ADE quivers and Lie theory

An honours thesis presented by

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Contents

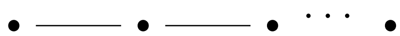
Chapter 1

Introduction

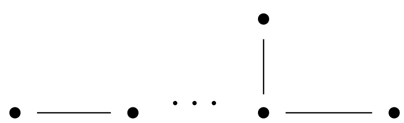
1.1 Motivation

1.1.1 ADE Classification

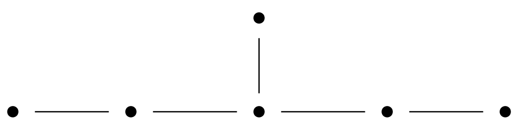
Dynkin diagrams of type ADE are unreasonably effective in classifying objects throughout mathematics. They look as follows:



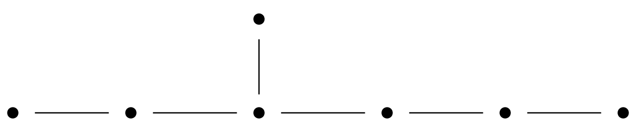
A_n



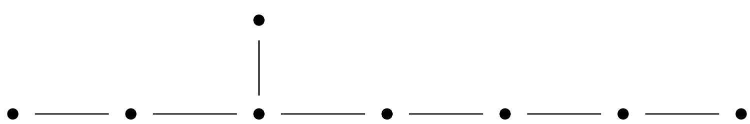
D_n



E_6



E_7



E_8

What makes these graphs special is that they are the only graphs with the property that their vertices admit a labelling so that twice the value at a vertex is the sum of the values at the adjacent vertices.

This property might seem obscure. But very many classification problems can be interpreted in terms of ADE Dynkin diagrams. Here are some examples.

1. **Simply laced Lie algebras/groups**. These are special examples of Lie groups, i.e. groups which also have the structure of a smooth manifold.

   $A_n \iff SU(n+1)$, the group of $n \times n$ unitary matrices w/ det $= 1$
   $D_n \iff Spin(2n)$, the universal cover of $SO(2n)$, the $2n \times 2n$ orthogonal matrices w/ det $= 1$
   $E_6, E_7, E_8 \iff$ Special exceptional groups which fall into no family.

   The ADE Lie groups play an immense role in mathematics and physics as natural symmetry groups. For instance, $Spin(n)$ is so called because it describes the notion of spin in quantum physics.

   The graphical perspective can feed back into useful study of the groups. For instance, D_3 is the same graph as A_3. So if our classification works, there should be an isomorphism $Spin(6) \simeq SU(4)$. Indeed, this is the case. But if we didn't have the graphs, it would be far from obvious that $SU(4)$ should be the universal cover of $SO(6)$. See [K2] for more details.

2. **Du Val singularities and the McKay correspondence**. $\mathbb{C}^2$ has a natural action of $SU(2)$, the group of two-by-two unitary matrices of determinant one. Let $G \subset SU(2)$ be a finite subgroup. Then the quotient, $\mathbb{C}^2//G$, is a singular complex surface. Up to analytic isomorphism, the singularities induced follow an ADE classification:

   $$\begin{aligned}
   A_n &\iff w^2 + x^2 + y^{n+1} = 0 \\
   D_n &\iff w^2 + y(x^2 + y^{n-2}) = 0 \\
   E_6 &\iff w^2 + x^3 + y^4 = 0 \\
   E_7 &\iff w^2 + x(x^2 + y^3) = 0 \\
   E_8 &\iff w^2 + x^3 + y^5 = 0
   \end{aligned}$$

   This also provides an ADE classification of finite subgroups of $SU(2)$. A_n corresponds to the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$. D_4 corresponds to the quaternion group. The $D_{>4}, E$ series correspond to covers of symmetry groups of Platonic solids.

   Note that the precise correspondence here is not with the ADE Dynkin diagrams as we have listed them above, but with the closely related *affine* ADE Dynkin diagrams. See [R1] for an explanation of how this arises.

3. **Minimal model conformal field theories**. A 2D conformal field theory is a 2D quantum field theory invariant under local conformal (angle-preserving) transformations.

   The states of a conformal field theory form a projective representation of the algebra of holomorphic local conformal transformations, and also a projective representation of the antiholomorphic local conformal transformations. So in fact, the states are a representation of the tensor product of these two (projectivised) algebras.

We say a 2D conformal field theory is a *minimal model* if its space of states is a sum of finitely many irreducible representations of this tensor product of algebras.

Distinct minimal models also fall into an ADE classification. For instance, A_n graphs correspond to 'diagonal models' built out of irreducible representations $\mathcal{R} \otimes \mathcal{R}$, i.e. built out of a tensor product of two of the same irreducible representations. See [G2] for more details.

1.1.2 Inverting the ADE classification for Lie algebras

The objects classified above clearly have more data than just that of an ADE Dynkin diagram. So however it is that we classified them, this classification should not be straightforwardly invertible.

However, we can ask what it takes to invert the ADE classification. For instance, given an ADE Dynkin diagram, can we construct an associated Lie algebra in some nice systematic way? Is the associated Lie algebra the ADE Lie algebra we wanted?

There is a classical answer, known as the *Chevalley-Serre relations*. Given a Dynkin diagram D, we can directly associate to it a Lie algebra with some preferred basis of generators $\{E_i, F_i, H_i\}_{i \in Vert(D)}$, whose commutation relations depend on the edges of the Dynkin diagram.

However, the choice of these generators is somewhat *ad hoc*. Can we do better? Using structures 'naturally associated to a graph', can we construct an ADE Lie algebra starting with an ADE Dynkin diagram? This question is the central one tackled in this thesis.

1.2 Overview

Take an ADE Dynkin diagram. It's a graph, so you can choose an orientation on it to turn it into a directed graph. So given a choice of orientation, we get an ADE Dynkin directed graph – what we'll call an ADE Dynkin quiver.

Importantly for us, directed graphs/quivers have a good notion of representation theory. A quiver representation is roughly the following data: to each vertex, attach a vector space; to each edge, attach a linear map between the vector spaces on the edge's vertices. Quiver representations form a category, which has surprising richness. In particular, the category of quiver representations of an ADE Dynkin diagram is very closely related to the associated ADE Lie algebra.

This undergraduate thesis is a survey of how the category of quiver representations of an ADE Dynkin quiver can be used to investigate combinatorial aspects of Lie theory.

1. First, we'll discuss how quiver representation categories categorify the idea of root systems. We will prove the classical Gabriel's theorem and its stronger derived variants, which roughly say that 'decategorified, the quiver representation category of a Dynkin quiver *is* the associated root system'.

   We'll look at the combinatorial insights this categorification provides. In particular, we'll use our category of quiver representations to define something called the Auslander-Reiten quiver, and we'll show how the combinatorics of the Auslander-Reiten quiver totally encode the data of the root system.

2. In the previous part, everything was done with reference to a choice of orientation on a Dynkin diagram. But all the results in the previous sections have been independent of this choice of orientation. Hence, one would like some construction of the categorical information which depends on no choice of orientation at all. We wil provide such a choiceless construction.

   Then we will discuss the notion of a t-structure and Bridgeland stability condition, which make precise the ways in which the information of an orientation can be preserved at the level of the derived category, and give an example where considering stability is natural.

3. Finally, we will use quiver theory to construct the Lie algebra itself, not just its root system, via the twisted Hall algebra of the category of Dynkin quiver represetntations. In fact, we will do a little better and construct the whole quantum associated to a Lie algebra. We will then briefly discuss what our categorical notion of BGP reflection, etc., imply for the quantum group. Our results here *will* rely on the orientation we started with.

1.2.1 Assumed mathematics

This thesis assumes some knowledge of category theory and homological algebra. Important assumed concepts are those of: a projective resolution; an adjoint functor; an abelian category; a derived category; a triangulated category; pushout and pullback; coproduct; $Ext^1(A, B)$. A good general reference is [GM]. See [M] for an excellent exposition of Ext^1 and projective resolutions. There is also some use of ∞-categories, a quick and informal introduction to which is given in section 2.5. For a more complete introduction, see chapter one of [L1].

This thesis does *not* assume much knowledge of ADE Lie theory; hardly any result relies on results in classical Lie theory.

One could take the Grothendieck group of a category of quiver representations to be the definition of the root lattice, and the Hall algebra of a category of quiver representations to be the definition of a quantum group/Lie algebra. Then our theorems mostly stand alone.

But this would be a very strange way to learn Lie theory. This thesis will be better motivated if you have seen the classical ADE Lie theory: see [K2] for a reference.

1.2.2 Acknowledgments

I would like to thank Prof. Arthur Jaffe for his excellent work as an advisor, for inviting me to wonderful group seminars, and for many exciting conversations about the relation between mathematics and physics. I would also like to thank Zhengwei Liu for introducing me to the study of quiver representations and suggesting that I investigate their relation to Lie theory. I also benefited immensely from conversations with Arnav Tripathy, William Norledge, Michele Tienni, Reuben Stern, and Serina Hu. Thank you to all of them.

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Chapter 2

Background on quiver representations and higher categories

2.1 Definition of quiver representations.

We are going to work with quivers. Less abstractly,

Definition 2.1.1. A **quiver** $\vec{Q}$ is a set of vertices, $Vert(\vec{Q})$, and directed edges between them.

This is almost the same as a directed graph. The difference is that directed graphs are defined to have at most one edge between any two vertices; a quiver can have as many edges between two vertices as you like.

Definition 2.1.2. A **quiver representation** of a quiver $\vec{Q}$ is

- A vector space V_i for each vertex i;
- A map $m_e : V_i \to V_j$ for each edge $e : i \to j$.

We are also going to define quivers via a more abstract perspective, where we define the category of quiver representations as some functor category.

The advantage of this perspective is twofold. First, notions like a morphism of quiver representations, or a subrepresentation of a quiver representation, are not entirely obvious. Rather than define them one-by-one, we will get them 'for free' from general functorial notions: e.g. a morphism of quiver representations will be a natural transformation, and a subrepresentation will be a subfunctor.

Second, we will define a lot of quivers by forgetting information about categories. Viewing quivers as free categories makes it clear that this is a natural operation.

Definition 2.1.3. The **base quiver category**, X_Q, is the category with

- Two objects, V the 'vertex object' and E the 'edge object'.
- Two nonidentity morphisms, $s : E \to V$, $t : E \to V$, called 'source' and 'target'.

Definition 2.1.4. A **quiver** is a functor $\vec{Q} : X_Q \to Set$. The **category of quivers**, $Quiv$, is the category of functors $X_Q \to Set$.

- There is a functor $Graph : Cat_{\text{small}} \to Quiv$, given by sending a category $\mathcal{C}$ to the quiver with vertices the objects and edges the morphisms.

- *Graph* admits a left adjoint functor $Free : Quiv \to Cat_{small}$ given by forming the free category generated by objects the vertices of a quiver and morphisms the edges.
- Consequently, we say a **quiver representation** of $\vec{Q}$, over a field k, is a functor

$$Free(\vec{Q}) \to Vect_k$$

- The **category of quiver representations of** $\vec{Q}$, $Rep_k(\vec{Q})$, is the category of such functors.

2.2 The Auslander-Reiten quiver.

Before we get into some examples of quivers, we will sign-post one important tool which will play a prominent role later on: the Auslander-Reiten quiver.

The idea is as follows: we have a functor, *Graph*, which makes a quiver from a category. But the resultant quiver has usually infinitely many vertices and infinitely many edges between each, and doesn't seem to tell us what was 'important' about the category we started with.

We want a way to get a quiver from a category which better distinguishes the important information in the category. Roughly, the Auslander-Reiten quiver remembers only those morphisms and vertices from which you can build all others. It often has finitely many edges and vertices, and in this case typically tells a lot about the underlying category.

We will see eventually that the Auslander-Reiten quiver can be used to give a totally combinatorial encoding of root system data, and to give a choiceless construction of the root system. Hence, a lot of this thesis will be devoted to tools to help calculate Auslander-Reiten quivers.

Definition 2.2.1. Fix a (possibly $(\infty, 1)-$) category $\mathcal{C}$.

- An object O is **indecomposable** if it is not equivalent to a nontrivial coproduct.
- A morphism $f : X \to Y$ is **irreducible** if f admits no right or left inverse and for any commuting diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & Y \\ & Z \arrow[ur, "\beta"'] & \end{tikzcd}$$

either α admits a left inverse or β admits a right inverse.

Definition 2.2.2. The **Auslander-Reiten quiver**, $AR(\mathcal{C})$, of a category $\mathcal{C}$ is the quiver with vertices the isomorphism classes of indecomposable objects, and edges isomorphism classes of irreducible morphisms between them.

Calculating the Auslander-Reiten quiver is often hard; we need a complete handle on the irreducible morphisms and indecomposable objects in our category. For any given quiver, the brute-force calculation of the Auslander-Reiten quiver is usually not too bad, but developing a general theory of what it should look like is more difficult. So we'll spend a lot of time dancing around the Auslander-Reiten quiver, building up the tools necessary to calculate it.

2.3 Dynkin quivers and an example

Almost exclusively, we will be interested in a special case of quivers, Dynkin quivers. Let D an ADE Dynkin diagram. We can fix an orientation on D to turn it into a **Dynkin quiver**, which we'll denote $\vec{D}^{\Omega}$. It has an associated category of quiver representations, $Rep_k(\vec{D}^{\Omega})$.

Example 2.3.1. 1. We can choose the 'standard' orientation on an A_n Dynkin diagram to make it into a quiver,

$$\vec{A}_n^{\text{std}} := \ 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \dots \longrightarrow n$$

To make contact with the precise definition discussed previously, this is the quiver

- With vertex set $A_n(V) := \{1, \dots, n\}$;
- With edge set $A_n(E) := \{1, \dots, n-1\}$;
- With $source(j) := j$;
- With $target(j) := j+1$.

