1}{3}, \frac{c_2}{3} - \frac{\sqrt{3}}{2}, 4\pi\}$. Find $N_1 > 0$ such that

$$\lambda \in H_{[c_1,c_2]}, \mathrm{Re}(\lambda) \geq N_1 \Rightarrow |\arg P(\lambda) - \frac{\pi}{2}| < \frac{\epsilon}{4} \mod 2\pi.$$

Finally, by Corollary 4.3.4, we can choose A (without loss of generality let $A \geq 1$) such that

$$A > \sup_{\lambda\in H_{[c_1,c_2]}} \Big|\frac{\partial \arg P(\lambda)}{\partial \mathrm{Im}(\lambda)}\Big| \qquad (*)$$

and

$$A > \frac{2}{\epsilon}\Big((N_1 - c_2') \sup_{H_{[c_1,c_2]},\mathrm{Re}(z)\leq N_1} \Big|\frac{\partial \arg P(\lambda)}{\partial \mathrm{Re}(\lambda)}\Big| + (c_2 - c_1) \sup_{H_{[c_1,c_2]},\mathrm{Re}(z)\leq N_1} \Big|\frac{\partial \arg P(\lambda)}{\partial \mathrm{Im}(\lambda)}\Big| + \pi\Big), \qquad (**)$$

where c_2' is the real coordinate of the intersection of ∂U_0 with $\{z|\mathrm{Im}(z) = c_2\}$.

Take any $a > A$. For $y \in [c_1 + \epsilon, c_2 - \epsilon]$, condition $(*)$ enables us to obtain $\gamma_{a,y}, f_{a,y}$ as defined in Corollary 4.4.2. Then, since $\Omega_a|_{\tilde{\Delta}_{\gamma_{a,y}}}$ has constant argument $-\frac{\pi}{2} - ay \mod 2\pi$, we obtain

$$\{\mathrm{Im}(z)|z \in \gamma_{a,y} \text{ and } \mathrm{Re}(z) \geq N_1\} \subset [y - \frac{\epsilon}{4a}, y + \frac{\epsilon}{4a}]$$

using an argument that appeared in the proof of Proposition 4.4.1. On the other hand, after choosing a lift of the argument function

$$\arg \overline{V}_{a,y}|_{\gamma_{a,y}\cap\{z|\mathrm{Re}(z)\leq N_1\}} : \gamma_{a,y} \cap \{z|\mathrm{Re}(z) \leq N_1\} \to S^1$$

to $\mathbb{R}$, $(**)$ implies that

$$\{\mathrm{Im}(z)|z \in \gamma_{a,y} \text{ and } \mathrm{Re}(z) \leq N_1\}$$

has diameter at most $\frac{\epsilon}{2}$. Thus, we conclude that

$$\gamma_{a,y} \subset H_{[y-\epsilon,y+\epsilon]}.$$

Finally, we observe that if $c_1 + \epsilon \leq y < y' \leq c_2 - \epsilon$, then $f_{a,y} < f_{a,y'}$. As y ranges over $[c_1 + \epsilon, c_2 - \epsilon]$, the intermediate value theorem guarantees the existence of a unique y_0 such that γ_{a,y_0} passes through p_0. Replace γ_{a,y_0} with $\gamma_{a,y_0} \cap \{z|\mathrm{Re}(z) \geq \mathrm{Re}(p_0)\}$ and we are done. □

In fact, for a given a, y_0 is explicitly given as follows. $\arg \Omega_a|_{\tilde{\Delta}_{\gamma_{a,y_0}}}$ has constant argument $-\frac{\pi}{2} - ay_0 \mod 2\pi$, which must also equal to the argument of the intergal

$$\int_{\tilde{\Delta}_{\gamma_{a,y_0}}} \Omega_a = \int_{\tilde{\Delta}_{\beta_1}} \Omega_a,$$

(recall that β_1 is the horizontal vanishing path from $+\infty$ to p_0) since $\tilde{\Delta}_{\beta_1}$ is obviously homologous to $\tilde{\Delta}_{\gamma_{a,y_0}}$.

