Marginal triviality of the scaling limits of 4D critical Ising and φ_4^4 models

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In $d = 4$ dimensions the *spin fluctuations* of Ising-type models, at their critical points are Gaussian in the scaling limit (infinite volume, vanishing lattice spacing).

The statement covers in particular the scaling limits of φ^4_4 fields constructed through the removal of a lattice cutoff.

The proofs are facilitated by the systems' random current representation. In it, the correlation functions' deviation from Wick's law are expressed in terms of intersection probabilities of random currents with prescribed sources.

This approach previously yielded such statements for $d > 4$. Their recent extension to the *marginal dimension* was enabled by a multiscale analysis of the critical clusters' intersections. *(Joint work with Hugo Duminil-Copin.)*

Field Theory $\Longleftrightarrow$ Stat Mech

Shared Math, but different goals and perspectives.

The search for a well defined field theory has had as its goal the construction of probability averages over *random distributions* $\varphi(x)$ for which the expectation value of functionals would have properties fitting the formal expression:

$$\langle F(\varphi)\rangle \approx \frac{1}{\text{norm}} \int F(\varphi)\, e^{-H(\varphi)} \prod_{x\in\mathbb{R}^d} d\varphi(x), \qquad \text{(EV)}$$

with $\qquad H(\varphi) :\approx (\varphi, A\varphi) + \int_{\mathbb{R}^d} :P(\varphi(x)):\ dx$

- $(\varphi, A\varphi)$ a positive definite and reflection positive quadratic form
- $P(\varphi(x))$ a polynomial (or a more general function)

$\Big[$ $:\varphi(x)^{2k}:$ interpreted heuristically as a local k-particle interaction. $\Big]$

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A non-interactive example

$$H(\varphi) = (\varphi, A\varphi) = \int_{\mathbb{R}^d} \left(k|\nabla\varphi|^2(x) + b|\varphi(x)|^2\right)\, dx.$$

This (quadratic/free) case can be made sense of, yielding a Gaussian FT. But even in this case the formulas need to be taken with a grain of salt ...

Functionals to which (EV) is intended to apply are based on the smeared averages
$$\boxed{T_f(\varphi) = \int_{\mathbb{R}^d} f(x)\varphi(x)dx}$$
with $f \in C_0(\mathbb{R}^d)$ (f continuous of compact support).

For products of such variables

$$\langle \prod_{j=1}^n T_{f_j}(\varphi)\rangle := \int_{(\mathbb{R}^d)^n} dx_1 \ldots dx_n \, S_n(x_1, \ldots, x_n) \, \prod_{j=1}^n f(x_j),$$

With the *"Schwinger functions"* $S_n(x_1, \ldots, x_n) \stackrel{\mathcal{D}}{=} \langle \prod_{j=1}^n \varphi(x_j)\rangle$.

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Gaussian fields are characterized by Wick's law:

$$S_{2n}(x_1, \dots, x_{2n}) = \sum_\pi \prod_{j=1}^n S_2(x_{\pi(2j-1)}, x_{\pi(2j)}) =: \mathcal{G}_n[S_2](x_1, \dots, x_{2n}) \quad (W)$$

where π ranges over pairing permutations of $\{1, \dots, 2n\}$.

The structure of such fields is simply determined by just $S_2(x_1, x_2)$. In physical terms (W) translates into the absence of interaction. Random fields of such structure have been referred to as *trivial*

(a misnomer, since not all about them is that simple).

To reach beyond the Gaussian case it is natural to explore the inclusion in $H(\varphi)$ of $\boxed{P(\varphi(x)) = \lambda :\varphi^4: -b :\varphi^2:}$. In *dim*. > 1 this creates problems.

The corresponding "φ^4_d" functional integral may be initially regularized through a pair of cutoffs:

ultraviolet (short dist.) cutoff: – restrict x to the vertices of $(a\mathbb{Z})^d$, $a \ll 1$,
infrared (long dist.) cutoff: – restrict the domain to $\Lambda_R := [-R, R]^d$, $R \gg 1$.

The goal is to construct the FT through the removal of these cutoffs: sending $R \nearrow \infty$ and $a \searrow 0$, with possible adjustments of the parameters (λ, b) so as to stabilize $S_2(x_1, x_2)$ and the other correlation functions.

