QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Tuesday September 2, 2014 (Day 1)

1. (AG) For any $0 < k < m \leq n \in \mathbb{Z}$, let $M \cong \mathbb{P}^{nm-1}$ be the space of nonzero $m \times n$ matrices mod scalars, and let $M_k \subset M$ be the subset of matrices of rank $k$ or less.

(a) Show that $M_k$ is closed in $M$ (in the Zariski topology).
(b) Show that $M_k$ is irreducible.
(c) What is the dimension of $M_k$?
(d) What is the degree of $M_1$?

2. (A) Let $S_3$ be the group of automorphisms of a 3-element set.

(a) Classify the conjugacy classes of $S_3$.
(b) Classify the irreducible representations of $S_3$.
(c) Write the character table for $S_3$.

3. (DG) Let $x, y, z$ be the standard coordinates on $\mathbb{R}^3$. Consider the unit sphere $S^2 \subset \mathbb{R}^3$.

1. Compute the critical points of the function $x|_{S^2}$. Show that they are isolated and non-degenerate.
2. Equip $S^2$ with the standard metric induced from $\mathbb{R}^3$. Compute the gradient vector field of $x|_{S^2}$. Compute the integral curves of this vector field.

4. (RA)

Find a solution for the heat equation

$$\frac{\partial}{\partial t} u(x, t) - \frac{\partial^2}{\partial x^2} u(x, t) = 0, \quad (t > 0, \quad 0 < x < 1),$$

with the initial condition $u(x, 0) = A$ where $A$ is a constant and the boundary conditions $u(0, t) = u(1, t) = 0, \quad t > 0$.

5. (AT)

(a) Show that a continuous map $f : X \to \mathbb{R}P^n$ factors through $S^n \to \mathbb{R}P^n$ if and only if the induced map $f^* : H^1(\mathbb{R}P^n, \mathbb{Z}/2) \to H^1(X, \mathbb{Z}/2)$ is zero.
(b) Show that a continuous map \( f : X \to \mathbb{CP}^n \) factors through \( S^{2n+1} \to \mathbb{CP}^n \) if and only if the induced map \( f^* : H^2(\mathbb{CP}^n; \mathbb{Z}) \to H^2(X, \mathbb{Z}) \) is zero.

6. (CA) Let \( f \) be a meromorphic function on a contractible region \( U \subset \mathbb{C} \), and let \( \gamma \) be a simple closed curve inside that region. Recall that the argument principle for a meromorphic function says that the integral

\[
\frac{1}{2\pi i} \oint_{\gamma} \frac{f'}{f}
\]

is equal to the number of zeroes minus the number of poles of \( f \) inside \( \gamma \).

(a) Prove Rouché's Theorem. That is, assume (1) \( f \) and \( g \) are holomorphic in \( U \), (2) \( \gamma \) is a simple, smooth, closed curve in \( U \), and (3) \(|f| > |g|\) on \( \gamma \). Then the number of zeroes of \( f + g \) inside \( \gamma \) is equal to the number of zeroes of \( f \) inside \( \gamma \). You may assume the Argument Principle.

(b) Show that for any \( n \), the roots of the polynomial

\[
\sum_{i=0}^{n} z^i
\]

all have absolute value less than 2.
1. (AT)
   (a) Let $X$ and $Y$ be compact, oriented manifolds of the same dimension $n$. Define the degree of a continuous map $f : X \to Y$.
   (b) What are all possible degrees of continuous maps $f : \mathbb{CP}^3 \to \mathbb{CP}^3$?

2. (A)
   (a) Show that every finite extension of a finite field is simple (i.e., generated by attaching a single element).
   (b) Fix a prime $p \geq 2$ and let $\mathbb{F}_p$ be the field of cardinality $p$. For any $n \geq 1$, show that any two fields of degree $n$ over $\mathbb{F}_p$ are isomorphic as fields.

3. (CA) Fix two positive real numbers $a, b > 0$. Calculate the value of the integral
   \[
   \int_{-\infty}^{\infty} \frac{\cos(ax) - \cos(bx)}{x^2} \, dx.
   \]

4. (AG) Let $C \subset \mathbb{P}^2$ be the smooth plane curve of degree $d > 1$ defined by the homogeneous polynomial $F(X,Y,Z) = 0$
   (a) If $p \in C$, find the homogeneous linear equation of the tangent line $T_p C \subset \mathbb{P}^2$ to $C$ at $p$.
   (b) Let $\mathbb{P}^{2*}$ be the dual projective plane, whose points correspond to lines in $\mathbb{P}^2$. Show that the Gauss map $g : C \to \mathbb{P}^{2*}$ sending each point $p \in C$ to its tangent line $T_p C \subset \mathbb{P}^{2*}$ is a regular map.
   (c) Let $C^* \subset \mathbb{P}^{2*}$ be the dual curve of $C$; that is, the image of the Gauss map. Assuming that the Gauss map is birational onto its image, what is the degree of $C^* \subset \mathbb{P}^{2*}$?

