QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Tuesday September 2, 2014 (Day 1)

1. (AG) For any $0 < k < m \leq n \in \mathbb{Z}$, let $M \cong \mathbb{P}^{mn-1}$ be the space of nonzero $m \times n$ matrices mod scalars, and let $M_k \subset M$ be the subset of matrices of rank k or less.

(a) Show that M_k is closed in M (in the Zariski topology).

(b) Show that M_k is irreducible.

(c) What is the dimension of M_k?

(d) What is the degree of M_1?

2. (A) Let S_3 be the group of automorphisms of a 3-element set.

(a) Classify the conjugacy classes of S_3.

(b) Classify the irreducible representations of S_3.

(c) Write the character table for S_3.

3. (DG) Let x, y, z be the standard coordinates on $\mathbb{R}^3$. Consider the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$.

1. Compute the critical points of the function $x|_{\mathbb{S}^2}$. Show that they are isolated and non-degenerate.
2. Equip $\mathbb{S}^2$ with the standard metric induced from $\mathbb{R}^3$. Compute the gradient vector field of $x|_{\mathbb{S}^2}$. Compute the integral curves of this vector field.

4. (RA)

Find a solution for the heat equation

$$\frac{\partial}{\partial t}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) = 0, \quad (t > 0, \quad 0 < x < 1),$$

with the initial condition $u(x, 0) = A$ where A is a constant and the boundary conditions $u(0,t) = u(1,t) = 0, \quad t > 0$.

5. (AT)

(a) Show that a continuous map $f : X \to \mathbb{RP}^n$ factors through $S^n \to \mathbb{RP}^n$ if and only if the induced map $f^* : H^1(\mathbb{RP}^n; \mathbb{Z}/2) \to H^1(X, \mathbb{Z}/2)$ is zero.

(b) Show that a continuous map $f : X \to \mathbb{CP}^n$ factors through $S^{2n+1} \to \mathbb{CP}^n$ if and only if the induced map $f^* : H^2(\mathbb{CP}^n; \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is zero.

6. (CA) Let f be a meromorphic function on a contractible region $U \subset \mathbb{C}$, and let γ be a simple closed curve inside that region. Recall that the argument principle for a meromorphic function says that the integral

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f}$$

is equal to the number of zeroes minus the number of poles of f inside γ.

(a) Prove Rouché's Theorem. That is, assume (1) f and g are holomorphic in U, (2) γ is a simple, smooth, closed curve in U, and (3) $|f| > |g|$ on γ. Then the number of zeroes of $f + g$ inside γ is equal to the number of zeroes of f inside γ. You may assume the Argument Principle.

(b) Show that for any n, the roots of the polynomial

$$\sum_{i=0}^{n} z^i$$

all have absolute value less than 2.

QUALIFYING EXAMINATION

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Wednesday September 3, 2014 (Day 2)

1. (AT)

(a) Let X and Y be compact, oriented manifolds of the same dimension n. Define the *degree* of a continuous map $f : X \to Y$.

(b) What are all possible degrees of continuous maps $f : \mathbb{CP}^3 \to \mathbb{CP}^3$?

2. (A)

(a) Show that every finite extension of a finite field is simple (i.e., generated by attaching a single element).

(b) Fix a prime $p \geq 2$ and let $\mathbb{F}_p$ be the field of cardinality p. For any $n \geq 1$, show that any two fields of degree n over $\mathbb{F}_p$ are isomorphic as fields.

3. (CA) Fix two positive real numbers $a, b > 0$. Calculate the value of the integral

$$\int_{-\infty}^{\infty} \frac{\cos(ax) - \cos(bx)}{x^2} dx.$$

4. (AG) Let $C \subset \mathbb{P}^2$ be the smooth plane curve of degree $d > 1$ defined by the homogeneous polynomial $F(X, Y, Z) = 0$

(a) If $p \in C$, find the homogeneous linear equation of the tangent line $\mathbb{T}_pC \subset \mathbb{P}^2$ to C at p.

(b) Let $\mathbb{P}^{2*}$ be the dual projective plane, whose points correspond to lines in $\mathbb{P}^2$. Show that the *Gauss map* $g : C \to \mathbb{P}^{2*}$ sending each point $p \in C$ to its tangent line $\mathbb{T}_pC \in \mathbb{P}^{2*}$ is a regular map.

(c) Let $C^* \subset \mathbb{P}^{2*}$ be the *dual curve* of C; that is, the image of the Gauss map. Assuming that the Gauss map is birational onto its image, what is the degree of $C^* \subset \mathbb{P}^{2*}$?