Remark 4.5.2. To sum up, we have shown that for $a \gg 0$, there exists three vanishing paths $\beta'_{-1}, \beta'_0, \beta'_1$ asymptotically horizontal at $+\infty$ such that $\tilde{\Delta}_{\beta'_i}$ are special Lagrangians with respect to the holomorphic volume $\Omega_a = e^{-aW}\Omega$ and some symplectic form $\tilde{\omega}$ (note that the construction following Conjecture 4.4.2 can be done in disjoint neighborhoods of $\tilde{\Delta}_{\beta'_i}$, respectively). Moreover, the larger a is, the closer each β'_i is to the original horizontal vanishing path β_i, and the farther apart the phases of $\tilde{\Delta}_{\beta'_i}$ are from each other.

Having evaded the notion of stability condition so far, we will finally point out some of its connection with our previous result. However, the nature of this remark is speculative and should not be taken very seriously. To any full strong exceptional collection $\mathcal{E} = \{E_0, E_1, E_2\}$ inside $\mathcal{D}^b\mathrm{Coh}(\mathbb{C}P^2)$, we can associate a space of algebraic stability conditions $\Theta_{\mathcal{E}}$. $\Theta_{\mathcal{E}}$ is parametrized by six-tuples of real numbers $(r_0, r_1, r_2, \phi_0, \phi_1, \phi_2)$ such that $r_i > 0$ for all i,

$$\phi_0 < \phi_1 < \phi_2 \text{ and } \phi_0 + 1 < \phi_2.$$

For each such tuple, there exists a unique stability condition $\sigma = (Z, \mathcal{P})$ such that i) each E_i is stable with phase ϕ_i and ii) $Z(E_i) = r_i e^{i\pi\phi_i}$. There is a subspace $\Theta_{\mathcal{E}}^{\mathrm{Pure}}$ (see [Li17]) corresponding to those σ satisfying

$$\phi_1 - \phi_0 \geq 1 \text{ and } \phi_2 - \phi_1 \geq 1.$$

For any stability condition $\sigma \in \Theta_{\mathcal{E}}^{\mathrm{Pure}}$, the only stable objects are $E_i[n], i = 0, 1, 2, n \in \mathbb{Z}$. In our situation, if we choose $a \gg 0$, the three phases $\phi_i = -\frac{\pi}{2} - ay_i, i = -1, 0, 1$ (where y_i is the height at $+\infty$ of the corresponding vanishing path) are very far apart and thus the (conjectural) stability condition corresponding to Ω_a is pure. Hence, among arbitrary Lagrangian connected sums out of $\tilde{\Delta}_{\beta'_i}$'s, we should expect $\tilde{\Delta}_{\beta'_i}$ to be the only special Lagrangians (or stable objects). This should agree with our intuition, at least for those connected sums that are fibered over vanishing paths: after all, the vertical distance a flow line of the vector field $\overline{V}_{a,y}$ can traverse is $O(\frac{1}{a})$, for a large.

On the other hand, we should expect wall crossing phenomenon to occur as we let $a \to 0$. In particular, we should see more stable objects as $\phi_{-1}, \phi_0, \phi_1$ get closer while still having $\phi_{-1} + 1 < \phi_1$. This corresponds to the fact that as we decrease a, $\frac{\partial \arg P(\lambda)}{\partial \mathrm{Im}(\lambda)}$ would appear more significant compared to a, and thus the vector field $\overline{V}_{a,y}$ would allow flow lines that traverse greater vertical distance. Thus, we should expect the study of special Lagrangians with respect to Ω_a to be intrinsically more interesting when a is small, but also more challenging, as many of the analytic results we have obtained earlier cease to hold.

5 Appendix

5.1 Some homological algebra

We first briefly review some basics of derived functors and derived categories. For a more detailed introduction, we refer the readers to Chapter 2 and 10 of [Wei94].

Let $\mathcal{A}$ be an abelian category with enough injectives, i.e. for every object $A \in \mathcal{A}$, there exists an injective object I and a monomorphism $A \hookrightarrow I$. In particular, this implies that every object A has an injective resolution $A \hookrightarrow I^\bullet$. As an example, the abelian category $\mathrm{Coh}X$ on a projective variety X has enough injectives.