2. A quiver representation assigns to every vertex a vector space V_i and every edge a linear map m_e: so a representation looks like

$$V_1 \xrightarrow{m_{1\to 2}} V_2 \xrightarrow{m_{2\to 3}} V_3 \xrightarrow{m_{3\to 4}} \dots \xrightarrow{m_{n-1\to n}} V_n$$

3. A morphism of quiver representations $f : V \to W$ is a morphism $f_i : V_i \to W_i$ at every vertex i so that the resulting diagram commutes. That is, a morphism looks like

$$\begin{tikzcd} V_1 \arrow[r, "m_{1\to 2}"] \arrow[d, "f_1"] & V_2 \arrow[r, "m_{2\to 3}"] \arrow[d, "f_2"] & V_3 \arrow[r, "m_{3\to 4}"] \arrow[d, "f_3"] & \dots \arrow[r, "m_{n-1\to n}"] & V_n \arrow[d, "f_n"] \\ W_1 \arrow[r, "m'_{1\to 2}"] & W_2 \arrow[r, "m'_{2\to 3}"] & W_3 \arrow[r, "m'_{3\to 4}"] & \dots \arrow[r, "m'_{n-1\to n}"] & W_n \end{tikzcd}$$

where the diagram is required to commute.

4. Now we can calculate $AR(Rep_k(\vec{A}_n^{\text{std}}))$. For simplicitly, we calculate $AR(Rep_{\mathbb{R}}(\vec{A}_2^{std}))$.

   (a) By linear algebra, every object is isomorphic to one of the form $\mathbb{R}^{n_1} \xrightarrow{id_{\mathbb{R}^m} \oplus 0} \mathbb{R}^{n_2}$.

   (b) Consequently, (up to isomorphism) indecomposable objects are

      i. $\mathbb{R} \xrightarrow{id} \mathbb{R}$

      ii. $\mathbb{R} \xrightarrow{0} 0$

      iii. $0 \xrightarrow{0} \mathbb{R}$

   (c) Irreducible morphisms in this case, up to isomorphism, are essentially all the nonisomorphism nonzero maps you can write down between indecomposable objects.

$$\begin{tikzcd} \mathbb{R} \arrow[r, "id"] \arrow[d, "id"] & \mathbb{R} \arrow[d, "0"] & 0 \arrow[r, "0"] \arrow[d, "0"] & \mathbb{R} \arrow[d, "id"] \\ \mathbb{R} \arrow[r, "0"] & 0 & \mathbb{R} \arrow[r, "id"] & \mathbb{R} \end{tikzcd}$$

(d) Now we can write down

$$
AR(Rep_{\mathbb{R}}(\vec{A}_2^{\mathrm{std}})) :=
\begin{tikzcd}
 & (\mathbb{R} \to \mathbb{R}) \arrow[dl] & \\
(\mathbb{R} \to 0) & & (0 \to \mathbb{R}) \arrow[ul]
\end{tikzcd}
$$

2.4 Quiver representations are modules over the path algebra

To really be able to do representation theory, we have to hope that $Rep_k(\vec{Q})$ is an abelian category, which is not immediately obvious from the way we defined it.

In fact, we'll go further and show that the category of quiver representations for any quiver is equivalently the category of modules over a k-algebra, the 'path algebra' $k\vec{Q}$ of $\vec{Q}$. Even better, we'll show that if $\vec{Q}$ is acyclic then every object of $Rep_k(\vec{Q})$ has a projective resolution of length two. This will be very useful when we move to the derived category.

Definition 2.4.1.

- Let $\vec{Q}$ a quiver. A **path of length n - 1** in $\vec{Q}$ is an inclusion

$$P_n : Rep_k(\vec{A}_n) \subset Rep_k(\vec{Q})$$

- The **path algebra** over a field k, $k\vec{Q}$, is the algebra generated by paths with multiplication by concatenation of paths, so given $P_n : Rep_k(\vec{A}_n) \subset Rep_k(\vec{Q}), P_m : Rep_k(\vec{A}_m) \subset Rep_k(\vec{Q})$, we let:

  1. If $P_n([1]) = P_m([m])$, let $P_nP_m : Rep_k(\vec{A}_{n+m-1}) \subset Rep_k(\vec{Q})$ be the concatenated path, such that $P_nP_m([j]) = P_m([j])$ if $j \leqslant m$ and $P_nP_m([j]) = P_n([j+1-m])$ if $j \geqslant m$.
  2. Otherwise, let $P_nP_m = 0$.

Graphically, this relation is clearer. Say you have a copy of $\vec{A}_3$, $1 \to 2 \to 3$. The relations say, for instance, that $(2 \to 3)(1 \to 2) = (1 \to 2 \to 3)$, $(2 \to 3)(2) = (2 \to 3)$, etc.

Example 2.4.2. Consider $\vec{A}_2, \{0\} \to \{1\}$. It has one path of length one, $P_1 = \{0\} \to \{1\}$, and two paths of length zero, $P_0^0 = \{0\}$ and $P_0^1 = \{1\}$. These elements generate $k\vec{A}_2$. The relations in $k\vec{A}_2$ are:

- P_1 ends at $\{1\}$ and starts at $\{0\}$ so $P_0^1P_1 = P_1$, $P_0^0P_1 = 0$, $P_1P_0^0 = P_1$, $P_1P_0^1 = 0$.
- Vertices start and end at the same point, so they square to themselves and are zero when multiplied by any other vertex. $(P_0^0)^2 = P_0^0$, $(P_0^1)^2 = P_0^1$, $P_0^0P_0^1 = 0 = P_0^1P_0^0$.

Proposition 2.4.3. *There is an equivalence of categories $k\vec{Q} - mod \simeq Rep_k(\vec{Q})$.*

Proof. Let $V \in Rep_k(\vec{Q})$, let $V_T := \oplus_{v \in V(\vec{Q})} V_v$ the direct sum of the vector spaces at each vertex. A path P_n induces a map $P_n([1]) \to P_n([n])$ by composing all maps along the path, i.e a map $m_{P_n} := V(P_n(e_{n-1})) \circ V(P_n(e_{n-2})) \circ \cdots \circ V(P_n(e_1))$. Let P_n act on V_T as m_{P_n}. Clearly these maps give V_T the structure of a $k\vec{Q}$-module.

Given a morphism $f : V \to W$ in $Rep_k(\vec{Q})$, send it to the morphism sending V_T to W_T. Because f is a natural transformation, this is compatible with the action of paths, so is a morphism of $k\vec{Q}$-modules.

In the other direction, let M a $k\vec{Q}$-module. Define a representation $\tilde{M}$ as follows:

- At vertices: there is a unique path P_i of length zero, containing only the vertex i. Let $\mathrm{im} P_i = \tilde{M}_i$, the vector space at vertex i.
- At edges: there is a unique path $P_{e:i\to j}$ of length one, containing an edge and its corresponding vertices. Let $m_e : \tilde{M}_i \to \tilde{M}_j$ be the map induced by the restriction of $P_{e:i\to j}$ to domain $\tilde{M}_i$.

Now let $f : M \to N$ a morphism of modules. Observe that $P_i f = f P_i$, so $\mathrm{im} P_i f = \mathrm{im} f P_i \subset \tilde{N}_i$. Hence $\tilde{f}_i : \tilde{M}_i \to \tilde{N}_i$ is a well-defined linear map. It suffices to show that the induced $\tilde{f}$ is a natural transformation, i.e. that $P_{i\to j}\tilde{f}_i = \tilde{f}_j P_{i\to j}$. This is clear because

$$\begin{aligned} P_{i\to j}\tilde{f}_i &= P_{i\to j}P_i f = P_{i\to j} f \\ \tilde{f}_j P_{i\to j} &= f P_j P_{i\to j} = f P_{i\to j} \end{aligned}$$

and f and $P_{i\to j}$ commute, because f is a morphism of modules.

Hence we have defined functors in both directions, which by computation are clearly inverse. $\square$

Now, we are going to use the perspective of the path algebra to study the projective objects in $k\vec{Q}-mod$, so equivalently in $Rep_k(\vec{Q})$. The projective objects will be especially useful when we move to the derived category, where objects are identified with their projective resolutions.

Recall

Definition 2.4.4. M is **projective** if the functor $V \to Hom(M, V)$ is exact. Equivalently, for every epimorphism $N \to N'$ and morphism $M \to N'$, there exists a morphism $M \to N$ making the following diagram commute:

$$\begin{tikzcd} & N \arrow[d, two heads] \\ M \arrow[r] \arrow[ur, dotted] & N' \end{tikzcd}$$

Definition 2.4.5. Let $i \in Vert(\vec{Q})$. Then $Simple(i) \in Ob(Rep_k(\vec{Q}))$ is the quiver representation which is k at vertex i and otherwise trivial.

Proposition 2.4.6. *Let i a vertex, and E_i the path including $\vec{A}_1$ as that vertex. Let $Proj(i) = (k\vec{Q})E_i$, i.e. the algebra of paths starting at the vertex i. Then $Proj(i)$ is projective and indecomposable. Furthermore, if $\vec{Q}$ is acyclic and $M \in k\vec{Q} - mod$ is projective, it is a direct sum of some indecomposable $\{Proj(i)\}$.*

Proof.
- First, I claim $Hom(Proj(i), V) \simeq Hom(k, V_i) \simeq V_i$. Given any map $\ell : k \to V_i$, and an edge $e : i \to j$, there is a unique map $\eta = m_e \circ \ell$ completing the commutative diagram

$$\begin{tikzcd} k \arrow[r, equal] \arrow[d, "\ell"] & k \arrow[d, "\eta"] \\ V_i \arrow[r, "m_e"] & V_j \end{tikzcd}$$

Inductively, any map $k \to V_i$ extends to a unique map $Proj(i) \to V$. In the other direction, there is a restriction map $Hom(Proj(i), V) \to Hom(k, V_i)$.

- Therefore, because the functor $V \to V_i$ is exact, $Hom(Proj(i), \_)$ is exact, so $Proj(i)$ is projective.

  Furthermore, $Hom(Proj(i), Proj(i)) = k$, so $Proj(i)$ is indecomposable: if $Proj(i) = X \oplus Y$, we would at least have $\dim Hom(Proj(i), Proj(i)) \geqslant \dim Hom(X, X) + \dim Hom(Y, Y) \geqslant 2$ because of the identity morphisms.

- Finally, let P a projective module. Let $P' = \oplus_i (\dim Hom(P, Simple(i)))Proj(i)$, where the prefactor $(\dim Hom(P, Simple(i)))$ denotes taking the direct sum of $Proj(i)$ $(\dim Hom(P, Simple(i)))$ times.

  Then any nonzero morphism $P \to Simple(i)$ induces a morphism $P \to Proj(i)$ such that

$$\begin{tikzcd} & Proj(i) \subset P' \arrow[d] \\ P \arrow[ur] \arrow[r] & Simple(i) \end{tikzcd}$$

Choosing a basis of such morphisms induces a map $P \to P'$. In the other direction, since every nonzero map into $Simple(i)$ is surjective, for every pair of surjections $Proj(i) \to Simple(i), P \to Simple(i)$ we get a map $P \to Proj(i)$ which, combined, induce an inverse map $P' \to P$. So $P' \simeq P$, and the conclusion follows.

□

Now we establish a length two projective resolution of any acyclic quiver representation.

Proposition 2.4.7. *Let $V \in k\vec{Q}$, $\vec{Q}$ acyclic. There is a short exact sequence*

$$0 \longrightarrow \oplus_{e:i\to j\in E(\vec{Q})} Proj(j) \otimes kP_e \otimes V_i \xrightarrow{d_1} \oplus_{i\in Vert(\vec{Q})} Proj(i) \otimes V_i \xrightarrow{d_0} V \longrightarrow 0$$

where $d_0 : p \otimes v \to P_p(v)$ and $d_1 : p \otimes h \otimes v \to ph \otimes v - p \otimes P_h(v)$.

Proof. There is clearly a surjective map $\oplus_{i\in Vert(\vec{Q})} Simple(i) \otimes V_i \to V$, which factors through $d_0 : \oplus_{i\in Vert(\vec{Q})} Proj(i) \otimes V_i \to V$ by taking simple representations as subspaces generated by paths at one vertex only. Hence d_0 is surjective. Furthermore d_1 is injective: otherwise, $\sum_n p_n h_n \otimes v_n = p_n \otimes P_{h_n}(v_n)$. But the paths of greatest degree in $p_n h_n$ are of length one longer than the paths of greatest degree p_n, so they cannot be equal.

Finally, $d_0 \circ d_1 = 0$ because $P_{ph}(v) = P_p P_h(v)$. □

2.5 Higher categories, cofibres and fibres

The literature on which this thesis is based is largely written by considering the derived category as a triangulated category. However, the derived category can be equipped with more structure: it is in particular a stable $(\infty, 1)$ category.

Using this extra structure makes the mathematics cleaner. The main improvement is that mapping cones are not functorial in triangulated categories, but they are in stable $(\infty, 1)$ categories. Because what follows extensively uses the concept of a mapping cone, the setting of higher category theory makes proofs much less *ad hoc* and eliminates much of the necessity for extensive checking of axioms found when using triangulated categories.

It also allows us to use techniques truly unavailable at the level of triangulated categories. For instance, we will explain how simple reflection can be categorified via the Grothendieck construction. This technique absolutely requires the use of higher categorical language.

Unfortunately, the language of $(\infty, 1)$ categories is likely more obscure to the reader than that of triangulated categories. As used in this thesis, stable-∞ categories can be considered, roughly, as triangulated categories in which cones are functorial and called 'cofibres', cocones are called 'fibres', the translation functor T is called 'suspension' and denoted Σ, and the inverse translation functor T^{-1} is called 'looping' and denoted Ω. Whenever we say 'the homotopy category' (of a stable-∞ category), we mean the underlying triangulated category.

Here is an informal 'user's guide' to these concepts.