Variants of this strategy employ: counter-terms, scale decompositions, renormalization group flow, regularity structures, ...

Heuristics, based on the analysis of small perturbations of the gaussian case, indicate that the approach should yield non-trivial limits for $d = 2, 3$.
(In the R-G terminology, the theory is "super-renormalizable" there).

For $d = 2, 3$ the challenge of constructing φ^4_d by such means has been met with considerable success (though not all goals have yet been reached).

The "Constructive FT" program yielded non-trivial scalar field theories (φ_d^4) over $\mathbb{R}^2$ and $\mathbb{R}^3$ (Glimm-Jaffe 70's, Osterwalder-Schrader '73, Guera-Rosen-Simon '75, Brydges-Fröhlich-Spencer '82)

However the constructive results' progression towards φ_4^4 was halted when it was proved that for dimensions $\boxed{d > 4}$ this approach yields only Gaussian fields (Aiz '81-'82, Fro '82).

Various partial results have indicated that a similar No-Go statement may hold true also for the critical dimension $d = 4$.

(Aiz-Gra '83, Ara-Car-Fro '83, Gaw-Kup '85, Har-Tas '87,..., Bau-Bry-Sla '14,...)

However a sweeping statement such as proven for $d > 4$ has remained open.

This gap was closed in our recent work :

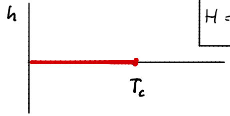
"Marginal triviality of the scaling limits of critical 4D Ising and φ_4^4 models"

Aiz. - Duminil Copin, https://arxiv.org/pdf/1912.07973.pdf

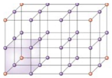
Reactions range over: ... *congratulations (!)*,... *this was expected*, ... *a non-trivial φ_4^4 may still exist*,... :=)

(cf: Fantoni-Klauder arXiv:2012.09991 (2020), and also Brezin-Aiz. post-seminar discussion at Rutgers Math-Phys seminar website).

The Ising model



$$\boxed{H = -\sum_{\langle x,y\rangle} J_{x-y}\,\sigma_x\sigma_y - h\sum_x \sigma_x}$$

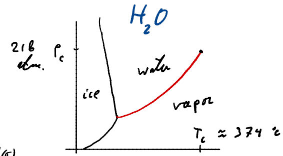


$G = \{V, E\}$

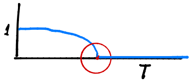
$\sigma: V \to \{-1,+1\}$

$$\langle F\rangle_{\beta,\Lambda} = \sum_{\sigma\in\Omega_\Lambda} F(\sigma)\,\frac{e^{-\beta H(\sigma)}}{\underset{\text{Norm.}\uparrow}{Z_\Lambda}}$$

$$\chi = \frac{\partial M}{\partial h}$$



$M(T,h) = \langle\sigma_x\rangle$



Thm: In any dimension $d>1$

1) $\exists\ T_c$ s.t.

$$\langle\sigma_x\sigma_y\rangle \simeq \begin{cases} A\,e^{-|x-y|/\xi} & T>T_c \quad \text{exp. dec.} \\ \to M(T)^2\ (\neq 0) & T<T_c \quad \text{LRO} \end{cases}$$

sharp transition

2) In the vicinity of T_c:

$M \sim (T_c-T)^{\hat\beta}$ $\longrightarrow$ $(T_c,0)$ $\longleftarrow$ $\chi := \frac{\partial M}{\partial h} \approx \frac{1}{(T-T_c)^\gamma}$

$M \approx h^{1/\delta}$ $\downarrow$

with:
$$\begin{cases} \hat\beta \leq 1/2 \\ \gamma \geq 1 \\ \delta \geq 3 \end{cases}$$

M-F bounds

saturated for $d>4$

Aiz. - Bar. - Fer. '87

$\xi(\beta, h) < \infty$ (finite correlation length) $\Longrightarrow$

volume averages of the fluctuations in local quantities, normalized by the usual CLT factor,

such as $\boxed{\frac{1}{\sqrt{|\Lambda|}} \sum_{u \in \Lambda} [\sigma_u - \langle \sigma_u \rangle]}$, are approximately Gaussian,

and for $\mathrm{diam}(\Lambda) \to \infty$ tend in distribution to normal random variables.