Definition 5.1.1. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between two abelian categories with $\mathcal{A}$ having enough injectives. We can define the *right derived functors* R^iF $(i \geq 0)$ as follows. For each $A \in \mathcal{A}$, choose an injective resolution $A \hookrightarrow I^\bullet$ and define

$$R^i F(A) = H^i(F(I^\bullet)).$$

Note that since $0 \to F(A) \to F(I_0) \to F(I_1)$ is exact, we always have $R^0 F \cong F$. It is an important fact that the definition of a right derived functor does not depend on the choice of the injective resolution. The proof of this is essentially the following: for any two injective resolutions of an object A, there exists a 'lift' from one to the other that is unique up to chain homotopy.

Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. We denote by $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ the homomorphism group of $\mathcal{O}_X$-modules, and $\mathscr{H}om(\mathcal{F}, \mathcal{G})$ the sheaf Hom construction. For fixed $\mathcal{F}$, $\mathrm{Hom}(\mathcal{F}, \cdot)$ is a left exact functor from $\mathrm{Mod}_{\mathcal{O}_X}$ to Ab, and $\mathscr{H}om(\mathcal{F}, \cdot)$ is a left exact functor from $\mathrm{Mod}_{\mathcal{O}_X}$ to $\mathrm{Mod}_{\mathcal{O}_X}$. Therefore, we may define their respective right derived functors as $\mathrm{Ext}^i(\mathcal{F}, \cdot)$ and $\mathscr{E}xt^i(\mathcal{F}, \cdot)$. As another example, the global section functor $\Gamma(X, \cdot)$ is left exact from $\mathrm{Mod}_{\mathcal{O}_X}$ to Ab, and its right derived functors are just the sheaf cohomologies H^i. It is a famous theorem that when X is a quasicompact and separated scheme and when $\mathcal{F} \in \mathrm{Coh}X$, the derived functor definition of sheaf cohomology agrees with Čech cohmology.

Now we describe the construction of (bounded)derived category of an abelian category $\mathcal{A}$, which consists of three stages:

1) We first consider the category of bounded cochain complexes $C^b(\mathcal{A})$, whose objects are cochain complexes $E^\bullet$ such that $H^i(E) = 0$ for all but finitely many i, and morphisms are cochain maps.

2) The homotopy category $K^b(\mathcal{A})$ is defined to have the same objects as $C^b(\mathcal{A})$, but two morphisms $f^\bullet, g^\bullet : E^\bullet \to F^\bullet$ are identified if they are homotopic, i.e. if there exists maps $h^i : E^i \to F^{i-1}$ such that $f_i - g_i = d \circ h^i - h^{i+1} \circ d$.

3) Finally, the bounded derived category $\mathcal{D}^b(\mathcal{A})$ is defined by 'inverting' all quasi-isomorphisms, i.e. chain maps that induce isomorphisms on each cohomology group. Formally, this process is called *localization* of a category, see [Wei94, Section 10.3].

In fact, $D^b(\mathcal{A})$ can be shown to be a triangulated category, equipped with the standard shift functor and whose distinguished triangles are given by the mapping cone construction.

Derived categories and derived functors, as their names suggest, are closely related. One way to motivate the derived functor construction from a derived category perspective is the following. If $F : \mathcal{A} \to \mathcal{B}$ is a functor between two abelian categoies, then F naturally extends to functors $C^b(F) : C^b(\mathcal{A}) \to C^b(\mathcal{B})$ and $K^b(F) : K^b(\mathcal{A}) \to K^b(\mathcal{B})$. The reason is that the relations defining chain complexes and homotopies are both functorial. In contrast, however, F does not itself define a functor from $\mathcal{D}^b(\mathcal{A})$ to $\mathcal{D}^b(\mathcal{B})$ unless F is exact. But is there a natural way to extend F to derived categories? The answer to this question is exactly(no pun intended) derived functors.

Before we give the construction, we need the following proposition about injective objects in an abelian category.

