QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday August 30, 2022 (Day 1)

1. (AG) Let V, W be complex vector spaces of dimensions $m \geq n \geq 2$, respectively. Let $\mathbb{P}\mathrm{Hom}(V, W) \cong \mathbb{P}^{mn-1}$ be the projective space of nonzero linear maps $\phi : V \to W$ modulo scalars. Further, let $\Phi \subset \mathbb{P}\mathrm{Hom}(V, W)$ be the subset of those linear maps ϕ which do not have full rank n. Prove that Φ is an irreducible subvariety of $\mathbb{P}^{mn-1}$ and find its dimension.

Solution: It's convenient to realize $\mathbb{P}\mathrm{Hom}(V, W)$ as the space of $n \times m$ matrices. If $A \in \Phi$ is a matrix of rank k, it can be written as $A = C_1 R_1$, where C_1, R_1 are matrices of sizes $n \times k$ and $k \times m$, respectively. Since $k \leq n-1$, we can also express $A = CR$, where C, R are matrices of sizes $n \times (n-1)$ and $(n-1) \times m$, respectively (add zeroes). Let $X \cong \mathbb{P}^{n(n-1)-1}$ and $Y \cong \mathbb{P}^{(n-1)m-1}$ be the projective spaces of $n \times (n-1)$ and $(n-1) \times m$ matrices. Our observation implies that the product morphism $X \times Y \to \Phi$ is surjective. The product $X \times Y$ of projective spaces is irreducible and complete, so the image Φ is an irreducible closed subset of $\mathbb{P}^{mn-1}$. It remains to find its dimension.

If A has rank $n-1$ and $A = CR = C'R'$ are two decompositions as above, then $C' = CG$ and $R' = G^{-1}R$, for some $G \in \mathrm{GL}(n-1, \mathbb{C})$. Moreover, let $\Phi' \subset \mathbb{P}^{mn-1}$ be the subset of matrices of rank $\leq n-2$; by a similar reasoning, this is closed, and clearly is a proper subset of Φ, therefore $\dim(\Phi) = \dim(\Phi \setminus \Phi')$. Take any $A \in \Phi \setminus \Phi'$, i.e. a matrix of rank $= n-1$. Its fiber under the surjective morphism $X \times Y \to \Phi$ is $\dim \mathrm{GL}(n-1, \mathbb{C}) = (n-1)^2 - 1$. Hence, $\dim(\Phi) = \dim(X \times Y) - ((n-1)^2 - 1) = [n(n-1) - 1] + [(n-1)m - 1] - [(n-1)^2 - 1] = mn - m + n - 2$.

2. (AT) Let S^n be the standard n-sphere

$$S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1\}$$

and let $S^k \subset S^n$ be the locus defined by the vanishing of the last $n-k$ coordinates $x_{k+1}, \ldots, x_n$. Assume $n - 1 > k > 0$.

1. Find the homology groups of the complement $S^n \setminus S^k$.
2. Suppose now that $T \subset S^n$ is the sphere defined by the vanishing of the first k coordinates; that is,

$$T = \{(0, \ldots 0, x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1\}.$$

What is the fundamental class of T in the homology group $H_{n-k-1}(S^n \setminus S^k)$?

Solution. The complement $S^n \setminus S^k$ is the same as the complement $\mathbb{R}^n \setminus \mathbb{R}^k \cong \mathbb{R}^k \times (\mathbb{R}^{n-k} \setminus \{0\})$, so this has the homotopy type of S^{n-k-1}; accordingly we have

$$H_m(S^n \setminus S^k) \cong \begin{cases} \mathbb{Z}, & \text{if } m = 0 \text{ or } m = n-k-1; \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, under the contraction of $S^n \setminus S^k$ to S^{n-k-1} the sphere T is carried isomorphically to S^{n-k-1}; so its fundamental class is the generator of $H_{n-k-1}(S^n \setminus S^k)$.

3. (CA) Compute

$$\int_0^{2\pi} \frac{1}{(3+\cos\theta)^2}\, d\theta$$

using contour integration.