Definition 2.5.1. A small $(\infty, 1)$-**category** $\mathcal{C}$, roughly, is

- A set of objects $Ob(\mathcal{C})$;
- A set of 1-morphisms, often just called 'morphisms', between objects which act just like morphisms in a normal category;
- A set of n-morphisms between $(n-1)$-morphisms which act just like morphisms in a category where the $(n-1)$-morphisms are objects, satisfying some coherence conditions;
- So that if $n > 1$, all n-morphisms are invertible.

Every morphism we will consider explicitly in this thesis will be a 1-morphism. The benefit of higher morphisms is that they encode homotopy equivalence. This allows maps to be unique 'up to homotopy' in a categorical setting, relaxing the strong constraints imposed by universal properties and functoriality in the ordinary categorical setting.

For instance, we can encode the notion of a triangle functorially as a cofibre sequence.

Definition 2.5.2.

- A **zero object** 0 is an object so that for every other object X, there are unique morphisms $X \to 0$, $0 \to X$. Unique morphism here means unique up to the action of n-morphisms for $n > 1$.
- A **cofibre sequence** associated to $f : X \to Y$ is a pushout diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

- A **fibre sequence** associated to $f : X \to Y$ is a pullback diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ 0 \arrow[r] & Y \end{tikzcd}$$

In the above, pushout and pullback mean the same things as in ordinary categories, except morphisms now need only be unique up to higher morphisms/'up to homotopy'.

Definition 2.5.3. A small $(\infty, 1)$**-category** $\mathcal{C}$ is **stable** if

- It has a zero object.
- A diagram is a cofibre sequence if and only if it is a fibre sequence.
- Every morphism is part of a cofibre sequence and a fibre sequence.

The notion of a stable $(\infty, 1)$ category is supposed to be an enhanced version of the notion of a triangulated category. In particular, we need an enhancement of the notion of the translation functor T.

Definition 2.5.4.

- ΣX, the **suspension** of X, is the pushout

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma X \end{array}$$

- ΩX, the **loop space** of X, is the pullback

$$\begin{array}{ccc} \Omega X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X \end{array}$$

Here Σ is a higher categorical version of T, and Ω is a higher categorical version of T^{-1}. By the universal property, clearly $\Sigma\Omega \simeq 1 \simeq \Omega\Sigma$.

How do we recover a triangulated category from an enhanced ∞-categorical structure?

Definition 2.5.5. Let $\mathcal{C}$ an $(\infty, 1)$-category. The **homotopy category**, $h\mathcal{C}$ is (roughly) the same set of objects and the same morphisms, with higher morphisms forgotten.

Warning. In our user's guide, we have swept huge subtleties under the rug involving what precisely objects and higher morphisms are. The definition of a homotopy category is especially imprecise.

Fact 2.5.6. *Let $\mathcal{C}$ a stable $(\infty, 1)$ category. Then the homotopy category, $h\mathcal{C}$ is a triangulated category.*

- *Σ is identified with the translation functor T;*
- *Ω is identified with the inverse translation functor T^{-1};*
- *Cofibre sequences $X \to Y \to Z$ are identified with distinguished triangles $X \to Y \to Z \to TZ$.*

Chapter 3

The root system via the category of quiver representations

3.1 Classical BGP reflection functors and Gabriel's theorem

There is a very good way to calculate the indecomposable objects of general ADE quiver categories, inspired by Lie theory. We will need to develop a categorified notion of 'simple reflection', called the Bernstein-Gelfand-Ponomarev (BGP) reflection functors, and show that all indecomposables are related by categorified 'simple reflection'. This sections covers that story.

It is absolutely better to consider *derived* BGP reflection functors, which are conceptually clearer and allow for stronger results. We will consider them shortly. This section is included as a 'warm-up' to gather intuition before we go to the derived world.

Definition 3.1.1. A quiver $\vec{Q}$ is **acyclic** if the only endomorphisms in $Free(\vec{Q})$ are identity morphisms. A graph G is **acyclic** if for every orientation Ω then $\vec{G}^{\Omega}$ is acyclic.

Pictorially, this means the graph has no cycles. In what follows, we will state most theorems in the generality of acyclic graphs or quivers. Note that, in particular, ADE Dynkin diagrams are acyclic.

Definition 3.1.2.

- A vertex i of a quiver $\vec{Q}$ is a **sink** if it is the source of no edge. A vertex i is a **source** if it is the target of no edge.
- If i is a source or sink of a quiver $\vec{Q}$, let $s_i\vec{Q}$ be the quiver with the direction of all arrows starting at i reversed.
- Let i a sink. There is a left exact functor $S_i^+ : Rep(\vec{Q}) \to Rep(s_i\vec{Q})$, which acts on objects by
  - For every vertex j, let
    - If $j \neq i$, $S_i^+ V_j = V_j$
    - If $j = i$, $S_i^+ V_i = \ker(\oplus_{e:k\to i} V_k \to V_i)$
  - For every edge e, let
    - If $target(e) \neq i$, let $S_i^+ m_e = m_e$.
    - If $target(e) = i$, let $S_i^+ m_e$ be the natural map $\ker(m_e) \to V_{source(e)}$.

If i is a source, there is a right exact functor S_i^- which is defined the same with arrows reversed. These functors are adjoint. We call the $S_i^\pm$ the **BGP reflection functors**.

Proposition 3.1.3. *Given an acyclic graph G with finitely many vertices and two orientations Ω_1 and Ω_2 on G, there exists a sequence so $\Omega_2 = s_{i_1} \dots s_{i_n} \Omega_1$, and all vertices flipped are sinks.*

Proof. Any such nontrivial graph always has a vertex attached to exactly one edge. Any vertex connected to exactly one edge is always a source or a sink. Therefore, the orientation of the edge attached to it can be flipped as desired. So WLOG it suffices to find s_i relating the restricted orientations Ω_1' and Ω_2' on the graph G' with all vertices of G connected to exactly one edge removed.

Because G' is still an acyclic graph, repeating this procedure and inducting on the number of vertices, eventually we'll reach the trivial graph for which all orientations are the same.

□

Proposition 3.1.4. *S_i^+ and S_i^- are adjoint.*

Proof. By explicit computation. □

Now, we are ready to use the BGP reflection functors to classify indecomposables.

Definition 3.1.5. Let $\vec{D}^\Omega$ an ADE quiver. Choose simple roots α_i for the root system corresponding to D. There is a map $\dim : Obj(Rep_k(\vec{D}^\Omega)) \to Gr(Rep_k(\vec{D}^\Omega))$, (the Grothendieck group of the category) given by

$$V \to \sum_{i \in Vert(D)} (\dim V_i)\alpha_i$$

Definition 3.1.6. $O \in Obj(Rep_k(\vec{D}^\Omega))$ is **simple** if it contains no nontrivial subrepresentation (precisely, no nontrivial subfunctor).

Theorem 3.1.7. ***(Gabriel's Theorem)****: The map* $\dim$ *induces a bijection between indecomposable representations of $\vec{D}^\Omega$ and the positive roots of the root lattice.*

Our strategy will be to classify simple representations, then show that we can build all indecomposable representations from them via BGP reflection.

There are three main propositions we need to prove to establish Gabriel's Theorem. We postpone their proofs to the end of the section to better show the main flow of the logic.

Proposition 3.1.8. *If $\vec{Q}$ is acyclic, then $Rep_k(\vec{Q})$ has simple objects isomorphic to the representations $Simple(i)$ which assign to the vertex i the vector space k, and trivial vector spaces to all other vertices, with all trivial edge maps.*

Furthermore, $\dim S_i^+ Simple(j) = s_i^+ \dim Simple(j)$*, except for $i = j$.*

That is, S_i^+ decategorified is almost simple reflection with respect to α_i.

Proposition 3.1.9. *If X is indecomposable in $Rep_k(\vec{D}^\Omega)$ and i is a sink, then either*

- *$S_i^+ X = 0$ and X is $Simple(i)$.*

- *$S_i^+ X$ is irreducible, $S_i^- S_i^+ X \simeq X$ and*
  1. $\dim S_i^+(X)_j = \dim V_j$ *for* $j \neq i$*;*
  2. $\dim S_i^+(X)_i = (\sum_{e:k\to i} \dim V_k) - \dim V_i$

  Consequently, indecomposability is preserved under BGP reflection.

Proposition 3.1.10. *If X is indecomposable, there is a sequence i_j so that $S_{i_m}^+ \dots S_{i_1}^+ X$ is simple.*

Now Gabriel's Theorem follows: every indecomposable is reached by BGP reflection from a simple object and is positive, consequently is a positive root. Because every positive root can be reached by iterated simple reflection on simple roots, this establishes the required bijection.

Here are the proofs of the preceding propositions.

Proof. (*Of Prop. 3.1.8*) Suppose V is a simple quiver representation. The subquiver on which V is nontrivial is also acyclic, therefore admits a sink i. Then there is a subrepresentation which assigns to i the vector space V_i and is otherwise trivial. Because V is simple, therefore i is the only nontrivial vertex and $\dim V_i = 1$.

The final identity now follows by definition. □

Proof. (*Of Prop. 3.1.9*) There is an isomorphism $Hom(S_i^+ X, S_i^+ X) \to Hom(S_i^- S_i^+ X, X)$ given by adjointness; call the image of the identity map ι. Now ι admits a left inverse, so by the splitting lemma there is a split exact sequence

$$0 \to S_i^- S_i^+ X \to X \to \operatorname{coker} \iota \to 0$$

Therefore, because X is indecomposable, either $S_i^- S_i^+ X = 0$ or $\operatorname{coker} \iota = 0$.

If $S_i^- S_i^+ X = 0$, then by definition of the $S_i^{\pm}$, X is necessarily only nontrivial at the vertex i. Because X is indecomposable, therefore $X = Simple(i)$.

Otherwise, $S_i^- S_i^+ X \simeq X$. Suppose $S_i^+ X = A \oplus B$. Then because S_i^- is additive (because it is left adjoint, hence preserves coproducts), then $X = S_i^-(A) \oplus S_i^-(B)$; one factor is zero by indecomposability, say $S_i^-(A) = 0$. Let ρ the image of the identity under the natural isomorphism $Hom(S_i^- S_i^+ X, S_i^- S_i^+ X) \to Hom(S_i^+ X, S_i^+ S_i^- S_i^+ X)$. It is an isomorphism, with ρ^{-1} the image of the identity under the natural isomorphism $Hom(S_i^- S_i^+ X, S_i^- S_i^+ X) \to Hom(S_i^+ S_i^- S_i^+ X, S_i^+ X)$. ρ induces a restricted isomorphisms $\rho_{|A}$. But $\rho_{|A}$ has range $S_i^+ S_i^-(A) = 0$. Hence $A = 0$ also. Therefore, S_i^+ is indecomposable.

It remains to establish the given formula on dimensions. By rank-nullity, it suffices to show that the map $(\oplus_{e:k\to i} V_k) \to V_i$ is surjective. Let U a complement to the image, and $\tilde{U}$ the quiver representation which has U at vertex i and is otherwise trivial. $\tilde{U}$ is clearly a direct summand of X, therefore $\tilde{U} = 0 \implies U = 0$. □

Proof. (*Of Prop. 3.1.10*) Given $\vec{D}^{\Omega}$ with n vertices, choose a sequence $i_1, \dots, i_n$ so that

- i_j is a sink for D with orientation $s_{i_{j-1}} \dots s_{i_1} \Omega$;
- No vertex appears twice.

This gives rise to a **Coxeter functor**, $\mathcal{C} := S^+_{i_1} \dots S^+_{i_n}$, which under the map dim is a Coxeter element C of the Weyl group. I claim for now without justification that C has finite order h and $C - 1$ is invertible. (See [K2] for classical proofs of these claims. Alternatively, we will discuss a categorification of Coxeter elements starting in section 3.6. We prove categorical versions of both of these statements, see prop. 3.6.7 and corr. 3.7.2.1, which imply this result.)

Now I claim for $v > 0$ in the root system, there exists k so $C^k v \leqslant 0$. We have $0 = (C-1)^{-1}(C^h - 1)v = C^{h-1}v + \dots + Cv + v$. Therefore at least one $C^k v \leqslant 0$. Apply this for $\dim X$. Then there exists minimal k so $C^k \dim X \leqslant 0$. So by the previous proposition, $\mathcal{C}^{k-1}X$ is simple. □

3.2 Moving to the derived category: homological prerequisites

There was a problem with Gabriel's theorem. We said that the BGP reflection functors were a categorified notion of simple reflection. Yet we had that

$$S_i^+ Simple(i) = 0$$

so the analogy was not precise; if the reflection functors really categorified simple reflection, we would have

$$S_i^+ Simple(i) = \text{``} - Simple(i)\text{''}$$

In particular, our simple reflections 'couldn't see the negative roots', because we had no good categorical notion of negativity. We will solve this problem by moving to a slightly larger category with a good categorified notion of negativity.

Before we can get there, we'll need some homological algebra prerequisites.

Definition 3.2.1. A **hereditary category** is an abelian category in which $Ext^2(A, B)$ vanishes for all A, B.

Note that, interpreting the $Ext^n(A, B)$ as right-derived functors of $A \to Hom(A, B)$, vanishing of $Ext^2(A, B)$ clearly implies vanishing of all higher Ext^n.

Proposition 3.2.2. *Categories of quiver representations are hereditary.*

Proof. We earlier (2.4.7) constructed an explicit projective resolution of length two for any quiver representation, hence Ext^2 must vanish. □

Lemma 3.2.3. *If $f : N \to M$ is a surjection in a hereditary category, then there is a surjection $Ext^1(X, N) \to Ext^1(X, M)$.*

Proof. Choose a projective resolution $0 \to P_1 \to P_0 \to X \to 0$. The surjection f induces a map of cochain complexes

$$\begin{tikzcd}
0 \arrow[r] & Hom(P_0, N) \arrow[r, "\partial_1^N"] \arrow[d, "f^\star"] & Hom(P_1, N) \arrow[r] \arrow[d, "f^\star"] & 0 \\
0 \arrow[r] & Hom(P_0, M) \arrow[r, "\partial_1^M"] & Hom(P_1, M) \arrow[r] & 0
\end{tikzcd}$$

which induces a surjection $Ext^1(X,N) = Hom(P_1,N)/im\partial_1^N \to Hom(P_1,M)/im\partial_1^M = Ext^1(X,M)$ because $f^\star$ is a surjection $Hom(P_1,N) \to Hom(P_1,M)$ such that $f^\star(im\partial_1^N) \subset im\partial_2^N$. □

Now we will prove the most important property of hereditary categories: namely, in a hereditary category $\mathcal{A}$, chain-complexes are always quasi-isomorphic to the direct sum of translates of objects of $\mathcal{A}$.

