QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday August 30, 2022 (Day 1)

1. (AG) Let V, W be complex vector spaces of dimensions $m \geq n \geq 2$, respectively. Let $\mathbb{P}\mathrm{Hom}(V, W) \cong \mathbb{P}^{mn-1}$ be the projective space of nonzero linear maps $\phi : V \to W$ modulo scalars. Further, let $\Phi \subset \mathbb{P}\mathrm{Hom}(V, W)$ be the subset of those linear maps ϕ which do not have full rank n. Prove that Φ is an irreducible subvariety of $\mathbb{P}^{mn-1}$ and find its dimension.

2. (AT) Let S^n be the standard n-sphere

$$S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1\}$$

and let $S^k \subset S^n$ be the locus defined by the vanishing of the last $n - k$ coordinates $x_{k+1}, \ldots, x_n$. Assume $n - 1 > k > 0$.

1. Find the homology groups of the complement $S^n \setminus S^k$.
2. Suppose now that $T \subset S^n$ is the sphere defined by the vanishing of the first k coordinates; that is,

$$T = \{(0, \ldots 0, x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1\}.$$

What is the fundamental class of T in the homology group $H_{n-k-1}(S^n \setminus S^k)$?

3. (CA) Compute

$$\int_0^{2\pi} \frac{1}{(3 + \cos\theta)^2}\, d\theta$$

using contour integration.

4. (A) Show that the symmetric group S_n has at least one Sylow p-subgroup which is a cyclic group of order p. (You may use the fact that for any prime p, there exists a prime in the interval $(p, 2p)$.)

5. (DG) Let $X = T^*\mathbb{C}^\times = \mathbb{C}^\times \times \mathbb{C}$, where we write z, w for holomorphic coordinates on the base and fiber, respectively. Find all time-1 periodic orbits of the vector field $V = \mathrm{Re}(zw\frac{\partial}{\partial z})$ – i.e., all points $x \in X$ such that the time-1 flow of x under V is equal to x.

6. (RA) Let $X_1, X_2, X_3, \ldots$ be independent and identically distributed random variables with finite expected value μ and finite nonzero variance. Let

$$\overline{X_n} = \frac{1}{n}(X_1 + \cdots + X_n).$$

Use Chebyshev's inequality to prove that $\overline{X_n}$ converges to μ in probability as $n \to \infty$.

QUALIFYING EXAMINATION

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Wednesday August 31, 2022 (Day 2)

1. (CA) Let $\Omega \subset \mathbb{C}$ denote the open set

$$\Omega = \{z : |z-1| > 1 \text{ and } |z-3| < 3\}.$$

Give a conformal isomorphism between Ω and the unit disk $D = \{z : |z| < 1\}$.

2. (A) Let $F = \mathbb{Q}(z)$ where

$$z = \cos\frac{2\pi}{13} + \cos\frac{10\pi}{13}.$$

i) Prove that $[F : \mathbb{Q}] = 3$ and $F/\mathbb{Q}$ is a Galois extension.

ii) Prove that if p is a prime and $p \neq 13$ then p is unramified in F, and that p is split in F if and only if $p \equiv \pm 1$ or $\pm 5 \bmod 13$.

3. (DG) Let $u \mapsto \tau(u)$, for $a < u < b$, be a smooth space curve in $\mathbb{R}^3$ with both its curvature and torsion nowhere zero. Assume that the parameter u is the arc-length of $u \mapsto \tau(u)$. Suppose $\sigma(v)$, for $c < v < d$, is a smooth function with $\sigma'(v)$ nowhere zero. Consider the surface S defined by

$$(u, v) \mapsto \vec{r}(u, v) = \tau(u) + \sigma(v)\tau'(u)$$

for $a < u < b$ and $c < v < d$. Compute the first and second fundamental forms of S in terms of $\tau(u)$ and $\sigma(v)$ and their derivatives. Determine the condition on the function $\sigma(v)$ so that the Gaussian curvature of the surface S is identically zero.

4. (RA) Let V be the vector space of continuous functions $[0, 1] \to \mathbb{R}$, and let $g : V \to \mathbb{R}$ be the linear functional $f \mapsto \int_0^1 x^{-1/3} f(x)\, dx$. For which $p \in (1, \infty)$ does g extend to a continuous functional $\overline{g} : L^p([0, 1]) \to \mathbb{R}$? For those p, what is the norm of this functional?