Solution: Making the substitution $z = e^{i\theta}$, so that $\cos\theta = \frac{z+z^{-1}}{2}$, we obtain

$$\begin{aligned} \int_0^{2\pi} \frac{1}{(3+\cos\theta)^2}\, d\theta &= \int_{|z|=1} \frac{1}{(3+\frac{z+z^{-1}}{2})^2} \frac{dz}{iz} \\ &= \frac{4}{i} \int_{|z|=1} \frac{z}{(z^2+6z+1)^2}\, dz. \end{aligned}$$

The polynomial z^2+6z+1 has simple roots $-3\pm2\sqrt{2}$. Only the root $-3+2\sqrt{2}$ lies in the unit disk, so the residue theorem implies that

$$\begin{aligned} \int_0^{2\pi} \frac{1}{(3+\cos\theta)^2}\, d\theta &= 8\pi \cdot \operatorname{res}_{z=-3+2\sqrt{2}} \left(\frac{z}{(z^2+6z+1)^2} \right) \\ &= 8\pi \lim_{z\to-3+2\sqrt{2}} \frac{\partial}{\partial z} \frac{z}{(z+3+2\sqrt{2})^2} \\ &= 8\pi \lim_{z\to-3+2\sqrt{2}} \left(\frac{1}{(z+3+2\sqrt{2})^2} - \frac{2z}{(z+3+2\sqrt{2})^3} \right) \\ &= 8\pi \frac{2\cdot 3}{(4\sqrt{2})^3} = \frac{3\sqrt{2}\pi}{16}. \end{aligned}$$

4. (A) Show that the symmetric group S_n has at least one Sylow p-subgroup which is a cyclic group of order p. (You may use the fact that for any prime p, there exists a prime in the interval $(p, 2p)$.)

Solution: By the fact in parentheses, if we let p be the largest prime $\leq n$, then $2p$ is greater than n. In other words, we have $\frac{n}{2} < p \leq n$. This ensures that $p \mid n!$ but $p^2 \nmid n!$, so we conclude that any p-Sylow subgroup must have order p and is therefore a cyclic group.

5. (DG) Let $X = T^*\mathbb{C}^\times = \mathbb{C}^\times \times \mathbb{C}$, where we write z, w for holomorphic coordinates on the base and fiber, respectively. Find all time-1 periodic orbits of the vector field $V = \mathrm{Re}(zw\frac{\partial}{\partial z})$ – i.e., all points $x \in X$ such that the time-1 flow of x under V is equal to x.

Solution: Let q be the locally-defined coordinate on the base given by $q = \log(z)$, so that we can rewrite the vector field as $\mathrm{Re}(w\frac{\partial}{\partial q})$. This vector field preserves the fibers of the map $T^*\mathbb{C}^\times \to \mathbb{C}_w$, so we may study each fiber individually. Write $q = \xi + i\theta$. If w has nonzero real component, then the vector field will have nontrivial $\frac{\partial}{\partial \xi}$ component, which acts as a translation and cannot have periodic orbits. On the other hand, if $w = ic \in i\mathbb{R}$, then the vector field is $-c\frac{\partial}{\partial \theta}$. In each such cylinder fiber $\{w = ic\}$, the vector field is a rotation of the cylinder, and the points in the fiber will return to themselves precisely if they complete an integral number of rotations.

Therefore, up to a normalization, the time-1 orbits of the vector field are precisely all the points in the fibers $\{z \in \mathbb{C}^\times, w \in 2\pi i\mathbb{Z}\}$.

6. (RA) Let $X_1, X_2, X_3, \ldots$ be independent and identically distributed random variables with finite expected value μ and finite nonzero variance. Let

$$\overline{X_n} = \frac{1}{n}(X_1 + \cdots + X_n).$$

Use Chebyshev's inequality to prove that $\overline{X_n}$ converges to μ in probability as $n \to \infty$.

Solution. Let the variance of X_i be σ^2. Then the expected value of $\overline{X_n}$ is is μ and the variance of $\overline{X_n}$ is $\frac{\sigma^2}{n}$. By Chebyshev's inequality

$$P\left(\left|\overline{X_n} - \mu\right| \geq \varepsilon\right) \leq \frac{\sigma^2}{n\varepsilon^2}$$

for any $\epsilon > 0$, which implies that $\overline{X_n}$ converges to μ in probability as $n \to \infty$.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Wednesday August 31, 2022 (Day 2)

1. (CA) Let $\Omega \subset \mathbb{C}$ denote the open set

$$\Omega = \{z : |z-1| > 1 \text{ and } |z-3| < 3\}.$$

Give a conformal isomorphism between Ω and the unit disk $D = \{z : |z| < 1\}$.