Proposition 3.2.4. *If $\mathcal{A}$ is hereditary, there is a roof diagram of quasi-isomorphisms between the chain complexes $X_\bullet$ and $\oplus_{n\in\mathbb{Z}} H^n(X_\bullet)[-n]$.*

Proof. By the previous lemma 3.2.3, the surjection $\partial^{n-1} : X^{n-1} \to \mathrm{im}\partial^{n-1}$ induces a surjection $Ext^1(H^nX, X^{n-1}) \to Ext^1(H^nX, \mathrm{im}\partial^{n-1})$. Hence, by the classical interpretation of Ext^1, there exists $\overline{X}^n$ and maps such that the diagram of short exact sequences (where ι indicates an inclusion map)

$$\begin{tikzcd}
0 \arrow[r] & X^{n-1} \arrow[r] \arrow[d, "\partial^{n-1}"] & \overline{X}^n \arrow[r] \arrow[d] & H^nX \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & im\partial^{n-1} \arrow[r, "\iota"] & X^n \arrow[r] & H^nX \arrow[r] & 0
\end{tikzcd}$$

commutes.

Let $\overline{X}_\bullet^n$ denote the complex $\cdots \to 0 \to X^{n-1} \to \overline{X}^n \to 0 \to \ldots$, where $\overline{X}^n$ is in degree n.

There is a clear morphism $\overline{X}_\bullet^n \to X_\bullet$, which has trivial maps except in degree n, where it has the map $\overline{X}^n \to X^n$ given in the commutative diagram above. By the short exact sequence given, this map is an isomorphism on the n-th homology group.

Hence there is a quasi-isomorphism $\oplus_{n\in\mathbb{Z}}\overline{X}_\bullet^n \to X_\bullet$. Furthermore clear quasi-isomorphisms $\overline{X}_\bullet^n \to H^n(X_\bullet)[-n]$ induce a roof diagram

$$\begin{tikzcd}
& \oplus_{n\in\mathbb{Z}}\overline{X}_\bullet^n \arrow[dl] \arrow[dr] & \\
\oplus_{n\in\mathbb{Z}}H^n(X_\bullet)[-n] & & X_\bullet
\end{tikzcd}$$

as desired. □

In ∞-categorical language, the aforementioned quasi-isomorphism presents a derived equivalence between $X_\bullet$ and $\oplus_{n\in\mathbb{N}}\Omega^n H^n(X_\bullet)$. We will now use this to prove that the two-periodic derived category also has nice properties.

Definition 3.2.5. Model $\mathcal{D}(\mathcal{A})$ as $Kom(\mathcal{A})$ (the category of chain complexes in $\mathcal{A}$) with weak equivalences inverted.

The two-periodic derived ∞-category $\mathcal{D}(\mathcal{A})/\Sigma^2$ is the full subcategory of $\mathcal{D}(\mathcal{A})$ with objects weakly equivalent to one of the form

$$\ldots \longrightarrow X_0 \xrightarrow{\alpha} X_1 \xrightarrow{\beta} X_0 \xrightarrow{\alpha} X_1 \xrightarrow{\beta} X_0 \longrightarrow \ldots$$

We call this the **root category** of our Dynkin quiver in the case that $Rep_k(\vec{D}^\Omega) = \mathcal{A}$.

Proposition 3.2.6. *$\mathcal{D}(\mathcal{A})/\Sigma^2$ is stable if $\mathcal{A}$ is hereditary.*

Proof. To show a subcategory of a stable category is stable, it suffices to show that the subcategory contains zero and all fibres and cofibres. $\mathcal{D}(\mathcal{A})/\Sigma^2$ contains zero clearly. Further, because $\mathcal{D}(\mathcal{A})/\Sigma^2$ is closed under Ω and $\Omega cofib(X \to Y) \simeq fib(X \to Y)$, it in fact suffices to show that the subcategory contains all cofibres.

Because $\mathcal{A}$ is hereditary, choose complexes equivalent to X and Y with trivial cochain maps by the previous proposition.

Then the cofibre sequence splits as

$$\begin{tikzcd} \oplus_{n\in\mathbb{Z}}\Sigma^{2n}(X_0 \oplus \Sigma X_1) \arrow[rr] \arrow[d] & & \oplus_{n\in\mathbb{Z}}\Sigma^{2n}(Y_0 \oplus \Sigma Y_1) \arrow[d] \\ 0 \arrow[r] & \oplus_{n\in\mathbb{Z}}\Sigma^{2n}(cofib(X_0 \to \Omega Y_1 \oplus Y_0 \oplus \Sigma Y_1) \oplus cofib(\Sigma X_1 \to Y_0 \oplus \Sigma Y_1 \oplus \Sigma^2 Y_0)) & \end{tikzcd}$$

Now $\oplus_{n\in\mathbb{Z}}\Sigma^{2n}(cofib(X_0 \to Y_0) \oplus \Sigma cofib(X_1 \to Y_1))$ is clearly in the subcategory, and is equivalent to $cofib(X \to Y)$. □

Remark 3.2.7. We defined the two-periodic derived category in terms of a modelling by chain complexes to be concrete. But we could have defined it more abstractly. Further, the category is stable even if the underlying abelian category isn't hereditary. The discussion surrounding prop. 4.1.6 will present this proof and definition in a more abstract and general context.

Finally, the theorem on equivalences in a derived category also nets a classification of its indecomposables:

Corollary 3.2.7.1. *Up to equivalence, indecomposable objects in the (two-periodic) derived category of a hereditary category are translates by Ω or Σ of indecomposable objects in the underlying hereditary category.*

In particular, in the Dynkin case Gabriel's theorem therefore identifies all the indecomposables in the root category.

3.3 Derived BGP reflection functors

The derived BGP reflection functors are the derived functors associated to the previously introduced BGP reflection functors. But whilst the normal BGP reflection functors were defined in a seemingly *ad hoc* way, the derived ones can be motivated more clearly.

Given a map $B \to C$, is there a way to 'reflect the arrow' functorially without losing information? That is, we have a map out of B and into C. Can we somehow get a map out of C or a map into B from this? In a stable $(\infty, 1)$ category, there are two natural candidates for orientation reversal.

We can take the cofibre map $C \to cofib(B \to C)$ induced by the pushout/pullback square

$$\begin{tikzcd} B \arrow[r] \arrow[d] & C \arrow[d] \\ 0 \arrow[r] & cofib(B \to C) \end{tikzcd}$$

or we could alternatively take the fibre map $fib(B \to C) \to B$ induced by the pushout/pullback square

$$\begin{tikzcd} fib(B \to C) \arrow[r] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & C \end{tikzcd}$$

These maps are mutually inverse, i.e. $fib(C \to cofib(B \to C)) = B$ and $cofib(fib(B \to C) \to B) = C$, because each internal square is pushout and pullback,

$$\begin{tikzcd} fib(B \to C) \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & C \arrow[r] & cofib(B \to C) \end{tikzcd}$$

hence they are information preserving.

More generally, given maps $\{B_i \to C\}_{i \in I}$ in a stable $(\infty, 1)$ category, we get maps $fib(\oplus_i B_i \to C) \to \oplus_i B_i \to B_i$, which can be inverted with $cofib$ to the original maps $B_i \to C$. We can do the same with arrows in the opposite direction.

Hence, derived BGP reflection functors are in some sense the natural invertible orientation-reversing maps of diagrams in a stable $(\infty, 1)$-category. This perspective will be made more precise in section 3.9.

Definition 3.3.1.
- Recall a vertex $i \in \vec{Q}$ is a **sink** if there are no edges pointing out of i;
- Recall a vertex $i \in \vec{Q}$ is a **source** if there are no edges pointing into i.
- If i is a sink, the **BGP reflection functor** RS_i^+ sends all maps associated to edges $j \to i$ to the maps $fib(\oplus_j V_j \to V_i) \to \oplus_j V_j \to V_j$.

  It is a functor $\mathcal{D}(Rep_k(\vec{Q})) \to \mathcal{D}(Rep_k(s_i\vec{Q}))$, where $s_i\vec{Q}$ is the quiver with the same underlying graph and the orientation of all arrows pointing into i flipped.
- If i is a source, the **BGP reflection functor** LS_i^- sends all maps associated to edges $i \to j$ to the maps $V_j \to \oplus_j V_j \to cofib(V_i \oplus_j V_j)j$.

  It is a functor $\mathcal{D}(Rep_k(\vec{Q})) \to \mathcal{D}(Rep_k(s_i\vec{Q}))$, where $s_i\vec{Q}$ is the quiver with the same underlying graph and the orientation of all arrows pointing out of i flipped.

Immediately, we have

Proposition 3.3.2. *If G is an acylic graph, Ω_1 and Ω_2 arbitrary orientations, then there is an equivalence of stable $(\infty, 1)$-categories $\mathcal{D}(Rep_k(\vec{G}^{\Omega_1})) \to \mathcal{D}(Rep_k(\vec{G}^{\Omega_2}))$ induced by iterated BGP reflection on sinks. In particular, this is true for Dynkin quivers which are all acyclic.*

We can also verify analogously to the non-derived case that

- Derived BGP functors preserve indecomposabilility *(Proof: suppose $RS_i^+ X = A \oplus B$. Then $LS_i^- A \oplus LS_i^- B \simeq X$. But if X is indecomposable, since LS_i^- is an equivalence, then A or B are zero. So $RS_i^+ X$ is indecomposable.)*

- If X is indecomposable, there is a sequence i_j so that $RS^+_{i_m} \dots RS^+_{i_1} X$ is simple. *(The same proof as Prop 3.1.10)*

Immediately, we get

Proposition 3.3.3. *The indecomposable objects of $\mathcal{D}(Rep(\vec{D}^\Omega))$ are in bijection with the roots of the root system corresponding to the Dynkin diagram D. The derived functors RS^-_i act as simple reflection and generate the Weyl group.*

3.4 The Categorical Root System Inner Product

We have the roots system and the Weyl group: there is one more piece of data needed to specify a root system; namely, an inner product of roots. Furthermore, we need a formal tool to get the whole root lattice, generalising the dim map of def. 3.1.5.

Definition 3.4.1. Let $\mathcal{C}$ a triangulated category. Then the **Grothendieck group**, or zero-th order K-theory, $K_0(\mathcal{C})$ is the group generated by isomorphism classes of objects of $\mathcal{C}$ subject to the relation that if $X \to Y \to Z \to TZ$ is a triangle, then

$$[X] + [Z] = [Y]$$

Remark 3.4.2. We can more generally define the Grothendieck group than just for triangulated categories. For an abelian category, we can impose relations $[A]+[C] = [B]$ for every exact sequence $0 \to A \to B \to C \to 0$. For a stable ∞-category, we can let $[A] + [C] = [B]$ for every fibre sequence $A \to B \to C$.

Note that the Grothendieck group of an abelian category is therefore the free abelian group generated by its (isomorphism classes of) indecomposable objects, so $K_0(\mathcal{D}(Rep_k(\vec{D}^\Omega)/T^2))$ is in fact the root lattice.

Now, the inner product of roots.

Definition 3.4.3. Fix a root category of an ADE Dynkin quiver. Define a bilinear form on isomorphism classes of indecomposables

$$\langle [X], [Y] \rangle := \dim RHom(X, Y) + \dim RHom(Y, X)$$

Note that $\dim RHom(X, Y) = \dim Hom(X, Y) - \dim Ext^1(X, Y)$.

Proposition 3.4.4. *BGP reflection is an isometry:*

$$\langle [RS^+_i X], [RS^+_i X] \rangle = \langle [X], [X] \rangle$$

Proof. $\dim RHom(X, Y)$ is preserved under RS^+_i because it's an equivalence, hence the bilinear form is also preserved. □

Corollary 3.4.4.1. *All roots are long, i.e. $\langle [X], [X] \rangle = 2$ for all indecomposables.*

Proof. $\dim Hom(Simple(i), Simple(i)) = \dim Hom_{k-alg}(k, k) = 1$ and

$$\dim Ext^1(Simple(i), Simple(i)) = \dim Ext^1(k, k) = 0$$

. Therefore $\langle [Simple(i)], [Simple(i)] \rangle = 2$. Now write any indecomposable as the BGP reflection of simple objects. □

Proposition 3.4.5. *The inner product is compatible with simple reflection in the usual way:* $[RS_i^+X] = [X] - \langle [X], [Simple(i)] \rangle [Simple(i)]$

Proof. This is clearly true if $X \simeq Simple(i)$. Otherwise,

$$fib(\oplus_{e:k\to i} V_k \to V_i) \simeq \ker(\oplus_{e:k\to i} V_k \to V_i)$$

,by definition. Consequently, we can apply the dimension formula of Prop. 3.1.9. It suffices to prove that $\sum_{e:k\to i} \dim X_k - \dim X_i = \dim X_i - \langle [X], [Simple(i)] \rangle$.