5. (DG) Let U the be upper half plane $U = \{(x, y) \in \mathbb{R}^2 | y > 0\}$ and introduce the Poincaré metric

$$g = y^{-2}(dx \otimes dx + dy \otimes dy).$$

Write the geodesic equations.

6. (RA)

(a) Define what is meant by an *equicontinuous* sequence of functions on the closed interval $[-1, 1] \subset \mathbb{R}$.

(b) Prove the Arzela-Ascoli theorem: that if $\{f_n\}_{n=1,2,\ldots}$ is a bounded, equicontinuous sequence of functions on $[-1,1]$, then there exists a continuous function f on $[-1,1]$ and an infinite subsequence $\Lambda \subset \{1,2,\ldots\}$ such that

$$\lim_{n\in\Lambda \text{ and } n\to\infty} \left(\sup_{t\in[-1,1]} |f_n(t) - f(t)| \right) = 0$$

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QUALIFYING EXAMINATION

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Department of Mathematics

Thursday September 4, 2014 (Day 3)

1. (DG) The symplectic group $Sp(2n, \mathbb{R})$ is defined as the subgroup of $Gl(2n, \mathbb{R})$ that preserves the matrix

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

where I_n is the $n \times n$ identify matrix. That is, it is composed of elements of $Gl(2n, \mathbb{R})$ that satisfy the relation

$$M^T \Omega M = \Omega.$$

(a) Show that every symplectic matrix is invertible with inverse $M^{-1} = \Omega^{-1} M^T \Omega$.

(b) Show that the square of the determinant of a symplectic metric is 1. (In fact, the determinant of a symplectic matrix is always 1, but you don't need to show this.)

(c) Compute the dimension of the symplectic group.

2. (RA) Suppose that σ is a positive number and f is a non-negative function on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} f(x)dx = 1; \quad \int_{\mathbb{R}} xf(x)dx = 0 \quad \text{and} \quad \int_{\mathbb{R}} x^2 f(x)dx = \sigma^2.$$

Let $\mathcal{P}$ denote the probability measure on $\mathbb{R}$ with density function f.

(a) Supposing that ρ is a positive number, give a non-trivial upper bound in terms of σ for the probability as measured by $\mathcal{P}$ of the subset $[\rho, \infty)$.

(b) Given a positive integer N, let $\{X_1, \ldots, X_N\}$ denote N independent random variables on $\mathbb{R}$, each with the same probability measure $\mathcal{P}$. Let S_N be the random variable on $\mathbb{R}^N$ given by

$$S_N = \frac{1}{N} \sum_{i=1}^{N} X_i$$

What are the mean and standard deviation of S_N?

(c) Let $\{X_1, X_2, \ldots, X_N\}$ be independent random variables on $\mathbb{R}$, each with the same probability measure $\mathcal{P}$, and let $P_N(x)$ denote the function on $\mathbb{R}$ given by the probability that

$$\frac{1}{\sqrt{N}} \sum_{k=1}^{N} X_k < x.$$

Given $x \in \mathbb{R}$, what is the limit as $N \to \infty$ of the sequence $\{P_N(x)\}$?

3. (AG) Let X be the blow-up of $\mathbb{P}^2$ at a point.

(a) Show that the surfaces $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and X are all birational.

(b) Prove that no two of the surfaces $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and X are isomorphic.

4. (AT) Suppose that G is a finite group whose abelianization is trivial. Suppose also that G acts freely on S^3. Compute the homology groups (with integer coefficients) of the orbit space $M = S^3/G$.

5. (CA) Recall that a function $u : \mathbb{R}^2 \to \mathbb{R}$ is called *harmonic* if $\Delta u := \partial_x^2 u + \partial_y^2 u = 0$. Prove the following statements using harmonic conjugates and standard complex analysis.

(a) Show that the average value of a harmonic function along a circle is equal to the value of the harmonic function at the center of the circle.

(b) Show that the maximum value of a harmonic function on a closed disk occurs only on the boundary, unless u is constant.

6. (A) Let G be a finite group.

(a) Let V be any $\mathbb{C}$-representation of G. Show that V admits a Hermitian, G-invariant inner product.

(b) Let N be a $\mathbb{C}[G]$-module which is finite-dimensional over $\mathbb{C}$, and let $M \subset N$ a submodule. Show that the inclusion splits.

(c) Consider the action of S_3 on $\mathbb{C}^3$ given by permuting the axes. Decompose $\mathbb{C}^3$ into irreducible S_3-representations.