5. (DG) Let $U$ the be upper half plane $U = \{(x,y) \in \mathbb{R}^2 | y > 0\}$ and introduce the Poincaré metric
   \[
   g = y^{-2}(dx \otimes dx + dy \otimes dy).
   \]
   Write the geodesic equations.

6. (RA)
   (a) Define what is meant by an equicontinuous sequence of functions on the closed interval $[-1, 1] \subset \mathbb{R}$.
(b) Prove the Arzela-Ascoli theorem: that if \( \{f_n\}_{n=1,2,...} \) is a bounded, equicontinuous sequence of functions on \([-1,1]\), then there exists a continuous function \( f \) on \([-1,1]\) and an infinite subsequence \( \Lambda \subset \{1,2,\ldots\} \) such that

\[
\lim_{n \in \Lambda \text{ and } n \to \infty} \left( \sup_{t \in [-1,1]} |f_n(t) - f(t)| \right) = 0
\]
1. (DG) The symplectic group $Sp(2n, \mathbb{R})$ is defined as the subgroup of $Gl(2n, \mathbb{R})$ that preserves the matrix
\[
\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}
\]
where $I_n$ is the $n \times n$ identity matrix. That is, it is composed of elements of $Gl(2n, \mathbb{R})$ that satisfy the relation
\[
M^T \Omega M = \Omega.
\]
(a) Show that every symplectic matrix is invertible with inverse $M^{-1} = \Omega^{-1} M^T \Omega$.
(b) Show that the square of the determinant of a symplectic metric is 1. (In fact, the determinant of a symplectic matrix is always 1, but you don’t need to show this.)
(c) Compute the dimension of the symplectic group.

2. (RA) Suppose that $\sigma$ is a positive number and $f$ is a non-negative function on $\mathbb{R}$ such that
\[
\int_{\mathbb{R}} f(x) dx = 1; \quad \int_{\mathbb{R}} x f(x) dx = 0 \quad \text{and} \quad \int_{\mathbb{R}} x^2 f(x) dx = \sigma^2.
\]
Let $\mathcal{P}$ denote the probability measure on $\mathbb{R}$ with density function $f$.
(a) Supposing that $\rho$ is a positive number, give a non-trivial upper bound in terms of $\sigma$ for the probability as measured by $\mathcal{P}$ of the subset $[\rho, \infty)$.
(b) Given a positive integer $N$, let $\{X_1, \ldots, X_N\}$ denote $N$ independent random variables on $\mathbb{R}$, each with the same probability measure $\mathcal{P}$. Let $S_N$ be the random variable on $\mathbb{R}^N$ given by
\[
S_N = \frac{1}{N} \sum_{i=1}^{N} X_i.
\]
What are the mean and standard deviation of $S_N$?
(c) Let $\{X_1, X_2, \ldots, X_N\}$ be independent random variables on $\mathbb{R}$, each with the same probability measure $\mathcal{P}$, and let $F_N(x)$ denote the function on $\mathbb{R}$ given by the probability that
\[
\frac{1}{\sqrt{N}} \sum_{k=1}^{N} X_k < x.
\]
Given $x \in \mathbb{R}$, what is the limit as $N \to \infty$ of the sequence $\{P_N(x)\}$?

3. (AG) Let $X$ be the blow-up of $\mathbb{P}^2$ at a point.
   (a) Show that the surfaces $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $X$ are all birational.
   (b) Prove that no two of the surfaces $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $X$ are isomorphic.

4. (AT) Suppose that $G$ is a finite group whose abelianization is trivial. Suppose also that $G$ acts freely on $S^3$. Compute the homology groups (with integer coefficients) of the orbit space $M = S^3/G$.

5. (CA) Recall that a function $u : \mathbb{R}^2 \to \mathbb{R}$ is called harmonic if $\Delta u := \partial^2 u + \partial^2 u = 0$. Prove the following statements using harmonic conjugates and standard complex analysis.
   (a) Show that the average value of a harmonic function along a circle is equal to the value of the harmonic function at the center of the circle.
   (b) Show that the maximum value of a harmonic function on a closed disk occurs only on the boundary, unless $u$ is constant.

6. (A) Let $G$ be a finite group.
   (a) Let $V$ be any $\mathbb{C}$-representation of $G$. Show that $V$ admits a Hermitian, $G$-invariant inner product.
   (b) Let $N$ be a $\mathbb{C}[G]$-module which is finite-dimensional over $\mathbb{C}$, and let $M \subset N$ a submodule. Show that the inclusion splits.
   (c) Consider the action of $S_3$ on $\mathbb{C}^3$ given by permuting the axes. Decompose $\mathbb{C}^3$ into irreducible $S_3$-representations.