Clearly $\dim Hom(X, Simple(i)) = \dim(X_i, k) = \dim X_i = \dim(k, X_i) = \dim Hom(Simple(i), X)$. Likewise, $\dim Ext^1(X, Simple(i)) = \dim \oplus_{e:k\to i} V_k$ and $\dim Ext^1(Simple(i), X) = 0$, where Ext groups are computed by projective resolution. Plugging these calculations in gets the desired result. □

Putting all this together gets

Theorem 3.4.6. *(Gabriel's theorem, enhanced) Let D a Dynkin diagram, Ω an orientation, and $\mathcal{R} = \mathcal{D}(Rep_k(\vec{D}^\Omega))/\Sigma^2$ the corresponding root category. The following data specify the root system corresponding to D:*

- *Roots the isomorphism classes of indecomposable objects of $\mathcal{R}$;*
- *Vector space generated by the Grothendieck group $K_0(\mathcal{R})$;*
- *Weyl group generated by the derived BGP reflection functors;*
- *An inner product of roots $\langle [X], [Y] \rangle := \dim RHom(X, Y) + \dim RHom(Y, X)$.*

3.5 The Auslander-Reiten quiver and combinatorial Lie theory

The previous section justifies the sense in which the 'root category' $\mathcal{D}(Rep_k(\vec{Q}))/\Sigma^2$ categorifies combinatorial Lie theory. Now we are going to show how the Auslander-Reiten quiver of this ∞-category gives a combinatorial quiver-theoretic interpretation of Lie theory.

Definition 3.5.1. Fix D a Dynkin diagram. Choose any orientation Ω. Let $\hat{D}_{cyc} := AR(\mathcal{D}(Rep_k(\vec{D}^\Omega)/\Sigma^2))$; call it the **periodic Auslander-Reiten quiver associated to** D.

By the work of the previous sections, the *vertices* of this periodic Auslander-Reiten quiver are in bijection with the roots of the root system; what are the *edges* of the Auslander-Reiten quiver? Before we proceed in general, we'll consider an instructive example.

Example 3.5.2. We will work in the homotopy category $D^b(\vec{A_2}^{std})/T^2$. Let's calculate $\hat{A}_{2cyc} = AR(D^b(Rep_{\mathbb{R}}(\vec{A_2}^{std})/T^2)$. We know the indecomposable objects are translates of the indecomposable objects we calculated for $Rep_{\mathbb{R}}(\vec{A_2}^{std})$ in Example 2.3.1. It suffices to calculate the irreducible morphisms.

Morphisms from objects X to Y are then in bijection with tuples (f_i, e_i), where $f_i : H^iX \to H^iY$ are morphisms and the $e_i \in Ext^1(H^nX, H^{n-1}Y)$. $Ext^1(0 \to \mathbb{R}, X) = 0$, therefore the irreducible morphisms $(0 \to \mathbb{R}) \to (\mathbb{R} \to \mathbb{R})$ and $T(0 \to \mathbb{R}) \to T(\mathbb{R} \to \mathbb{R})$ from Ex. 2.3.1 are preserved.

Likewise, $Ext^1(X, 0 \to \mathbb{R}) = 0$, so the irreducible morphism $(\mathbb{R} \to \mathbb{R}) \to (0 \to \mathbb{R})$ and its translate are preserved. However, $\dim Ext^1(\mathbb{R} \to 0, 0 \to \mathbb{R}) = 1$. $\dim Ext^1(\mathbb{R} \to 0, \mathbb{R} \to \mathbb{R}) = 1$ also, but one can show this factors via the morphism $(0 \to \mathbb{R}) \to (\mathbb{R} \to \mathbb{R})$. So the Auslander-Reiten quiver looks like

$$\begin{tikzcd}
 & T(0 \to \mathbb{R}) \arrow[dl] & (\mathbb{R} \to 0) \arrow[l] & \\
T(\mathbb{R} \to \mathbb{R}) \arrow[dr] & & & (\mathbb{R} \to \mathbb{R}) \arrow[ul] \\
 & T(\mathbb{R} \to 0) \arrow[r] & (0 \to \mathbb{R}) \arrow[ur] &
\end{tikzcd}$$

In general, we can calculate the Auslander-Reiten quiver combinatorially. This calculation will take some work.

Definition 3.5.3. Let D an ADE Dynkin diagram with associated Coxeter number h. We can associate a quiver, $D \times \mathbb{Z}$, with

- Vertex set $V(D \times \mathbb{Z}) = D \times \mathbb{Z}$
- An edge $(v, j) \to (w, j+1)$ whenever $v - w$ is an edge of the Dynkin diagram.

Because Dynkin diagrams are bipartite, $D \times \mathbb{Z}$ has two identical connected components. Let $\hat{D}$ a connected component of the quiver $D \times \mathbb{Z}$; we call it the **Auslander-Reiten quiver** associated to D.

Proposition 3.5.4. *$AR(\mathcal{D}(Rep_k(\vec{D}^\Omega))) = \hat{D}$.*

We know the indecomposable objects in $\mathcal{D}(Rep_k(\vec{D}^\Omega))$; it suffices to calculate the irreducible morphisms. Here's the idea: we'll calculate the irreducible morphisms of $AR(Rep_k(\vec{D}^\Omega))$, then we'll identify an operation τ which 'translates' it. Formally developing this idea will take up the next section.

3.6 Auslander-Reiten translation

We are going to introduce lots of notions of duality, which don't quite commute. By composing them, we will get an interesting autoequivalence, the Auslander-Reiten translation τ, which will in fact be a *Serre functor*, and will give us the tool of almost-split sequences. Many of our arguments appeal to projective resolution, and we omit some of the detailed checking formally required to avoid getting bogged down in the commutative algebra details. See e.g. [K5] for a more careful treatment.

Definition 3.6.1.
- Let $\Lambda := k\vec{Q}$, for $\vec{Q}$ Dynkin. (More generally, what we'll say works for a Noetherian ring which is a k-algebra);
- There is a duality functor, $(\_)^\star := Hom_\Lambda(\_, \Lambda) : mod\Lambda \to mod\Lambda^{op}$;

- Given a projective presentation $P^1 \to P^0 \to X$, let $TrX := cofib((P^1 \to P^0)^\star)$, the **Auslander-Bridger transpose**. In a hereditary category with enough projectives and injectives, this gives an equivalence $\mathcal{D}(mod\Lambda) \to \mathcal{D}(mod\Lambda^{op})$.
- There is an equivalence $D := Hom_k(\_, k) : mod\Lambda \to mod\Lambda^{op}$.
- Let $\tau := D \circ Tr : \mathcal{D}(mod\Lambda) \to \mathcal{D}(mod\Lambda)$, the **Auslander-Reiten translation**.

Remark 3.6.2. Recall we defined $Proj(i) := k\vec{Q}E_i$, the algebra of paths starting the vertex i. We could equivalently have defined $Inj(i) := E_i k\vec{Q}$, the algebra of paths ending at the vertex i, and we could show that the $Inj(i)$ are a full list of the indecomposable injectives in our quiver the same way as we did in Prop. 2.4.6 for $Proj(i)$, for instance because $Inj(i)$ is the projective indecomposable starting at vertex i for the opposite quiver $\vec{Q}^{op}$.

Proposition 3.6.3. *1. τ is an additive equivalence, with inverse $\tau^{-1} := Tr \circ D$.*

2. *If $X \in mod\Lambda$ is indecomposable, considered as a complex in degree zero, then τX lowers the degree of X if X is projective, sending projective objects to lower-graded injective objects, and otherwise preserves grading.*
3. *$\Sigma\tau$ is a **Serre functor**: i.e. $Hom(X, Y) \simeq Hom(Y, \Sigma\tau X)^\star$ naturally.*

Proof. 1. Because Tr and D are dualities.

2. Suppose X is projective, $X = Proj(i) = P$. Then $\tau P = DTrP = Dcofib(\star(0 \to P_1)) = D\Sigma P_1^\star = \Omega DP_1^\star$. We can compute by definition $DProj(i)^\star = Inj(i)$, hence $DP_1^\star$ is injective. So $\tau Proj(i) = \Omega Inj(i)$. Hence, τ lowers grading on projective objects, and sends projective objects to injective ones.

   Otherwise, let $P_0 \to P_1$ a projective resolution of X. $\tau X = DTrX = Dcofib(\star(P_0 \to P_1)) = Dcoker(\star(P_0 \to P_1)) = kerD \star (P_0 \to P_1)$, hence has grading zero.
3. Hence, on projective objects $\Sigma\tau Proj(i) \simeq Inj(i)$. Hence to establish Serre duality, taking projective resolutions, it suffices to establish isomorphisms $Hom(Proj(i), Proj(j)) \simeq Hom(Proj(j), Inj(i))^\star$, which are clear because both vector spaces are generated by the paths $i \to j$.

□

Using Auslander-Reiten translation, we are ready to study almost-split sequences.

Definition 3.6.4. A short exact sequence $0 \to A \to B \to C \to 0$ is **almost split** if

1. A and C are indecomposable;
2. The sequence is not split;
3. Given any indecomposables X, Y, and nonisomorphisms $A \to X$, $Y \to C$, the maps factor

$$\begin{tikzcd}
 & & & Y \arrow[d] \arrow[dl, dotted] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[dl, dotted] & C \arrow[r] & 0 \\
 & X & & &
\end{tikzcd}$$

Proposition 3.6.5. *If X is a nonprojective indecomposable, then there is $\tilde{X}$ and an almost split exact sequence $0 \to \tau X \to \tilde{X} \to X \to 0$.*

Proof. In this case, Serre duality tells us that $Ext^1(X, \tau X) \simeq DEnd(X) \simeq k$. By the classical interpretation of Ext^1, therefore there exists $\tilde{X}$ and a short exact sequence $0 \to \tau X \to \tilde{X} \to X \to 0$ which is not split. Further, τX is indecomposable because τ is invertible and additive.

Now suppose we have Y indecomposable with a map into X. Now $Hom(Y, X) \simeq Hom(X, \Sigma\tau Y)^\star$. Choosing an identification of the dual $Hom(X, \Sigma\tau Y)^\star \simeq Hom(X, \Sigma\tau Y)$, we get a map to $Hom(X, \Sigma\tau Y) \to Hom(\tilde{X}, \Sigma\tau Y)$; composing all this nets a map $Hom(Y, X) \to Hom(Y, \tilde{X})$ (choosing an identification of the dual once more.)

Tracing the exact maps and using the fact that an indecomposable cannot split, we can show that the composition $Hom(Y, X) \to Hom(Y, \tilde{X}) \to Hom(Y, X)$ is an isomorphism. Then choose identifications of the dual so that this isomorphism is the identity map. □

Note that this proposition is still true if X is projective, except that the objects are no longer necessarily complexes in degree zero.

Also note that in the process of our proof, we showed that

Corollary 3.6.5.1. *If $A \to X$ is an irreducible morphism and $0 \to A \to B \to C \to 0$ is an almost split exact sequence, X is a direct summand of B.*

We are almost done; we just need a bit more handle on irreducible maps.

Proposition 3.6.6. *If $M \to N$ is irreducible, the functorially induced map $\tau M \to \tau N$ is also. Furthermore, if N is not projective, there is an irreducible morphism $\tau N \to M$.*

Proof. Irreducibility is preserved because τ is an additive equivalence.

Now given the morphism $M \to N$, the factoring property of the almost split exact sequence $0 \to \tau N \to \tilde{N} \to N \to 0$ implies the existence of a morphism $M \to \tilde{N}$, which by the irreducibility of $M \to N$ must admit a section. Hence $\tilde{N} \simeq M \oplus \tilde{N}'$.

Now, projection induces a morphism $\tau N \to M$. $\tau N \to M$ is irreducible, because if it factors with some $\tau N \to Z \to M$ the lifting property of almost split exact sequences implies there exists a map $M \to Z$ which is a one-sided inverse to $Z \to M$. □

Lemma 3.6.7. *There is no $0 \neq X$ indecomposable in $k\vec{Q} - mod$ so $\tau^k X \simeq X$.*

Proof. Because τ lowers degree on projective objects, take a projective resolution and compute.

Equivalently, prop. 3.6.8 below establishes τ as a categorified Coxeter element. We can appeal to the classical fact that the orbit of every positive root under a Coxeter element contains a nonpositive root, hence for some ℓ we must have $\tau^\ell X$ with lower grading than X. □

Corollary 3.6.7.1. *For $\vec{Q}$ Dynkin, every irreducible object in $Rep_k(\vec{Q})$ is the image of iterated inverse Auslander-Reiten translation of a projective object.*

Proof. For a Dynkin quiver, $\dim Hom(Simple(i), Simple(i)) = 1$ clearly. Now BGP reflection preserves the dimension of Hom-spaces, so for any indecomposable X $\dim Hom(X, X) = 1$. Hence, we can apply the previous proposition, to show that τ has no fixed points.

Now Dynkin quivers have finitely many indecomposables in fixed degree and τ only lowers degree on projective objects, hence apply the pigeonhole principle. □

Consequently, we can apply Auslander-Reiten translation to build up the Auslander-Reiten quiver. We will discuss this further shortly.

Note that there is an alternate description of Auslander-Reiten translation for quiver representation categories in terms of simple reflection. This justifies calling Auslander-Reiten translation a 'Coxeter functor', for $RS^+_{i_m} \dots RS^+_{i_1}$ decategorified is a Coxeter element.

Proposition 3.6.8. *Let $\vec{Q}$ Dynkin; consider $\mathcal{D}(k\vec{Q} - mod)$. There is a sequence $\{i_j\}$ so that $RS^+_{i_m} \dots RS^+_{i_1} \simeq \tau$.*

Proof. Let n the number of vertices. There exists a sequence $i_1, \dots, i_n$, all i_j distinct, such that at each step i_j is a sink. Hence, under $RS^+_{i_n} \dots RS^+_{i_!}$ every edge has orientation reversed twice. Therefore $C := RS^+_{i_n} RS^+_{i_{n-1}} \dots RS^+_{i_1}$ is an (invertible) endofunctor on $\mathcal{D}(Rep_k(\vec{Q}))$; call this a Coxeter functor.

Now we can calculate $CProj(i) = \Omega Inj(i)$. Choose a sink v on which the representation $Proj(i)$ is nontrivial. If $v \neq i$, there is a unique edge j on which $Proj(i)$ is nontrivial, and the associated map is $id : k \to k$. Hence $fib(k \to k) = 0$. Inductively, if $j \neq i$ and there is a path $i \to \dots \to j$, $(CProj(i))_j = 0$.