Solution: The conformal transformation $z \mapsto \frac{1}{z}$ sends Ω to the open strip $S = \{z : \frac{1}{6} < \mathrm{Re}(z) < \frac{1}{2}\}$. The conformal map $z \mapsto e^{3\pi i(z-\frac{1}{6})}$ transforms the strip S to the upper half-plane, and $z \mapsto \frac{z-i}{z+i}$ sends the upper half plane to the open unit disk.

Composing these, we obtain the desired conformal isomorphism

$$z \mapsto \frac{e^{3\pi i(\frac{1}{z}-\frac{1}{6})} - i}{e^{3\pi i(\frac{1}{z}-\frac{1}{6})} + i}.$$

2. (A) Let $F = \mathbb{Q}(z)$ where

$$z = \cos\frac{2\pi}{13} + \cos\frac{10\pi}{13}.$$

i) Prove that $[F : \mathbb{Q}] = 3$ and $F/\mathbb{Q}$ is a Galois extension.

ii) Prove that if p is a prime and $p \neq 13$ then p is unramified in F, and that p is split in F if and only if $p \equiv \pm 1$ or $\pm 5 \bmod 13$.

Solution. i) Let w be the 13th root of unity $\exp(2\pi i/13)$, so

$$2z = w + w^{-1} + w^5 + w^{-5};$$

and let $K = \mathbb{Q}(w)$. Then $[K : \mathbb{Q}]$ is the 13th cyclotomic extension, so it is Galois and we can identify $\mathrm{Gal}(K/\mathbb{Q})$ with the group $G = (\mathbb{Z}/13\mathbb{Z})^*$ so that each $c \in (\mathbb{Z}/13\mathbb{Z})^*$ acts by $w \mapsto w^c$. Now $H := \{\pm 1, \pm 5\} \subset G$ is a subgroup (note that $5^2 = -1$ in $\mathbb{Z}/13\mathbb{Z}$), so $c \in G$ fixes z if and only if $c \in H$ [this uses the fact that the minimal polynomial of z is $(z^{13}-1)/(z-1)$], whence F is the subfield of K fixed by H. Thus by a fundamental theorem of Galois theory $F/\mathbb{Q}$ is Galois with group G/H, which proves the claim because $[G : H] = 3$.

ii) If $p \neq 13$ then p is unramified in K (the discriminant of the minimal polynomial of w is ± 1 times a power of 13 — in fact it is 13^{11}). Then p splits in F if and only if the p-Frobenius element of $\mathrm{Gal}(K/\mathbb{Q})$ is in H. But this Frobenius element is identified with the residue of $p \bmod 13$ under our identification of $\mathrm{Gal}(K/\mathbb{Q})$ with $G = (\mathbb{Z}/13\mathbb{Z})^*$, because w goes to w^p. Thus p splits if and only if $p \bmod 13 \in H = \{\pm 1, \pm 5\}$, QED.

3. (DG) Let $u \mapsto \tau(u)$, for $a < u < b$, be a smooth space curve in $\mathbb{R}^3$ with both its curvature and torsion nowhere zero. Assume that the parameter u is the arc-length of $u \mapsto \tau(u)$. Suppose $\sigma(v)$, for $c < v < d$, is a smooth function with $\sigma'(v)$ nowhere zero. Consider the surface S defined by

$$(u, v) \mapsto \vec{r}(u, v) = \tau(u) + \sigma(v)\tau'(u)$$

for $a < u < b$ and $c < v < d$. Compute the first and second fundamental forms of S in terms of $\tau(u)$ and $\sigma(v)$ and their derivatives. Determine the condition on the function $\sigma(v)$ so that the Gaussian curvature of the surface S is identically zero.