In contrast, where the correlation length diverges, $\lim_{\beta \to \beta_c} \xi(\beta)$:

1. The typical size of the bulk fluctuations of local variables may diverge on the scale of $\sqrt{|\Lambda|}$.
2. Under a corrective rescaling (to stabilize the second moment) the large scale fluctuation may attain a non-trivial distribution.
3. It makes sense to consider the scaling limit of the local fluctuation fields, expressed in terms similar to the random fields described above.
4. For the scaling description of scale L, the rescaled correlation function is

   with $\boxed{S_n^{(L)}(x_1, \ldots, x_n) = \tau(L) \langle \sigma_{[x_1 L], \ldots, [x_n L]} \rangle}$

   $\tau(L)$ adjusted so that as $L \to \infty$ the function has a non-zero finite limit.

Dual perspectives $\begin{cases} L \to \infty & \text{on the lattice scale} \\ a \to 0 & \text{on the “bulk scale”} \end{cases}$

Note: $S_2^{(L)}(0, 2x) / S_2^{(L)}(0, x) \approx e^{-L\xi} \quad \Longrightarrow$

Criticality of the lattice approximation is essential for any continuum theory based on local interactions.

Correlation functions:

$$S_4(x_1,\dots,x_4) = \langle \sigma_{x_1}\dots\sigma_{x_n}\rangle_\Lambda = \sum_{\sigma\in\{-1,1\}^\Lambda} \sigma_{x_1}\dots\sigma_{x_n}\, e^{-\beta H_\Lambda(\sigma)}/Z_\Lambda$$

Ursel 4-point function (at $h=0$): with $\langle i,j\rangle \equiv \langle \sigma_{x_i}\sigma_{x_j}\rangle$

$$U_4(x_1,\dots,x_4) := \langle \sigma_{x_1}\dots\sigma_{x_4}\rangle - [\langle 1\,2\rangle\langle 3\,4\rangle + \langle 1\,3\rangle\langle 2\,4\rangle + \langle 1\,4\rangle\langle 2\,3\rangle]$$

A measure of deviation from the Gaussian behavior

$$0 \underset{\text{(Lebowitz)}}{\leq} \frac{-U_4(x_1,\dots,x_4)}{S_4(x_1,\dots,x_4)} \underset{\text{(Glimm-Jaffe)}}{\leq} 2$$

A scale-dependent renormalized coupling constant can be based on this ratio.

Scaling limits are of interest where $\xi\to\infty$. “Triviality” corresponds to:

1. this ratio $\to 0$ at distances $(\min_{i,j}\{|x_i-x_j|\}) \gg \xi$
2. the corresponding statements hold for all (even) n.

Theorem (Newman, Aiz.) $(1) \iff (2)$ – for Ising and φ^4 systems, any $d<\infty$.

A related phenomenon is that for $d\geq 4$ the model's critical exponents take their mean-field values. (Logarithmic corrections expected in $4D$...)

Recall: in the scaling limit we keep truck of the correlation functions $S_n(x_1, ..., x_n)$ which yield the distribution of random variables of the form

$$T_{F,L}(\sigma) := \frac{1}{\sqrt{\Sigma_L}} \sum_{x \in \mathbb{Z}^d} F(\frac{x}{L})\, \sigma_x \ , \qquad \text{with} \quad \Sigma_L := \left\langle \left(\sum_{x \in \Lambda_L} \sigma_x \right)^2 \right\rangle$$

and F ranging over compactly supported continuous functions).

A common feature of the Ising model and the φ^4 (lattice) functional integral is that their states are given by Gibbs probability measures of the form

$$e^{-\beta H(\sigma)} \prod_x \rho(d\sigma)/Z \quad , \qquad H(\sigma) = J \sum_{u \approx v} (\sigma_u - \sigma_v)^2 \, ,$$

with $\rho(d\sigma)$ in the Griffiths-Simon class.

Theorem In $d = 4$ dimensions any random field reachable as an ∞-volume scaling limit of finite Ising or φ^4 systems at $\beta \leq \beta_c$, for which

i) $0 < |S_2(x, y)| < \infty$ for $|x - y| \neq 0$, ii) $S_2(0, x) \underset{x \to \infty}{\longrightarrow} 0$,

is a Gaussian field, i.e. its Schwinger functions S_n satisfy (W).