Conversely, suppose j lies on a path $j \to \dots \to i$. $fib(0 \to k) = \Omega k$, so inductively $(CProj(i))_j = \Omega k$ and the maps on the path are the identity.

Finally, suppose there is no path $j \to i$ or $i \to j$. Then $fib(0 \to 0) = 0$; inductively, $(CProj(i))_j = 0$.

So, putting this all together, $CProj(i) = \Omega Inj(i)$, as desired. Now by the same argument as the last part of prop. 3.6.3, C is a Serre functor. By the Yoneda lemma, all Serre functors are isomorphic, so $C \simeq \tau$. $\square$

3.7 Proof of the combinatorial description of the Auslander-Reiten quiver, and the theory of height functions

We are going to do one final example calculation before doing the proof.

Example 3.7.1. Consider the quiver $\vec{A}_3$,

$$1 \longrightarrow 2 \longrightarrow 3$$

which has indecomposable projectives $k \to k \to k$, $0 \to k \to k$, $0 \to 0 \to k$.

1. The commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & k \arrow[d, "id"] \\
0 \arrow[r] \arrow[d] & k \arrow[r, "id"] \arrow[d, "id"] & k \arrow[d, "id"] \\
k \arrow[r, "id"] & k \arrow[r, "id"] & k
\end{tikzcd}$$

Makes it clear that there is a sequence of irreducible morphisms $(0 \to 0 \to k) \to (0 \to k \to k) \to (k \to k \to k)$.

2. Now we can apply inverse Auslander-Reiten translation. We have, viewing opposite representations on the quiver as representations on the opposite quiver

$$\tau^{-1}(0 \to 0 \to k) = TrD(0 \to 0 \to k) = Tr(0 \leftarrow 0 \leftarrow k)$$

We have a projective resolution $0 \to (k \leftarrow k \leftarrow 0) \to (k \leftarrow k \leftarrow k) \to (0 \leftarrow 0 \leftarrow k) \to 0$, hence because $Proj(i)^\star = Proj(i)^{op}$,

$$\tau^{-1}(0 \to 0 \to k) = cofib((0 \to 0 \to k) \to (0 \to k \to k)) = 0 \to k \to 0$$

3. Likewise, we can compute $\tau^{-1}(0 \to k \to k) = (k \to k \to 0)$ and $\tau^{-1}(k \to k \to k) = (k \to 0 \to 0)$.

4. Earlier, we showed that if $A \to X$ is irreducible, then X is a summand of B in the almost split exact sequence $0 \to A \to B \to \tau^{-1}C \to 0$. Hence we know all morphisms and we can write down the Auslander-Reiten quiver $AR(Rep_k(\vec{A}_3))$,

$$\begin{tikzcd}[column sep=tiny]
 & & (0 \to k \to 0) \arrow[dr] & & \\
 & (0 \to k \to k) \arrow[ur] \arrow[dr] & & (k \to k \to 0) \arrow[dr] & \\
(0 \to 0 \to k) \arrow[ur] & & (k \to k \to k) \arrow[ur] & & (k \to 0 \to 0)
\end{tikzcd}$$

Now, we are going to exactly mimic the ideas we used in the above example more abstractly to prove the previous proposition. First, we are going to calculate the projective indecomposables and the irreducible morphisms between them. Then we will use translation invariance under BGP reflection to construct the entire Auslander-Reiten quiver. Then we will define a grading on the Auslander-Reiten quiver to show that our calculation gives the same result as the combinatorial description does.

Theorem 3.7.2. *Let $\hat{D}$ be a connected component of the quiver $D \times \mathbb{Z}$, with D Dynkin. Then $AR(\mathcal{D}(Rep_k(\vec{D}^\Omega)) = \hat{D}$.*

Proof. If there is an edge $i \to j$, there is a clear irreducible morphism $Proj(j) \to Proj(i)$ which is the identity map between vertices other than j and the 0 map on j.

Because $\dim Hom(Proj(j), Proj(k)) \leqslant 1$, these constitute a full list of irreducibles between projective indecomposables: if $\dim Hom(Proj(j), Proj(k)) = 1$, and j and k are not adjacent, there is an intermediate vertex i such that any nontrivial map factors $Proj(j) \to Proj(i) \to Proj(k)$, contradicting its irreducibility.

So by invariance of the Auslander-Reiten quiver under BGP reflection, the quiver is composed of translates of copies of $\vec{D}^\Omega$s.

We can construct morphisms other than those between our projective slice by repeatedly applying inverse Auslander-Reiten translation.

Functoriality of τ^{-1} means that when applied to our copy of $\vec{D}^\Omega$, it produces another copy of $\vec{D}^\Omega$. The connecting irreducible morphisms factor through the sequences $0 \to X \to \tilde{X} \to \tau^{-1}X \to 0$.

Suppose $X = Proj(i)$. If there is an edge $j \to i$, by corollary 3.6.5.1, the irreducible map $Proj(i) \to Proj(j)$ implies $Proj(j)$ is a direct summand of $\tilde{X}$ and there is an irreducible morphism $Proj(j) \to \tau^{-1}Proj(i)$.

By translation invariance under BGP reflection, all such arrows must be of this form; hence, this constructs the Auslander-Reiten quiver.

Now we would like to show that $AR(\mathcal{D}(Rep_k(\vec{D}^\Omega)) = \hat{D}$. Grade $AR(\mathcal{D}(Rep_k(\vec{D}^\Omega))$ by $D \times \mathbb{Z}$ as follows. Choose a sink, v, of our projective slice of $\vec{D}^\Omega \subset AR(\mathcal{D}(Rep_k(\vec{D}^\Omega))$, and declare that it has grading $(i, 0)$, where i is the vertex in D to which it corresponds.

Then

- If there is a path $v \to v'$, and the minimum path $v \to v'$ is of length ℓ, label v' by (j, ℓ) where v' is the image of Auslander-Reiten translation of $j \in \vec{D}^\Omega$, considered as the projective slice $\subset AR(\mathcal{D}(Rep_k(\vec{D}^\Omega))$ discussed above;
- If there is a path $v' \to v$, and the minimum path $v' \to v$ is of length ℓ, label v' by $(j, -\ell)$;
- If there are no paths $v \to v'$ or $v' \to v$, declare v' to have grading $(j, 0)$.

This grading is well-defined because the Auslander-Reiten quiver is acyclic because $\tau^k X \not\cong X$, per prop. 3.6.7.

It suffices to show that if $i - j$ in D, then there is an irreducible morphism $(i, n) \to (j, n+1)$. By translation invariance, we can assume $(i, n) = Proj(i)$. If $j \to i \in \vec{D}^\Omega$, then $Proj(i) \to Proj(j)$ constitutes the necessary morphism. If $i \to j \in \vec{D}^\Omega$, then $Proj(i) \to \tau^{-1}Proj(j)$ suffices.

Therefore $AR(\mathcal{D}(Rep_k(\vec{Q})) = \hat{D}$. □

Corollary 3.7.2.1. *Hence, $AR(\mathcal{D}(Rep_k(\vec{Q}))/\Sigma^2)$ is a connected component of $D \times \mathbb{Z}_{2h}$, where h is the order of τ on indecomposables in $\mathcal{D}(Rep_k(\vec{Q}))/\Sigma^2$.*

Definition 3.7.3. h is the **Coxeter number**.

To justify that this matches up with the usual definition of the Coxeter number, it suffices to appeal to the classical theorem that a root system has $|D|h$ roots (indecomposable objects).

This further justifies that τ can be identified as a **Coxeter functor**, as previously discussed in prop 3.6.8.

Note also, labelling $\hat{D}$ as a connected component of $D \times \mathbb{Z}$, we have

Corollary 3.7.3.1. $\tau(i, n) = (i, n-2)$

3.8 Combinatorial theory of the Auslander-Reiten quiver

Now that we have combinatorial handle on the Auslander-Reiten quiver, we can decategorify our insights coming from the quiver representation category and hope to express combinatorial Lie theory in terms of the combinatorics of the Auslander-Reiten quiver.

First, we will explicitly describe the root system inner product in terms of the edges of the Auslander-Reiten quiver, demonstrating that the Auslander-Reiten quiver pictorially encodes all data of a root system.

Second, we will develop the theory of height functions, which let us encode ordinary Dynkin quivers as subquivers of the Auslander-Reiten quiver. We will explain how a height functions gives us a choice of simple roots. This allows us to more precisely view the Auslander-Reiten quiver as 'all possible orientations on the Dynkin diagram glued together'.

3.8.1 Root system in terms of the Auslander-Reiten quiver

Theorem 3.8.1. *Let D a Dynkin diagram, $(D\times\mathbb{Z}_{2h})_0$ the corresponding periodic Auslander-Reiten quiver. The following data constitute a root system:*

- *Roots the vertices of $\hat{D}_{cyc}$;*
- *An Euler form $\langle , \rangle_E$ defined by the equations:*

$$\langle (i,n),(j,n)\rangle = \delta_{ij} \tag{3.1}$$

$$\langle (i,n),(j,n+1)\rangle = \text{The number of paths } (i,n)\to(j,n+1) \tag{3.2}$$

$$\langle (k,m),(i,n)\rangle = [\sum_{j \text{ connected to } i \text{ in } D} \langle (k,m),(j,n+1)\rangle] - \langle (k,m),(i,n+2)\rangle \tag{3.3}$$

- *A root system inner product $\langle X,Y\rangle_R := \langle X,Y\rangle_E + \langle Y,X\rangle_E$;*
- *A vector space generated by the roots under the relations $(i,n+h)+(i,n)=0$;*

Note that the number of paths $(i,n)\to(j,n+1)$ is, in particular, one or zero.

Proof. Given previous results, it remains to prove the formula for the Euler form.

To calculate this, fix an overlying category $Rep_k(\vec{D}^{\Omega})$.

Now equations 3.1 and 3.2 follow by definition of the grading we used to give the combinatorial description of $\hat{D}_{cyc}$ in Theorem 3.7.2. Finally, because translation $(i,n)\to(i,n+2)$ is decategorified inverse Auslander-Reiten translation, equation 3.3 follows by the existence of the unique split exact sequence between X and τ^{-1} (see 3.6.5), and that every indecomposable inbetween X and $\tau^{-1}X$ is a direct summand of the middle term of the split exact sequence by corr. 3.6.5.1.

∎

Example 3.8.2. Take the example of the quiver $\vec{A}_3$, $1\to 2\to 3$ which has Coxeter number 4. We'll label objects as $X_i(j)$, where $i\in Vert(D)$ and $j\in\mathbb{Z}_{2h}$.

The Auslander-Reiten quiver has 12 vertices,

$$\begin{tikzcd}
 & & & X_1(1) \arrow[drr] & & & \\
 & X_2(0) \arrow[urr] \arrow[drr] & & & & X_2(2) \arrow[ddl] \arrow[ddr] & \\
 & & & X_3(1) \arrow[urr] & & & \\
X_1(7) \arrow[uur] & & X_3(7) \arrow[uul] & & X_3(3) \arrow[ddr] & & X_1(3) \arrow[ddl] \\
 & & & X_3(5) \arrow[dll] & & & \\
 & X_2(6) \arrow[uul] \arrow[uur] & & & & X_2(4) \arrow[ull] \arrow[dll] & \\
 & & & X_1(5) \arrow[ull] & & &
\end{tikzcd}$$

$\langle X_2(0), X_1(1)\rangle = 1 = \langle X_2(0), X_3(1)\rangle$ and $\langle X_1(1), X_2(2)\rangle = 1 = \langle X_3(1), X_2(2)\rangle$ determine the root system; we can get the rest by induction.

This prescription makes calculating a lot easier than the alternative of calculating the categorical Auslander-Reiten translation of all objects and then the dimensions of their respective $RHom$-spaces.

3.8.2 A quick introduction to height functions.

The Auslander-Reiten quiver forgot the orientation on our Dynkin diagram. What data, combinatorially, remembers the orientation we imposed? The key observation is that, for all possible orientations Ω on D, the Auslander-Reiten quiver has $\vec{D}^{\Omega}$ as a subquiver.

Definition 3.8.3. Let D an ADE Dynkin diagram. A **height function** is a function $h : Vert(D) \to \mathbb{Z}$ or $Vert(D) \to \mathbb{Z}_{2h}$, such that if i is adjacent to j in D then $h(i) - h(j) = \pm 1$.

A height function induces an orientation on D in a straightforward way.

Definition 3.8.4. Given a height function h on a Dynkin diagram D, define a quiver $\vec{D}_h$ such that $i \to j$ if $h(j) - h(i) = 1$, and $j \to i$ if $h(i) - h(j) = 1$.

But it also does better: it defines $\vec{D}_h$ as a subquiver of the Auslander-Reiten quiver. If $h : D \to \mathbb{Z}_{2h}$, then the graph of h presents $\vec{D}_h \subset \hat{D}_{cyc}$. By doing so, a height function chooses a set of simple roots: namely, the vertices of the subquiver $\hat{D}_h$.

We'll use height functions to understand how we can give a 'choiceless' construction of the the category $D^b(Rep_k(\vec{D}^{\Omega}))/T^2$.

3.9 A purely formal theory of BGP functors

This section makes precise the way in which BGP functors are natural orientation-reversing maps, by building them from a totally formal and general equivalence of stable-∞ categories. Our treatment is a much abbreviated version of the one given in [DJW2], which goes far beyond only the case of classical BGP reflection functors.

To do so, it extensively uses the general tool of the Grothendieck construction, which we will avoid to make the section more accessible. Instead, we'll explicitly construct what we could have instead constructed much more abstractly.

This section lies outside the main development of this thesis and can be skipped.

Definition 3.9.1. Let $\mathcal{F} : \mathcal{B} \to \mathcal{A}$ an exact functor of ∞-categories.