Solution. Since the parameter u is the arc-length of $u \mapsto \tau(u)$, the length of $\tau'(u)$ is identically 1 so that $\tau''(u)$ is perpendicular to $\tau'(u)$. The first fundamental form $I = Edu^2 + 2Fdudv + Gdv^2$ is given by

$$\begin{aligned} E &= \vec{r}_u \cdot \vec{r}_u = \big(\tau'(u) + \sigma(v)\tau''(u)\big) \cdot \big(\tau'(u) + \sigma(v)\tau''(u)\big) = 1 + \sigma(v)^2 \left(\tau''(u) \cdot \tau''(u)\right) \\ F &= \vec{r}_u \cdot \vec{r}_v = \big(\tau'(u) + \sigma(v)\tau''(u)\big) \cdot \sigma'(v)\tau'(u) = \sigma'(v) \\ G &= \vec{r}_v \cdot \vec{r}_v = \sigma'(v)\tau'(u) \cdot \sigma'(v)\tau'(u) = \sigma'(v)^2 \end{aligned}$$

so that

$$DG - F^2 = \sigma(v)^2\sigma'(v)^2 \left(\tau''(u) \cdot \tau''(u)\right).$$

To compute the unit normal vector $\vec{n}$, we compute

$$\vec{r}_u \times \vec{r}_v = \big(\tau'(u) + \sigma(v)\tau''(u)\big) \times \sigma'(v)\tau'(u) = -\sigma(v)\sigma'(v)\tau'(u) \times \tau''(u)$$

to form

$$\vec{n} = -\tau'(u) \times \frac{\tau''(u)}{\|\tau''(u)\|}$$

because $\|\vec{r}_u \times \vec{r}_v\| = \sqrt{EG - F^2}$. It means that

$$\vec{n},\ \tau'(u),\ \frac{\tau''(u)}{\|\tau''(u)\|}$$

form an orthonormal frame. To obtain the coefficients L, M, N of the second fundamental form $II = Ldu^2 + 2Mdudv + Ndv^2$, we compute the partial derivatives of the radius vector $\vec{r}$ of order 2,

$$\begin{aligned} \vec{r}_{uu} &= \tau''(u) + \sigma(v)\tau'''(u), \\ \vec{r}_{uv} &= \sigma'(v)\tau''(u), \\ \vec{r}_{vv} &= \sigma''(v)\tau'(u), \end{aligned}$$

so that

$$\begin{aligned}
L = \vec{r}_{uu}\cdot\vec{n} &= \big(\tau''(u)+\sigma(v)\tau'''(u)\big)\cdot\vec{n} \\
&= -\frac{\sigma(v)(\tau''(u)\times\tau'''(u)}{||\tau''||} \\
M = \vec{r}_{uu}\cdot\vec{n} &= \big(\sigma'(v)\tau''(u)\big)\cdot\vec{n} = 0, \\
N = \vec{r}_{uu}\cdot\vec{n} &= \big(\sigma''(v)\tau'(u)\big)\cdot\vec{n} = 0.
\end{aligned}$$

The Gaussian curvature

$$\frac{LN-M^2}{EG-F^2}$$

is always zero for any function $\sigma(v)$.

Remark. With use of the more general function $\sigma(v)$ instead of $\sigma(v) = v$, this problem is simply the statement, in disguise, that the tangent developable surface has zero Gaussian curvature.

4. (RA) Let V be the vector space of continuous functions $[0,1]\to\mathbb{R}$, and let $g : V \to \mathbb{R}$ be the linear functional $f \mapsto \int_0^1 x^{-1/3} f(x)\,dx$. For which $p \in (1,\infty)$ does g extend to a continuous functional $\overline{g} : L^p([0,1]) \to \mathbb{R}$? For those p, what is the norm of this functional?

Solution. Let $q = p/(p-1)$, so $1/p + 1/q = 1$. Using the isometric identification of $L^q([0,1])$ with the dual of $L^p([0,1])$, we see that there is such a continuous functional $\overline{g}$ if and only if $x^{-1/3}$ is an L^q function, in which case $\|\overline{g}\|$ is the L^q norm of that function. Now $x^{-1/3}$ is in $L^q([0,1]) \Leftrightarrow \int_0^1 x^{-q/3}\,dx < \infty \Leftrightarrow -q/3 > -1 \Leftrightarrow q < 3 \Leftrightarrow p > 3/2$. For such p, the the L^q norm of $x^{-1/3}$ is

$$\left(\int_0^1 x^{-q/3}\,dx\right)^{1/q} = \left(\frac{3}{3-q}\right)^{1/q}.$$

[We do not require the rewriting of $3/(3-q)$ as $(3p-3)/(2p-3)$, or of $1/q$ as $(p-1)/p$.]