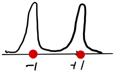
$$Z_{Ising} = \int \dots \int e^{-\beta\left[\sum_{\langle u,v\rangle} J_{u,v}(\sigma_u - \sigma_v)^2/2 + h\sum_u \sigma_u\right]} \prod_u \delta(\sigma_u^2 - 1) d\sigma_u$$

$$Z_\varphi \approx \int \dots \int e^{-\beta J \sum_u \left[J|\nabla\varphi|^2 + h\varphi_u\right]} \prod_u \rho_\lambda(d\varphi)\,; \quad \boxed{\rho_\lambda(d\varphi) = \frac{e^{-\lambda(\varphi^2-1)^2 + b\varphi^2} d\varphi}{\text{Norm.}}}$$

$\varphi \Rightarrow$ Ising: (elementary)

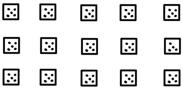
$$\delta(\sigma_u^2 - 1) d\sigma_u = \lim_{\lambda\to\infty} \rho_\lambda(d\varphi)$$

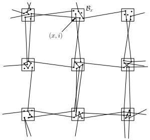
D-lim
$\lambda \to \infty$



Ising $\Rightarrow \varphi$: (the Griffiths-Simon representation)

$\rho_\lambda(d\varphi_u)$ = the limiting distribution of the block spin $\varphi_u = \sum_{x\in B_u} \sigma_x$ under the mean-field interaction, with the ultra-local coupling adjusted to $T \approx T_c$





∴ The generalized Ising model as the proverbial "grain of rice" ...)

A loop-soup representation (aka "high temp. expansion")

$Z = \sum_{\Gamma} W(\Gamma)$; Γ :

Γ: a multigraph decomposable into loops.

Γ represented by a flux function

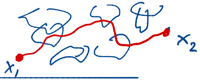
$n : \mathcal{E} \to \mathbb{Z}_+ \qquad n = (n_b)_{b \in \mathcal{E}}$

with no "sources" $\partial n = \emptyset$

$\partial n := \{ x \in \mathbb{Z}^d \mid (-1)^{\sum_{b \ni x} n_b} = -1 \}$

Correlation functions' "random current representation"

$$\langle \sigma_{x_1} \sigma_{x_2} \rangle = \frac{\sum_{\partial n = \{x_1, x_2\}} W(n)}{\sum_{\partial n = \emptyset} W(n)}$$

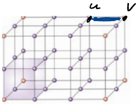


$=$

path expansion

$$= \sum_{\gamma : x_1 \to x_2} \rho(\gamma)$$

More explicitly:



$b \leftrightarrow (u,v)$

$$Z_\Lambda = \sum_{\sigma_\Lambda} e^{\beta \sum_{b=\{u,v\}} J_{uv}\sigma_u\sigma_v}$$

$$= \sum_{\underline{n}} w(\underline{n}) \sum_{\sigma_\Lambda} \prod_u \sigma_u^{\left(\sum_{b\ni u} n_b\right)}$$

$$e^{\beta J_b \sigma_u\sigma_v} = \sum_{n_b=0}^{\infty} \frac{(\beta J_b)^{n_b}}{n_b!}(\sigma_u\sigma_v)^{n_b}$$

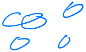
define: $\underline{n} = \{n_b\}$; $n_b = 0,1,2,\dots$

$$\boxed{Z_\Lambda = 2^{|\Lambda|}\cdot \sum_{\underline{n}:\,\partial\underline{n}=\emptyset} w(\underline{n})}$$

weight: $\boxed{w(\underline{n}) = \prod_b \frac{(\beta J_b)^{n_b}}{n_b!}}$

$$\partial\underline{n} := \left\{x \;\middle|\; (-1)^{\sum_{b\ni x} n_b} = -1\right\}$$

$$\left\langle \prod_{j=1}^{2m}\sigma(x_j)\right\rangle = \frac{\sum_{\partial\underline{n}=\{x_1,\dots,x_{2m}\}} w(\underline{n})}{\sum_{\partial\underline{n}=\emptyset} w(\underline{n})} = \frac{\sum_{\partial\underline{n}=\{x_1,\dots,x_{2m}\}} w(\underline{n})}{\sum_{\partial\underline{n}=\emptyset} w(\underline{n})}$$



$$U_4(x_1, \dots, x_4) := \langle \sigma_{x_1} \cdots \sigma_{x_4} \rangle - [\langle 1\,2\rangle\langle 3\,4\rangle + \langle 1\,3\rangle\langle 2\,4\rangle + \langle 1\,4\rangle\langle 2\,3\rangle]$$