Let $\mathcal{L}_\star(\mathcal{F})$ the category defined by the pullback in $(\infty,1)Cat$, the category of $(\infty,1)$ categories:

$$\begin{tikzcd}
\mathcal{L}_\star(\mathcal{F}) \arrow[r] \arrow[d] & Fun(\vec{A}^2, \mathcal{A}) \arrow[d] \\
\mathcal{B} \arrow[r, "\mathcal{F}"] & Fun(\{0\}, \mathcal{A}) \simeq \mathcal{A}
\end{tikzcd}$$

And let $\mathcal{L}^\star(\mathcal{F})$ the category defined by the pullback

$$\begin{tikzcd}
\mathcal{L}_\star(\mathcal{F}) \arrow[r] \arrow[d] & Fun(\vec{A}^2, \mathcal{A}) \arrow[d] \\
\mathcal{B} \arrow[r, "\mathcal{F}"] & Fun(\{1\}, \mathcal{A}) \simeq \mathcal{A}
\end{tikzcd}$$

Proposition 3.9.2. *There are inverse equivalences $S^- : \mathcal{L}^\star(\mathcal{F}) \to \mathcal{L}_\star(\mathcal{F})$, $S^+ : \mathcal{L}_\star(\mathcal{F}) \to \mathcal{L}^\star(\mathcal{F})$.*

Proof. Define S^-, S^+ by the Cartesian cube below, where because $cofib$ and fib are inverse S^- and S^+ are also.

$$\begin{tikzcd}
& \mathcal{B} \arrow[rr, "\mathcal{F}"] \arrow[dd, equal] & & Fun(\{1\}, \mathcal{A}) \arrow[dd, equal] \\
\mathcal{L}^\star(\mathcal{F}) \arrow[ur] \arrow[rr] \arrow[dd, shift right, "S^-"'] & & Fun(\vec{A}^2 = \{0 \to 1\}, \mathcal{A}) \arrow[ur] \arrow[dd, shift right, "cofib"'] & \\
& \mathcal{B} \arrow[rr, dotted] & & Fun(\{1\}, \mathcal{A}) \\
\mathcal{L}_\star(\mathcal{F}) \arrow[ur, dotted] \arrow[rr] \arrow[uu, shift right, "S^+"'] & & Fun(\vec{A}^2 = \{1 \to 2\}, \mathcal{A}) \arrow[ur] \arrow[uu, shift right, "fib"'] &
\end{tikzcd}$$

□

This phenomenon is very general, and somehow can be thought of as 'generalised BGP reflection', in the following sense. $\mathcal{L}_\star(\mathcal{F})$ is roughly the category with objects $(B, \mathcal{F}(B) \to A)$, where $B \in \mathcal{B}$ and $A \in \mathcal{A}$, and $\mathcal{L}^\star(\mathcal{F})$ is the category with objects $(B, A \to \mathcal{F}(B))$.

So we can view $\mathcal{L}_\star(\mathcal{F})$ as a category of diagrams with a new 'source' ($\mathcal{B}$) glued in along $\mathcal{F}$, and $\mathcal{L}^\star(\mathcal{F})$ as the same category of diagrams except $\mathcal{B}$ is glued in as a sink with one arrow out of it.

We will be interested in a special case of the theorem, namely regular BGP reflection.

Note that regular BGP reflection applies to a vertex with many arrows out of (or into) it. So we will need a formal tool which lets us acquire these many arrows. The tool will be a special type of quiver:

Definition 3.9.3. Call a quiver $\vec{Q}$ **bipartite on edges** if there exists a decomposition $Vert(\vec{Q}) = Q_0 \sqcup Q_1$, where every edge of $\vec{Q}$ has source in Q_0 and target in Q_1.

We are going to define some useful categories, which are ordinarily constructed via the Grothendieck construction. But in this special case, we will instead define them via pushouts:

Definition 3.9.4. Let $\vec{Q}$ bipartite on edges, $\mathcal{F} : Free(\vec{Q}) \to (\infty, 1)Cat$ a functor. Then the upper and lower Grothendieck constructions, $\int_Q \mathcal{F}$ and $\int^Q \mathcal{F}$, are the pushouts

$$\begin{tikzcd} \coprod_{\alpha: q_0 \to q_1} \mathcal{F}(q_0) \arrow[r, "\mathcal{F}(m_\alpha)"] \arrow[d] & \coprod_{q_1 \in Q_1} \mathcal{F}(q_1) \arrow[d] \\ \coprod_{q_0 \in Q_0} \mathcal{F}(q_0) \times q_0 \backslash Free(\vec{Q}) \arrow[r] & \int_Q \mathcal{F} \end{tikzcd} \qquad \begin{tikzcd} \coprod_{\alpha: q_0 \to q_1} \mathcal{F}(q_0) \arrow[r, "\mathcal{F}(m_\alpha)"] \arrow[d] & \coprod_{q_1 \in Q_1} \mathcal{F}(q_1) \arrow[d] \\ \coprod_{q_0 \in Q_0} \mathcal{F}(q_0) \times Free(\vec{Q})^{op}/q_0 \arrow[r] & \int_Q \mathcal{F} \end{tikzcd} \tag{3.4}$$

where $q_0 \backslash Free(\vec{Q})$ and $Free(\vec{Q})/q_0$ are the under- and over- slice categories: that is, an object of $q_0 \backslash Free(\vec{Q})$ is a vertex q with a map $q_0 \to q$, and a morphism $q \to q'$ is the subset of morphisms in $Free(\vec{Q})$ which make the following diagram commute:

$$\begin{tikzcd} q \arrow[rr] & & q' \\ & q_0 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Proposition 3.9.5. *Let $\vec{Q}$ a finite quiver bipartite on edges, $\mathcal{F} : Free(\vec{Q}) \to Cat_\infty$ a functor. Let $\mathcal{D}$ the derived stable ∞-category of k-vector spaces. There is an exact functor $F : Hom(\coprod_{v \in Q_1} \mathcal{F}(v), \mathcal{D}) \to Hom(\coprod_{v \in Q_0} \mathcal{F}(v), \mathcal{D})$, and an equivalence*

$$\mathcal{L}^\star(F) \simeq Hom(\int_Q \mathcal{F}, \mathcal{D}) \leftrightarrows Hom(\int^Q \mathcal{F}, \mathcal{D}) \simeq \mathcal{L}_\star(F)$$

Proof. Define F as the composition of functors

$$\begin{tikzcd} Hom(\coprod_{q_1 \in Q_1} \mathcal{F}(v), \mathcal{D}) \arrow[d, "\{(\mathcal{F}(e))^\star\}_e"] & \\ \oplus_{q_1 \in Q_1} \oplus_{\alpha: q_0 \to q_1} Hom(\mathcal{F}(q_0), \mathcal{D}) \arrow[r] & \oplus_{q_0 \in Q_0} Hom(\mathcal{F}(q_0), \mathcal{D}) \simeq Hom(\coprod_{q_0 \in Q_0} \mathcal{F}(q_0), \mathcal{D}) \end{tikzcd}$$

Taking $X \to Hom(X, \mathcal{D})$ transforms the diagrams of eq. 3.4 into pullback diagrams. It suffices to show that $\mathcal{L}_\star(F)$ satisfies the universal property of $Hom(\int_Q \mathcal{F}, \mathcal{D})$, which can be done by tracing definitions. $\square$

Now we will apply this result in a very special case.

Definition 3.9.6. Let $\vec{K}_d$ be the quiver with two vertices, 0 and 1, and d edges pointing from $0 \to 1$. It is bipartite on edges.

One can compute

Fact 3.9.7. *Let $Free(\vec{Q})$ a quiver considered as a free ∞-category, and $f_1, \ldots, f_d : 0 \to Free(\vec{Q})$ functors corresponding to classifying vertices v_i. Then $\int_{\vec{K}_d} f$ is the (free category associated to the) quiver with a vertex adjoined with edges into the v_i, and $\int^{\vec{K}_d} f$ is the quiver with a vertex adjoined with edges out of the v_i.*

Now definitionally $Hom(Free(\vec{Q}), \mathcal{D}) \simeq \mathcal{D}(Rep_k(\vec{Q}))$. We call the induced equivalences S^+, S^- between $\mathcal{D}(Rep_k(\int_{K_d} f))$ and $\mathcal{D}(Rep_k(\int^{K_d} f)$ the **BGP reflection functors**.

Remark 3.9.8. This construction is satisfying in that it justifies our earlier intuition that BGP reflection functors are somehow a natural way to reverse arrows. Practically it is overkill for defining ordinary BGP reflection functors. But it would be very interesting if one could use similar formal techniques to directly define the Serre functor.

Chapter 4

Choiceless and choiceful ways to construct the root category

For a Dynkin diagram D, the corresponding root categories $D^b(Rep_k(\vec{D}^\Omega))/T^2$ are *equivalent* for all choices of Ω, related by the derived BGP reflection functors.

One can ask: does the orientation we started with matter, and how can we keep track of it? The answer is that the data of an orientation is equivalentlty that of a 't-structure', as we will discuss in the next section.

Then, the results we have established so far do not depend on the orientation, just on the root category. Can we construct a representative of the root category, defined without a preferred choice of orientation/t-structure, equivalent to all these categories? The answer is yes, and we will construct this category in two different ways.

Then we will study a natural enhancement of the notion of t-stucture, known as Bridgeland stability, and construct the root category in a way such that it acquires a preferred notion of Bridgeland stability.

4.1 What structure preserves orientation?

The choice of orientation on a Dynkin diagram is equivalent to a choice of simple roots for the root system, for they correspond to representations $Simple(i)$ of the quiver $\vec{D}^\Omega$. We know the set of simple representations is not preserved by BGP reflection. Yet in Lie theory, we know that a distinguished set of simple roots is very significant. Once we've made a choice of orientation, we should keep track of it.

We would like a structure which allows us to do so. The key idea is that though the derived categories $\mathcal{D}(Rep_k(\vec{D}^\Omega))$ are equivalent for different Ω, the categories $Rep_k(\vec{D}^\Omega)$ are generally not: for instance, decategorified, they correspond to different choices of simple roots.

Hence, the choice of an orientation is equivalently the choice of a distinguished abelian subcategory $\mathcal{A} \subset \mathcal{D}(Rep_k(\vec{D}^\Omega))$. We can formalise this with the idea of a t-structure.

Definition 4.1.1. Let $\mathcal{C}$ a triangulated category. A **t-structure** on $\mathcal{C}$ is a pair of full subcategories $\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0}$ such that

1. Let $X \in \mathcal{C}_{\geqslant 0}$, $Y \in \mathcal{C}_{\leqslant 0}$. Then $Hom(X, T^{-1}Y) = 0$.
2. $T\mathcal{C}_{\geqslant 0} \subset \mathcal{C}_{\geqslant 0}$, $T^{-1}\mathcal{C}_{\leqslant 0} \subset \mathcal{C}_{\leqslant 0}$.

Given a t-structure, let $\mathcal{C}_{\geqslant n} := T^n\mathcal{C}_{\geqslant 0}$ and $\mathcal{C}_{\leqslant -n} := T^{-n}\mathcal{C}_{\leqslant 0}$.

A t-structure on a stable-∞ category $\mathcal{C}$ is a t-structure on its homotopy category. In this case let $\mathcal{C}_{\geqslant n}$ denote the full subcategory spanned by objects which descend in the homotopy category to objects of $h\mathcal{C}_{\geqslant n}$.

Definition 4.1.2. The **heart** of a stable ∞-category $\mathcal{C}$ equipped with a t-structure, $\mathcal{C}^{\heartsuit}$, is the full subcategory $\mathcal{C}_{\leqslant 0} \cap \mathcal{C}_{\geqslant 0} \subset \mathcal{C}$.

Example 4.1.3. Say $\mathcal{A}$ is an abelian category with enough projectives, $D(\mathcal{A})$ its associated triangulated derived category. There is a natural t-structure, given by letting

1. $D(\mathcal{A})_{\geqslant 0}$ be those objects X such that $H^i(X) = 0$ for $i > 0$;
2. $D(\mathcal{A})_{\leqslant 0}$ be those objects X such that $H^i(X) = 0$ for $i < 0$.

In this case, $\mathcal{D}(\mathcal{A})^{\heartsuit} \simeq \mathcal{A}$ as abelian categories.

Hence, the natural t-structure induced from $D(Rep_k(\vec{D}^{\Omega}))$ distinguishes $Rep_k(\vec{D}^{\Omega})$ as its heart.

Note that t-structure also distinguishes a notion of cohomology;

Definition 4.1.4.
- View a triangulated category $\mathcal{C}$ as chain complexes, with weak equivalences the quasi-isomorphisms. Let $\tau^{\leqslant 0} : \mathcal{C} \to \mathcal{C}_{\leqslant 0}$ the functor truncating chain complexes in degree > 0, $\tau^{\geqslant 0}$ the functor truncating chain complexes in degree < 0.
- Define $H^n := \tau^{\leqslant 0}\tau^{\geqslant 0}\Sigma^n$, the **n-th cohomology functor**.

We won't immediately apply the idea of a t-structure; but knowing what it is allows us to understand the limitations on how we might construct a choiceless root category. In particular, we *won't* be able to construct it as the derived category of some nice abelian category.

4.1.1 The two-periodic case

Note that our definition 3.2.5 of $D^b(\mathcal{A})/T^2$ implicitly relied on the natural t-structure coming from the derived category, for we modelled $D^b(\mathcal{A})/T^2$ as some subcategory of complexes in $\mathcal{A}$ up to weak equivalence. But we can give a definition independent of t-structure.

Definition 4.1.5. Let $\mathcal{T}$ a stable-∞ category. Define a subcategory, $\mathcal{T}/\Sigma^2$, as the full subcategory of objects $\mathcal{O}$ so that $\Sigma^2\mathcal{O} \simeq \mathcal{O}$.

Proposition 4.1.6. *$\mathcal{T}/\Sigma^2$ is a stable-∞ category.*

Proof. A subcategory of a stable ∞ category is stable if it contains zero and is closed under fibres and cofibres. The category clearly contains 0 and is closed under Ω, so it suffices to check if $\mathcal{T}/\Sigma^2$ contains all cofibres.