Proposition 5.1.2. *Let $\mathcal{A}$ be an abelian category. Then the following hold:*
1) If $A^\bullet \in \mathcal{D}^b(\mathcal{A})$ and $I^\bullet$ a bounded complex consists of injectives, then

$$\mathrm{Hom}_{\mathcal{D}^b(\mathcal{A})}(A^\bullet, I^\bullet) = \mathrm{Hom}_{K^b(\mathcal{A})}(A^\bullet, I^\bullet).$$

2) Suppose $\mathcal{A}$ has enough injectives, then

$$\mathcal{D}^b(\mathcal{A}) \cong K^b(\mathcal{I}),$$

where $K^b(\mathcal{I})$ is the full subcategory of $K^b(\mathcal{A})$ whose objects are bounded complexes consisting of injectives.

For a proof of this proposition, see [Wei94, Section 10.4]. In 2), the equivalence of categories is given by sending a complex to the total complex of its Cartan-Eilenberg resolution. Given this proposition, we can define the derived functor of $F : \mathcal{A} \to \mathcal{B}$ as the composition

$$\mathcal{D}^b(\mathcal{A}) \cong K^b(\mathcal{I}) \to K^b(\mathcal{B}) \to \mathcal{D}^b(\mathcal{B}),$$

where the middle map is $K^b(F)$, and the last map is simply passing to the localization. It can be easily verified that for $A^\bullet \in D^b(\mathcal{A})$ concentrated in degree 0, then the definition of derived functor we just gave is the same as the 'naive' definition given at the beginning of this section.

Another useful application of Proposition 5.1.2 is the following lemma.

Lemma 5.1.3. *For $A \in \mathcal{A}$, $A[i]$ denote the complex whose $(-i)$-th entry is A and zero everywhere else. Then, for any $E, F \in \mathcal{A}$, we have*

$$\mathrm{Hom}_{\mathcal{D}^b(\mathcal{A})}(E, F[i]) = \begin{cases} 0 & \textit{for } i < 0 \\ \mathrm{Ext}^i(E, F) & \textit{for } i \geq 0 \end{cases}$$

5.2 Some A_∞ homological algebra

In this subsection, we give the definition of an A_∞-category and a very informal discussion of mapping cones and twisted complexes. This will serve as an algebraic model for the Fukaya-Seidel category defined in Section 2. For a detailed introduction to A_∞-categories, we refer the readers to Chapter 1 of [Sei08].

Definition 5.2.1. Fix a coefficient field k (in this paper we always assume $k = \mathbb{C}$). A *non-unital* A_∞-category is a k-linear category $\mathcal{A}$ consisting of a set of objects $\mathrm{Ob}\mathcal{A}$, a graded vector space $\hom_{\mathcal{A}}(X_0, X_1)$ for any pair of objects, and compositions for each $d \geq 1$,

$$\mu^d : \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1) \to \hom_{\mathcal{A}}(X_0, X_d)[2-d]$$

satisfying the A_∞-relations

$$\sum_{m=1}^{d} \sum_{n=0}^{d-m} (-1)^* \mu^{d-m+1}(a_d, \cdots, a_{n+m+1}, \mu^m(a_{n+m}, \cdots, a_{n+1}), a_n, \cdots, a_1) = 0,$$

where $* = n + \deg(a_1) + \cdots + \deg(a_n)$.

In particular, $(\mu^1)^2 = 0$ and we may define the *cohomological category* $H(\mathcal{A})$ to have the same objects as $\mathcal{A}$, but with morphism spaces $H(\hom_{\mathcal{A}}(X_0, X_1), \mu^1)$, and compositions

$$[a_2] \cdot [a_1] = (-1)^{\deg(a_1)} [\mu^2(a_2, a_1)].$$

In particular, $H(\mathcal{A})$ is a linear graded category, but possibly non-unital. The definitions of A_∞-functors and natural transformations, which we omit here, can be found in [Sei08, (1b),(1d)].