Some relevant properties of the Ising models on $\mathbb{Z}^d$

1. Positivity: $S_n(x_1, \dots, x_n) \geq 0$ (Griffiths)
2. Compatibility: $S_4(x_1, x_2, x_3, x_4) \gtrless \frac{1}{3} \cdot \max_{\text{perm. } \pi} S_2(x_{\pi(1)}, x_{\pi(2)})\, S_2(x_{\pi(3)}, x_{\pi(4)})$
3. "Infrared bound": $\boxed{S_2(x,y) \leq \dfrac{C(\beta)}{|x-y|^{d-2}}}$ $\forall \beta \leq \beta_c$ (Fröhlich-Simon-Spencer '76, Sokal '81)
4. "Tree diagram bound": $\boxed{|U_4(x_1, \dots, x_4)| \leq 2 \sum_y \langle 1\,y\rangle\langle 2\,y\rangle\langle 3\,y\rangle\langle 4\,y\rangle}$ (Aiz '81)

$\therefore$ A rough dimensional estimate (proven for both Ising and φ^4):

$$\frac{|U_4(x_1,x_2,x_3,x_4)|}{S_4(x_1,x_2,x_3,x_4)} = O\left(\frac{L^d\, S(L)^4}{S(L)^2}\right) \leq O\left(\frac{L^d}{L^{(d-2)2}}\right) = O\left(\frac{1}{L^{d-4}}\right) \quad \text{(AHA!!)}$$

(More explicit statements for $d > 4$ were provided in Aiz. '81-82, and Fro. '82.)

OK, but what about $d = 4$?!

Remark: This estimate is in line with K. Wilson's renormalization group calculation, but it is proven beyond the range of perturbative expansions around the Gaussian fixed point.

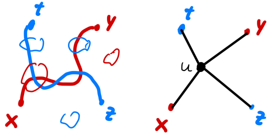
In general Ising spin models U_4 admits the following exact representation (Aiz '81):

$$\frac{|U_4^{(\beta)}(x,y,z,t)|}{\langle\sigma_x\sigma_y\sigma_z\sigma_t\rangle_\beta} \le \frac{|U_4^{(\beta)}(x,y,z,t)|}{\langle\sigma_x\sigma_y\rangle\langle\sigma_z\sigma_t\rangle_\beta} = 2\,\mathbf{P}_\beta^{xy,zt}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)\cap\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(z)\neq\emptyset]\,.$$

Intersection probability is often trickier to evaluate than the intersection's mean size. However in general, for any $A, B \subset \Lambda$:

$$\mathbf{Pr}(A\cap B\neq\emptyset) \le \sum_{u\in\Lambda}\mathbf{Pr}(u\in A\cap B) = \mathbf{E}(|A\cap B|) \qquad (X)$$

The above tree diagram bound is based on this relation.



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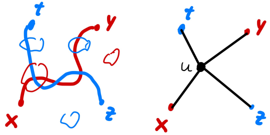


A more faithful relation is:

$$\mathbf{Pr}(A \cap B \neq \emptyset) = \frac{\mathbf{E}(|A \cap B|)}{\mathbf{E}(|A \cap B| : A \cap B \geq 1)}$$

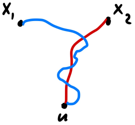
Thus, for a better upper bound on the LHS, one needs a lower bound on the conditional expectation on the right, $\mathbf{E}(|A \cap B| : A \cap B \geq 1) \geq \; ??$

An aside: notice the implications of (X) for 2D and, potentially, 3D.