5. (AG) Let $X \subset \mathbb{P}^n$ be any hypersurface of degree $d \geq 2$, and $\Lambda \subset X \subset \mathbb{P}^n$ a k-plane in $\mathbb{P}^n$ contained in X.

1. Show that if $k \geq n/2$, then X is necessarily singular.
2. If $k = n/2$ and $X \subset \mathbb{P}^n$ is a general hypersurface containing a k-plane, describe the singular locus of X.

Solution. For the first, suppose that $\Lambda = \{[X_0, \ldots, X_k, 0, \ldots, 0]\}$ is defined by the vanishing of the last $n-k$ coordinates; suppose that X is the zero locus of the homogeneous polynomial $F(X_0, \ldots, X_n)$. At any point of Λ, we have

$$\frac{\partial F}{\partial X_i} = 0 \quad \text{for} \quad i = 1, \ldots, k$$

and since $k \geq n-k$, by Bezout the remaining partial derivatives $\frac{\partial F}{\partial X_i}$ with $i = k+1, \ldots, n$ must have a common zero in Λ; thus X is singular. This also answers the second question: if X is general, then since the partial derivatives $\frac{\partial F}{\partial X_i}$ are general polynomials of degree $d-1$ on $\mathbb{P}^k$, X will have $(d-1)^k$ singular points.

6. (AT)

(a) Given compact oriented manifolds M and N, both of dimension n, define the degree of a continuous map $f : M \to N$.

(b) What are the possible degrees of continuous maps $\mathbb{CP}^4 \to \mathbb{CP}^4$? Justify your answer.

Solution:

(a) The orientations determine isomorphisms $H^n(M;\mathbb{Z}) \cong \mathbb{Z}$ and $H^n(N;\mathbb{Z}) \cong \mathbb{Z}$, and the degree is given by the image of $1 \in \mathbb{Z}$ under the map $\mathbb{Z} \cong H^n(N;\mathbb{Z}) \xrightarrow{f^*} H^n(M;\mathbb{Z}) \cong \mathbb{Z}$.

(b) The degree may be any integer of the form λ^4 for $\lambda \in \mathbb{Z}$. The cohomology ring of $\mathbb{CP}^4$ is given by $H^*(\mathbb{CP}^4;\mathbb{Z}) \cong \mathbb{Z}[x]/(x^5)$ with x in degree 2. Given a continuous map $f : \mathbb{CP}^4 \to \mathbb{CP}^4$, let $\lambda \in \mathbb{Z}$ be such that $f^*(x) = \lambda x$. Then we must have $f^*(x^4) = f^*(x)^4 = \lambda^4 x^4$. This implies that the degree of such a map must be of the form λ^4.

To show that every integer of the form λ^4 is the degree of a continuous map $f : \mathbb{CP}^4 \to \mathbb{CP}^4$, we begin by noting that λ may be assumed nonnegative without loss of generality. Then the map

$$[Z_1 : Z_2 : Z_3 : Z_4 : Z_5] \mapsto [Z_1^\lambda : Z_2^\lambda : Z_3^\lambda : Z_4^\lambda : Z_5^\lambda]$$

does the job.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday September 1, 2022 (Day 3)

1. (DG) Let X be a compact Riemannian manifold.

(a) Let ξ_i be a smooth 1-form on X which is both d-closed and d^*-closed. Let Δ denote the Laplacian. Denote by $|\xi|$ the pointwise norm of ξ. Denote by $|\nabla\xi|$ the pointwise norm of the covariant differential $\nabla\xi$ of ξ. Use the notation Ricci for the Ricci tensor of X. Prove the following identity of Bochner on X

$$\frac{1}{2}\Delta\left(|\xi|^2\right) = |\nabla\xi|^2 + \mathrm{Ricci}(\xi,\xi)$$

by directly computing $\Delta\left(|\xi|^2\right)$ and appropriately contracting the commutation formula for $\nabla_\alpha\nabla_\beta\xi - \nabla_\alpha\nabla_\beta\xi$ with ξ to yield the Ricci term.