In order for the HMS conjecture to make sense, we need to suitably derive the Fukaya category so that the induced triangulated structure can be identified with that on the derived category of coherent sheaves. Roughly speaking, an *exact triangle* in an A_∞-category $\mathcal{A}$ is a diagram

$$\begin{tikzcd}
X_0 \arrow[rr, "f"] & & X_1 \arrow[dl, "g"] \\
& X_2 \arrow[ul, "h", "{[1]}"'] &
\end{tikzcd}$$

where $f \in \hom^0_{\mathcal{A}}(X_0, X_1), g \in \hom^0_{\mathcal{A}}(X_1, X_2), h \in \hom^1_{\mathcal{A}}(X_2, X_0)$ and that X_2 is quasi-isomorphic to a *mapping cone* of $f : X_0 \to X_1$. We won't give the precise definition of an abstract mapping cone here, which requires introducing the A_∞-modules and the Yoneda embedding. An important thing to note is that while exact triangles in an ordinary triangulated category are additional structures, exact triangles in the A_∞-setting are already determined by the A_∞-operations (see [Sei08, Lemma 3.7]).

Nonetheless, a priori mapping cones only exist as A_∞-modules. We want to enlarge $\mathcal{A}$ to some A_∞-category $\mathrm{Tw}(\mathcal{A})$ (called its *twisted complexes*) such that $\mathrm{Tw}(\mathcal{A})$ is both closed under mapping cones and not too unreasonable to work with.

Definition 5.2.2. A *twisted complex* (E, δ^E) consists of
1) a formal direct sum $E = \bigoplus_{i=1}^N E[k_i]$ of shifted objects of $\mathcal{A}$ and 2) a strictly lower triangular differential $\delta^E \in \mathrm{End}(E)$, i.e. a collection of maps $\delta^E_{ij} \in \hom^{k_j - k_i + 1}_{\mathcal{A}}(E_i, E_j), i < j$, satisfying

$$\sum_{d \geq 1} \sum_{i = i_0 < i_1 < \cdots < i_d = j} \mu^d(\delta^E_{i_{k-1} i_k}, \cdots, \delta^E_{i_0 i_1}) = 0$$

for all $1 \leq i < j \leq N$. If $E = \bigoplus E_i[k_i]$ and $E' = \bigoplus E'_j[k'_j]$, then an element of $\hom^d(E, E')$ is given by a collection of $a_{ij} \in \hom^{k'_j - k_i + d}(E_i, E'_j)$.

Given $d \geq 1$, twisted complexes $(E_0, \delta^0), \cdots, (E_d, \delta^d)$ and morphisms $a_i \in \hom(E_{i-1}, E_i)$, we set

$$\mu^d_{\mathrm{Tw}(\mathcal{A})}(a_d, \cdots, a_1) = \sum_{j_0, \cdots, j_d \geq 0} \mu^{d + j_0 + \cdots + j_d}(\delta^d, \cdots, \delta^d, a_d, \cdots, \delta^1, \cdots, \delta^1, a_1, \delta^0, \cdots, \delta^0),$$

where each δ^i appears j_i times. This sum is finite since the δ^i's are strictly lower triangular.

The set of all twisted complexes together with the $\mu^d_{\mathrm{Tw}(\mathcal{A})}$'s turns out to be a triangulated A_∞-category such that
1) there is an embedding $\mathcal{A} \hookrightarrow \mathrm{Tw}(\mathcal{A})$;
2) $\mathrm{Tw}(\mathcal{A})$ is generated by the full subcategory $\mathcal{A}$;
3) $\mathrm{Tw}(\mathcal{A})$ contains all mapping cones.

In fact, the mapping cone is given by an explicit formula analogous to that in the usual derived category of an abelian category. Given $(E, \delta), (E', \delta') \in \mathrm{Tw}(\mathcal{A})$, and a morphism $f \in \hom(E, E')$ such that $\mu^1_{\mathrm{Tw}(\mathcal{A})}(f) = 0$, the mapping cone of f is the twisted complex

$$\mathrm{Cone}(f) = \left(E[1] \oplus E', \begin{pmatrix} \delta & 0 \\ f & \delta' \end{pmatrix} \right).$$

For proofs of the above claims, see [Sei08, Section 1.3].

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