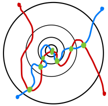
The bunching of points of intersection

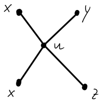
$$\mathbf{E}_{\beta}^{xy,zt}\left(|\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x) \cap \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(z)| \,\middle|\, |\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x) \cap \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(z)| \ni u\right)$$

In $4D$, if all the points are all at distances $\approx L$ then intersections occur (with uniformly positive mean number) on each scale of distances from u, up to L.
A random walk analogy:



$$\mathbf{E}\left(|\gamma_1 \cap \gamma_2|\right) = \sum_{\substack{x \in \mathbb{Z}^d \\ 1 < |x-u| < L}} \frac{1}{|x-u|^{(d-2)2}} \approx \log L$$





The random currents' "switching lemma" (a combinatorial identity which goes back to GHS '70), combined with the stochastic geometric picture of the phase transition which was developed in Aiz. '82, yields the identity:

$$\frac{|U_4^{(\beta)}(x,y,z,t)|}{\langle\sigma_x\sigma_y\rangle\langle\sigma_z\sigma_t\rangle_\beta} = 2\,\mathbf{P}_\beta^{xy,zt}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)\cap\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(z)\neq\emptyset]$$

$$= 2\,\frac{\mathbf{E}_\beta^{xy,zt}\left(|\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)\cap\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(z)|\right)}{\mathbf{E}_\beta^{xy,zt}\left(|\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)\cap\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(z)|\;\Big|\;|\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x)\cap\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(z)|\geq 1\right)}$$

The tree diagram bound is obtained by applying diagrammatic inequalities to the numerator, and bounding the denominator from above by 1. The improvement is based on a two track argument (for $d = 4$):

1. If, for some scale of distances, $S_2(x,y)\approx\frac{1}{|x-y|^{d-2+\eta}}$ with $\eta > 0$, then the (AHA) estimate for that scale improves into $O\left(\frac{1}{L^{d-4+2\eta}}\right)$ - OK!
2. Assuming $S_2(x,y)\leq\frac{C}{|x-y|^{d-2}}$, a multi-scale analysis is used to show that for $d = 4$ the above denominator is shown to exceed Const.$(\log L)^c$.

An unconditional proof is obtained by proving the prevalence of "regular scales" for which one of the above estimates applies.

Combining the analysis of U_4 with multi-spin correlation relations we prove:

Proposition: There exist $c, C > 0$ such that for the n.n.f. Ising model on $\mathbb{Z}^4$, every $\beta \le \beta_c$, every $L \le \xi(\beta)$, and every test function $f \in C_0(\mathbb{R}^4)$,

$$\Big| \Big\langle \exp[z\, T_{f,L}(\sigma) - \tfrac{z^2}{2} \langle T_{f,L}(\sigma)^2 \rangle_\beta] \Big\rangle_\beta \; - \; 1 \Big| \; \le \; \frac{C \|f\|_\infty^4 r_f^{12}}{(\log L)^c} \, z^4$$

where $\|f\|_\infty := \max\{|f(x)| : x \in \mathbb{R}^4\}$ and is r_f the smallest $r \ge 1$ such that f vanishes outside $[-r, r]^4$.

Since, by the Infrared Bound, for any such function

$$C\, r_f^2 \|f\|_\infty^2 \ge \langle T_{f,L}(\sigma)^2 \rangle_\beta \ge c_f > 0,$$

uniformly in $\beta \le \beta_c$ and L, the above estimate implies that for $L \gg 1$ the distribution of $T_{f,L}(\sigma)$ is approximately Gaussian of variance $\langle T_{f,L}(\sigma)^2 \rangle_\beta$.

Technical details, and an extensive (though not exhaustive) reference list of other rigorous works on the subject can be found in:

• M. Aizenman, Hugo Duminil-Copin: *"Marginal triviality of the scaling limits of critical 4D Ising and φ_4^4 models"*. (2019 preprint, arXiv:1912.07973).

The R-C stochastic geometric representation, and the analysis it enables, have been found useful also below the upper critical dimension (Aiz. '82).

A recent application to quasi-two-dimensional Ising spin systems can be found in:

• M. Aizenman, Hugo Duminil-Copin, Vincent Tassion, Simone Warzel. *"Emergent planarity in two-dimensional Ising models with finite-range Interactions"*, Invent. Math. **216**, 661 (2019).

Thank you for your attention.