5. (AG) Let $X \subset \mathbb{P}^n$ be any hypersurface of degree $d \geq 2$, and $\Lambda \subset X \subset \mathbb{P}^n$ a k-plane in $\mathbb{P}^n$ contained in X.

1. Show that if $k \geq n/2$, then X is necessarily singular.
2. If $k = n/2$ and $X \subset \mathbb{P}^n$ is a general hypersurface containing a k-plane, describe the singular locus of X.

6. (AT)

(a) Given compact oriented manifolds M and N, both of dimension n, define the degree of a continuous map $f : M \to N$.

(b) What are the possible degrees of continuous maps $\mathbb{CP}^4 \to \mathbb{CP}^4$? Justify your answer.

QUALIFYING EXAMINATION

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Thursday September 1, 2022 (Day 3)

1. (DG) Let X be a compact Riemannian manifold.

(a) Let ξ_i be a smooth 1-form on X which is both d-closed and d^*-closed. Let Δ denote the Laplacian. Denote by $|\xi|$ the pointwise norm of ξ. Denote by $|\nabla\xi|$ the pointwise norm of the covariant differential $\nabla\xi$ of ξ. Use the notation Ricci for the Ricci tensor of X. Prove the following identity of Bochner on X

$$\frac{1}{2}\Delta\left(|\xi|^2\right) = |\nabla\xi|^2 + \mathrm{Ricci}(\xi, \xi)$$

by directly computing $\Delta\left(|\xi|^2\right)$ and appropriately contracting the commutation formula for $\nabla_\alpha\nabla_\beta\xi - \nabla_\alpha\nabla_\beta\xi$ with ξ to yield the Ricci term.

(b) Assume that the Ricci curvature is positive semidefinite everywhere on X and is strictly positive at at least one point of X. By integrating Bochner's identity in (**a**) over X to prove that every harmonic 1-form on X must be identically zero. Here harmonic means d-closed and d^*-closed.

2. (RA) Suppose $w : [0, 1] \to (0, \infty)$ is a continuous function.

i) Prove that there exist unique monic polynomials $p_0, p_1, p_2, \ldots \in \mathbb{R}[x]$ such that each p_n has degree n and $\int_0^1 w(x)\, p_m(x)\, p_n(x)\, dx = 0$ for all $m, n \geq 0$ such that $m \neq n$.

ii) Prove that for each $n > 0$ the four polynomials $p_{n-1}, p_n, xp_n, p_{n+1}$ are linearly dependent.

3. (AG) Let $\Gamma \subset \mathbb{P}^n$ be any closed algebraic variety.

1. Define the *Hilbert function* $h_\Gamma(m)$.
2. If $\Gamma = D \cap E \subset \mathbb{P}^2$ is the transverse intersection of plane curves D, E of degrees d and e, what is the Hilbert function of Γ?

4. (AT) Let $G = \mathbb{Z}/m$ denote a finite cyclic group of **odd** order m. Suppose that we are given a free action of G on S^3. Compute the homology groups with integer coefficients of of the orbit space $M = S^3/G$.

5. (CA) Let $f(z)$ be an entire function. Assume that for any $z_0 \in \mathbb{R}$, at least one coefficient in the analytic expansion $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ around z_0 is equal to zero, i.e. $c_n = 0$, for some $n \in \mathbb{Z}_{\geq 0}$. Prove that f is a polynomial.

6. (A) Let k be the finite field $\mathbb{Z}/13\mathbb{Z}$; let C be the subgroup $\{1, 5, 8, 12\}$ of k^*; and let G be the group of 52 permutations of k of the form $g_{a,b} : x \mapsto ax + b$ where $a \in C$ and $b \in k$. Let (V, ρ) be the permutation representation of G acting on complex-valued functions on k, and χ its associated character.

1. i) Determine $\chi(g_{a,b})$ for all $a \in C$ and $b \in g$, and prove that $\langle \mathbf{1}, \chi \rangle = 1$ and $\langle \chi, \chi \rangle = 4$. Here $\mathbf{1}$ is the character of the trivial 1-dimensional representation V_1 of G.
2. Deduce that V is the direct sum of four pairwise non-isomorphic irreducible representations of G.