(b) Assume that the Ricci curvature is positive semidefinite everywhere on X and is strictly positive at at least one point of X. By integrating Bochner's identity in (**a**) over X to prove that every harmonic 1-form on X must be identically zero. Here harmonic means d-closed and d^*-closed.

Solution. **(a)** We use the convention of summing over an index which occurs both as a subscript and a superscript. Let g_{ij} be the Riemannian metric of X with Riemannian curvature tensor R^ℓ_{ijk} and Ricci curvature

$$R_{ij} = R^i_{ijk}.$$

Direct computation of the Laplacian of $|\xi|^2$ yields

$$\frac{1}{2}\Delta(|\xi|^2) = \frac{1}{2}g^{rs}\nabla_s\nabla_r(g^{ab}\xi_a\xi_b) = g^{rs}g^{ab}\nabla_r\xi_a\nabla_s\xi_b + g^{rs}g^{ab}(\nabla_s\nabla_r\xi_a)\xi_b.$$

(The factor $\frac{1}{2}$ occurs from differentiating a quadratic expression.) Contracting the formula of commutation of covariant differentiation

$$\nabla_k\nabla_j\xi_i - \nabla_j\nabla_k\xi_i = \xi_\ell R^\ell_{ijk}$$

with $g^{ij}g^{km}\xi_m$ yields

$$(*)\qquad g^{ij}g^{km}\xi_m\nabla_k\nabla_j\xi_i - g^{ij}g^{km}\xi_m\nabla_j\nabla_k\xi_i = g^{ij}g^{km}\xi_m\xi_\ell R^\ell_{ijk} = R^{\ell m}\xi_\ell\xi_m.$$

Use $\nabla_j \xi_i = \nabla_i \xi_j$ (from $d\xi = 0$) to change the first term of $(*)$ to

$$g^{ij} g^{km} \xi_m \nabla_k \nabla_i \xi_j = g^{km} \xi_m \nabla_k \left(g^{ij} \nabla_i \xi_j \right) = 0$$

where $g^{ij} \nabla_i \xi_j = 0$ comes from $d^* \xi = 0$. Again, use $\nabla_j \xi_i = \nabla_i \xi_j$ (from $d\xi = 0$) to change the second term of $(*)$ to

$$g^{ij} g^{km} \xi_m \nabla_j \nabla_k \xi_i = g^{ij} g^{km} \xi_m \nabla_j \nabla_i \xi_k$$

which with a change of indices becomes $g^{rs} g^{ab} (\nabla_s \nabla_r \xi_a) \xi_b$ so that we can conclude that

$$\frac{1}{2} \Delta(|\xi|^2) = g^{rs} g^{ab} \nabla_r \xi_a \nabla_s \xi_b - R^{\ell m} \xi_\ell \xi_m.$$

Thus,

$$\frac{1}{2} \Delta \left(|\xi|^2 \right) = |\nabla \xi|^2 + \mathrm{Ricci}(\xi, \xi)$$

if $d\xi = 0$ and $d^* \xi = 0$. (The sign and notation convention for the components of the curvature tensor follows Bochner's original choice.)

(b) Integrating over X yields

$$\int_X |\nabla \xi|^2 + \int_X \mathrm{Ricci}(\xi, \xi) = 0.$$

If the Ricci curvature is positive semidefinite everywhere, then $\nabla \xi = 0$ and ξ is parallel. If the Ricci curvature is strictly positive at some point, ξ has to vanish at that point and the parallel property of ξ implies that ξ is identically zero.

2. (RA) Suppose $w : [0, 1] \to (0, \infty)$ is a continuous function.

i) Prove that there exist unique monic polynomials $p_0, p_1, p_2, \ldots \in \mathbb{R}[x]$ such that each p_n has degree n and $\int_0^1 w(x)\, p_m(x)\, p_n(x)\, dx = 0$ for all $m, n \geq 0$ such that $m \neq n$.

ii) Prove that for each $n > 0$ the four polynomials $p_{n-1}, p_n, x p_n, p_{n+1}$ are linearly dependent.

Solution. i) Since w is a continuous real-valued function on a compact set, w attains its infimum; since w takes values in $(0, \infty)$, this infimum is positive. In particular it follows that $(p, q) := \int_0^1 w(x)\, p(x)\, q(x)\, dx$ defines an inner product on $\mathbb{R}[x]$. We can now argue by induction. Base case: p_0 must be 1. For $n > 0$, assume we have proven existence and uniqueness of p_m for $0 \leq m < n$. These are linearly independent (the coefficient matrix is triangular with 1's on the diagonal), and thus span the n-dimensional vector space $\mathcal{P}_{n-1}$ of polynomials of degree at most $n-1$. Therefore the condition that $(p_m, p_n) = 0$ for each $m < n$ means p_n is the orthogonal complement of $\mathcal{P}_{n-1}$ in $\mathcal{P}_n$. This complement has dimension $(n+1) - n = 1$, and intersects $\mathcal{P}_{n-1}$ trivially (if p is in the intersection then $(p, p) = 0$ so $p = 0$), so its nonzero elements have nonzero x^n coefficients. There is thus a unique choice of p_n for which that coefficient is 1.

ii) Since $\{p_m : m \leq n+1\}$ is a basis for $\mathcal{P}_{n+1}$, we can write $xp_n = \sum_{m=0}^{n+1} a_m p_m$. We claim $a_m = 0$ for $m < n-1$. Indeed each a_m is determined by $a_m(p_m, p_m) = (xp_n, p_m)$, but $(xp_n, p_m) = (p_n, xp_m)$ which is a positive multiple of the coefficient of p_n in the expansion of xp_m with respect to the orthogonal basis $p_0, p_1, p_2, \ldots$ for $\mathbb{R}[x]$. If $m < n-1$ then $\deg xp_m < n$ so p_n does not occur in this expansion, **QED**.

3. (AG) Let $\Gamma \subset \mathbb{P}^n$ be any closed algebraic variety.

1. Define the *Hilbert function* $h_\Gamma(m)$.
2. If $\Gamma = D \cap E \subset \mathbb{P}^2$ is the transverse intersection of plane curves D, E of degrees d and e, what is the Hilbert function of Γ?

Solution. For the first part, the Hilbert function $h_\Gamma(m)$ is defined to be the codimension, in the space S_m of homogeneous polynomials of degree m on $\mathbb{P}^n$, of the mth graded piece $I(\Gamma)_m$ of the homogeneous ideal $I(\Gamma)$.

For the second, if D and E are the curves given by homogeneous polynomials F and G, then the homogeneous ideal $I(\Gamma)$ is generated by F and G; that is, we have a surjective map

$$S_{m-d} \oplus S_{m-e} \xrightarrow{(F,G)} I(\Gamma)_m.$$

The kernel of this map, moreover, is simply the image of the inclusion $S_{m-d-e} \hookrightarrow S_{m-d} \oplus S_{m-e}$ given by sending $A \in S_{m-d-e}$ to $(GA, -FA)$. Counting dimensions, we have

$$h_\Gamma(m) = \binom{m+2}{2} - \binom{m-d+2}{2} - \binom{m-e+2}{2} + \binom{m-d-e+2}{2}$$

(note that this is valid for all m, if we adopt the convention that the binomial coefficient $\binom{a}{b}$ is 0 when $a < b$).

4. (AT) Let $G = \mathbb{Z}/m$ denote a finite cyclic group of **odd** order m. Suppose that we are given a free action of G on S^3. Compute the homology groups with integer coefficients of of the orbit space $M = S^3/G$.

Solution: By definition, M is a compact connected smooth manifold of dimension 3, and since S^3 is simply connected $\pi_1 M \cong G = \mathbb{Z}/m$. It follows that $H_0(M;\mathbb{Z}) \cong \mathbb{Z}$ and $H_1(M;\mathbb{Z}) \cong (\pi_1 M)_{\mathrm{ab}} \cong \mathbb{Z}/m$.

Since m is odd, all maps $\pi_1 M \cong \mathbb{Z}/m \to \mathbb{Z}/2$ are zero, so that M must be orientable. It follows from Poincaré duality that $H_2(M;\mathbb{Z}) \cong H^1(M;\mathbb{Z}) \cong \mathrm{Hom}(\pi_1 M, \mathbb{Z}) = 0$ and that $H_3(M;\mathbb{Z}) \cong H^0(M;\mathbb{Z}) \cong \mathbb{Z}$. Finally, the homology groups in degrees ≥ 4 vanish for dimension reasons.

5. (CA) Let $f(z)$ be an entire function. Assume that for any $z_0 \in \mathbb{R}$, at least one coefficient in the analytic expansion $f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$ around z_0 is equal to zero, i.e. $c_n = 0$, for some $n \in \mathbb{Z}_{\geq 0}$. Prove that f is a polynomial.

Solution: By contradiction, assume f is not a polynomial. Observe that the set of roots of any nonzero entire function is countable. Indeed, this follows because the number of roots inside any compact subset of $\mathbb{C}$ is finite (otherwise the roots would accumulate at some point, implying that the entire function is identically zero). Therefore the set Z_0 of zeroes of f is countable. Moreover, since $f(z)$ is not a polynomial, then all higher derivatives $f^{(n)}(z)$ are nonzero and entire. Then the set Z_n of zeroes of $f^{(n)}$ is also countable, for any $n \in \mathbb{Z}_{\geq 1}$. It follows that $\bigcup_{n\geq 0} Z_n$ is countable.

But by assumption, for any $z_0 \in \mathbb{R}$, there exists some $n \in \mathbb{Z}_{\geq 0}$ such that $f^{(n)}(z_0) = n! \cdot c_n = 0$. This implies that $\mathbb{R} \subset \bigcup_{n\geq 0} Z_n$, and so $\bigcup_{n\geq 0} Z_n$ is uncountable. We have reached a contradiction, as needed.

6. (A) Let k be the finite field $\mathbb{Z}/13\mathbb{Z}$; let C be the subgroup $\{1, 5, 8, 12\}$ of k^*; and let G be the group of 52 permutations of k of the form $g_{a,b} : x \mapsto ax + b$ where $a \in C$ and $b \in k$. Let (V, ρ) be the permutation representation of G acting on complex-valued functions on k, and χ its associated character.

i) Determine $\chi(g_{a,b})$ for all $a \in C$ and $b \in g$, and prove that $\langle \mathbf{1}, \chi \rangle = 1$ and $\langle \chi, \chi \rangle = 4$. Here $\mathbf{1}$ is the character of the trivial 1-dimensional representation V_1 of G.

ii) Deduce that V is the direct sum of four pairwise non-isomorphic irreducible representations of G.

Solution. i) The character of a permutation representation takes any permutation to its number of fixed points. If $a = 1$ then $g_{a,b}$ fixes all elements of k if $b = 0$, and none otherwise; so $\chi(g_{1,b}) = 13$ or 0 according as $b = 0$ or $b \neq 0$.

If $a \neq 1$ then there is a unique fixed point, so $\chi(g_{1,b}) = 1$ for all b. Thus

$$\langle \mathbf{1}, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) = \frac{1}{52}(1 \cdot 13 + 12 \cdot 0 + 39 \cdot 1) = 1,$$

$$\langle \chi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^2 = \frac{1}{52}(1 \cdot 13^2 + 12 \cdot 0^2 + 39 \cdot 1^2) = 4$$

as claimed.

ii) Let $V = \oplus_i V_i^{\oplus n_i}$ be a decomposition of V as a direct sum of irreducible representations, with the V_i pairwise distinct and V_1 the trivial representation. Then $n_0 = \langle \mathbf{1}, \chi \rangle = 1$ and $\sum_i n_i^2 = \langle \chi, \chi \rangle = 4$. Therefore $\sum_{i \neq 0} n_i^2 = 4 - 1 = 3$. Since $2^2 = 4 > 3$, this means that each multiplicity n_i is either 0 or 1, so there are three $i \neq 0$ such that $n_i = 1$ and all other multiplicities are zero.