Σ commutes with taking cofibres up to unique isomorphism, because we have a diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & cofib(X \to Y) \arrow[r] \arrow[d] & \Sigma X \arrow[r] \arrow[d] & 0 \arrow[d] \\
 & 0 \arrow[r] & \Sigma Y \arrow[r] & \Sigma cofib(X \to Y)
\end{tikzcd}$$

So $\Sigma cofib(X \to Y) \simeq cofib(\Sigma X \to \Sigma Y)$. Hence $\Sigma^2 cofib(X \to Y) \simeq cofib(\Sigma^2 X \to \Sigma^2 Y) \simeq cofib(X \to Y)$, so the category contains all cofibres and is stable. $\square$

As defined, the notion of t-structures is *not* very well suited to a two-periodic stable ∞-categories.

Suppose you had a t-structure on a triangulated category. Then in this case, $\mathcal{C}_{\geqslant 0} = T^2\mathcal{C}_{\geqslant 0} \subset T\mathcal{C}_{\geqslant 0} \subset \mathcal{C}_{\geqslant 0}$. So $\mathcal{C}_{\geqslant 0} = T\mathcal{C}_{\geqslant 0}$, and $\mathcal{C}_{\leqslant 0} = T\mathcal{C}_{\leqslant 0}$. Therefore if $X \in \mathcal{C}_{\geqslant 0}, Y \in \mathcal{C}_{\leqslant 0}$, then $Hom(X, Y) = 0$, because $Y = T\tilde{Y}$ for some $\tilde{Y}$.

In particular, therefore if $X \in \mathcal{C}_{\geqslant 0} \cap \mathcal{C}_{\leqslant 0}$, then $Hom(X, X) = 0$. So $\mathcal{C}^{\heartsuit}$ would be necessarily *empty*. Therefore, while the notion of a t-structure captures the notion of orientation for $D^b(Rep_k(\vec{D}^{\Omega}))$, it is ill-suited to describe $D^b(Rep_k(\vec{D}^{\Omega}))/T^2$.

That is not to say that moving to the two-periodic subcategory forgets the original choice of orientation. In general the question of what data of the t-structure remains after moving to the two-periodic subcategory is complicated; we will deal only with our very specific case of interest.

Proposition 4.1.7. *If $\mathcal{A}$ is hereditary, $\mathcal{D}^b(\mathcal{A})/T^2$ has a natural $\mathbb{Z}/2\mathbb{Z}$ grading, inherited from the natural t-structure.*

Proof. Model $\mathcal{D}^b(\mathcal{A})/T^2$ as two-periodic objects in $Kom(\mathcal{A})$ with weak equivalences inverted. It has a natural grading coming from the underlying category of chain complexes.

That is, declare $0 \neq A \in \mathcal{D}^b(\mathcal{A})/T^2$ to have degree 0 if A is weakly equivalent to a complex which is zero in odd degrees, and $0 \neq A \in \mathcal{D}^b(\mathcal{A})/T^2$ to have degree 1 if it is weakly equivalent to a complex zero in even degrees.

Now because $\mathcal{A}$ is hereditary, applying prop. 3.2.4, every object in $\mathcal{D}^b(\mathcal{A})/T^2$ is a direct sum of graded objects, as desired. □

Remark 4.1.8. Note that though the $\mathbb{Z}/2\mathbb{Z}$ grading lets us pick out a distinguished abelian subcategory, we no longer have any 'Hom-vanishing' condition. That is, if $\deg X = 0$ and $\deg Y = 1$, we can still have $Hom(X, TY) \neq 0$.

For instance, in $D_b(Rep_k(1 \to 2))/\Sigma^2$,

$$Hom(Simple(1), TSimple(2)) \simeq Ext^1(Simple(1), Simple(2))$$

, which has dimension one.

4.2 Orientation-free root category via the Auslander-Reiten quiver

The first way in which we will construct the root category without orientation is from the Auslander-Reiten quiver of the root category.

The idea, due to [KT3], is as follows. The Auslander-Reiten quiver $\hat{D}_{cyc}$ associated to the root category of $\vec{D}^{\Omega}$ contains every possible orientation on D as a subquiver. Clearly, a represention of $\hat{D}^{cyc}$ can contain much more data than can one of $\vec{D}^{\Omega}$. However, we can look for a subcategory of representations of $\hat{D}_{cyc}$, which are those 'determined by their value on any Dynkin subquiver of $\hat{D}_{cyc}$'. What should such a subcategory look like?

In the previous section, we were able to describe the root system inner product in terms of the Auslander-Reiten quiver by the data of the inner product on a Dynkin subquiver, and then some relation involving decategorified Auslander-Reiten translation. One idea, which will turn out to be successful, is to use a categorified version of this relation to define our candidate subcategory.

This definition will require some preliminaries.

Definition 4.2.1. Let $\vec{Q}$ a quiver with at most one edge between any two vertices, and no self-loops. Define an endofunctor $m : Rep_k(\vec{Q}) \to Rep_k(\vec{Q})$ as follows.

Let V a representation. Denote the vector space at a vertex q by V_q and the map at an edge e by m_e.

- At a vertex q, let $mV_q := \bigoplus_{e:q\to q'} V_{q'}$;
- At an edge $e : q \to q'$, let $m(m_e) : mV_q \to mV_{q'}$ be the map

$$V_{q'} \subset \bigoplus_{e:q\to q''} V_{q''} \xrightarrow{\bigoplus_{e:q\to q'} m_e} \bigoplus_{e:q'\to q'''} V_{q'''}$$

Now let $\mu : V \to V'$ a map of quiver representations. Let $m(\mu) : mV_q = \bigoplus_{e:q\to q'} V_{q'} \to mV'_q = \bigoplus_{e:q\to q''} V'_{q''}$ be the direct sum of the maps $\mu : V_{q'} \to V'_{q''}$.

The assignment is functorial; given a morphism $\mu : V \to V'$ of quiver representations, the induced map $m(\mu) : mV_q \to mV'_q$ is given by the direct sums of the maps $\mu : V_{q'} \to V'_{q'}$, and likewise on edges.

More informally, what is m? On any path, it 'pushes all vector spaces one back', so e.g. $m(V_1 \to V_2 \to V_3) = V_2 \to V_3 \to 0$. On cyclic paths, it rotates the path backward by one unit:

$$m\left(\begin{tikzcd} & V_1 \arrow[dr] & \\ V_3 \arrow[ur] & & V_2 \arrow[ll] \end{tikzcd}\right) = \begin{tikzcd} & V_2 \arrow[dr] & \\ V_1 \arrow[ur] & & V_3 \arrow[ll] \end{tikzcd}$$

Definition 4.2.2. Suppose there is at most one edge between any two vertices in $\vec{Q}$, and no self-loops. Then there is a distinguished map, $s_m : V \to mV$, given on vertices by

$$(s_m)_q : V_q \to mV_q = \bigoplus_{e:q\to q'} V_{q'}$$
$$(s_m)_q = \bigoplus_{e:q\to q'} m_e$$

This is a map of quiver representations, because the square

$$\begin{tikzcd} V_q \arrow[r, "m_{q\to q'}"] \arrow[d, "\bigoplus_{e:q\to q''} m_e"] & V_{q'} \arrow[d, "\bigoplus_{e:q'\to q'''} m_e"] \\ \bigoplus_{e:q\to q''} V_{q''} \arrow[r, "\bigoplus_{e:q'\to q'''} m_e"] & \bigoplus_{e:q'\to q'''} V_{q'''} \end{tikzcd}$$

commutes.

Definition 4.2.3. Now let $\vec{Q}$ be the Auslander-Reiten quiver. We know τ, decategorified, is an automorphism of the underlying quiver. This induces a distinguished endofunctor $\tilde{\tau} : Rep_k(\vec{Q}) \to Rep_k(\vec{Q})$, given just by shifting vertices $(i, n) \to (i, n-2)$.

There is a morphism $t_m : mV \to \tilde{\tau}V$, given by sending $mV_q \to \tilde{\tau}V_q$ by the map $\bigoplus_{q'\to\tau q} : \bigoplus_{q':q\to q'} V_{q'} \to V_{\tau q}$.

Remark 4.2.4. Though we defined the periodic Auslander-Reiten quiver as the Auslander-Reiten quiver of some category, and choosing that category required choosing an orientation, the combinatorial description 3.8.1 allows us to construct the periodic Auslander-Reiten quiver with no choice of orientation needed.

Example 4.2.5. Note that even though we informally described m as roughly 'shift one to the left' and $\tilde{\tau}$ as 'shift two to the left', m^2 is not the same as $\tilde{\tau}$. The simplest example for which they disagree is for the Auslander-Reiten quiver of $\vec{A}_3$, which we calculated in example 3.8.2.

A quiver representation X looks like

$$X = \begin{tikzcd}
 & & & X_1(1) \arrow[drr] & & & \\
 & X_2(0) \arrow[urr] \arrow[drr] & & & & X_2(2) \arrow[dl] \arrow[dr] & \\
 & & & X_3(1) \arrow[urr] & & & \\
X_1(7) \arrow[ur] & & X_3(7) \arrow[ul] & & X_3(3) \arrow[dr] & & X_1(3) \arrow[dl] \\
 & & & X_3(5) \arrow[dll] & & & \\
 & X_2(6) \arrow[ul] \arrow[ur] & & & & X_2(4) \arrow[ull] \arrow[dll] & \\
 & & & X_1(5) \arrow[ull] & & &
\end{tikzcd}$$

Hence we have $\tilde{\tau}X$

$$\tilde{\tau}X = \begin{tikzcd}
 & & & X_1(3) \arrow[drr] & & & \\
 & X_2(2) \arrow[urr] \arrow[drr] & & & & X_2(4) \arrow[dl] \arrow[dr] & \\
 & & & X_3(3) \arrow[urr] & & & \\
X_1(1) \arrow[ur] & & X_3(1) \arrow[ul] & & X_3(5) \arrow[dr] & & X_1(5) \arrow[dl] \\
 & & & X_3(7) \arrow[dll] & & & \\
 & X_2(0) \arrow[ul] \arrow[ur] & & & & X_2(6) \arrow[ull] \arrow[dll] & \\
 & & & X_1(7) \arrow[ull] & & &
\end{tikzcd}$$

However, mX looks like

$$mX = \begin{tikzcd}
 & & & X_2(2) \arrow[drr] & & & \\
 & X_1(1) \oplus X_3(1) \arrow[urr] \arrow[drr] & & & & X_1(3) \oplus X_3(3) \arrow[ddl] \arrow[ddr] & \\
 & & & X_2(2) \arrow[urr] & & & \\
X_2(0) \arrow[uur] & & X_2(0) \arrow[uul] & & X_2(4) \arrow[ddr] & & X_2(4) \arrow[ddl] \\
 & & & X_2(6) \arrow[dll] & & & \\
 & X_3(7) \oplus X_1(7) \arrow[uul] \arrow[uur] & & & & X_3(5) \oplus X_1(5) \arrow[ull] \arrow[dll] & \\
 & & & X_2(6) \arrow[ull] & & &
\end{tikzcd}$$

Therefore, m^2X is

$$\begin{tikzcd}
 & & & X_1(3) \oplus X_3(3) \arrow[drr] & & & \\
 & X_2(2) \oplus X_2(2) \arrow[urr] \arrow[drr] & & & & X_2(4) \oplus X_2(4) \arrow[ddl] \arrow[ddr] & \\
 & & & X_1(3) \oplus X_3(3) \arrow[urr] & & & \\
X_1(1) \oplus X_3(1) \arrow[uur] & & X_1(1) \oplus X_3(1) \arrow[uul] & & X_1(5) \oplus X_3(5) \arrow[ddr] & & X_1(5) \oplus X_3(5) \arrow[ddl] \\
 & & & X_1(7) \oplus X_3(7) \arrow[dll] & & & \\
 & X_2(0) \oplus X_2(0) \arrow[uul] \arrow[uur] & & & & X_2(6) \oplus X_2(6) \arrow[ull] \arrow[dll] & \\
 & & & X_1(7) \oplus X_3(7) \arrow[ull] & & &
\end{tikzcd}$$

Which is not the same as $\tilde{\tau}X$. The reason why they do not agree is that when we shift by one, we 'forget' which vertex the object we are shifting came from. So m^2 corresponds to some symmetrised version of $\tilde{\tau}$.

Now, we are ready to describe our special subcategory. It is going to be the subcategory where the representation V is somehow related to the representations shifted by m and $\tilde{\tau}$.

Definition 4.2.6. An object X of $\mathcal{D}^b(Rep_k(\hat{D}_{cyc}))$ satisfies the **fundamental relation** if the induced maps $X \to mX \to \tilde{\tau}X$ define a cofibre sequence.

Let $\mathcal{D}_D \subset \mathcal{D}^b(Rep_k(\hat{D}_{cyc}))$ be the full subcategory with objects satisfying the fundamental relation.

The idea of this definition is that we can compute the represention X up to isomorphism if we know it on a Dynkin subquiver, by taking cofibres.

Proposition 4.2.7. *$\mathcal{D}_D$ is a stable $(\infty, 1)$-category.*

Proof. Because $\mathcal{D}_D$ is a subcategory of a stable $(\infty, 1)$-category, it suffices to show that $\mathcal{D}_D$ is closed under finite limits and colimits. In particular, it suffices to show m and $\tilde{\tau}$ are exact functors. $\tilde{\tau}$ is an exact functor because it is an equivalence. m is exact because it is exact on every vertex. □

Now the functor $\rho_h : \mathcal{D}^b(Rep_k(\hat{D}_{cyc})) \to \mathcal{D}^b(Rep_k(\vec{D}^{\Omega}))$, defined by pullback along a height function h, restricts to a functor $\mathcal{D}_D \to \mathcal{D}^b(Rep_k(\vec{D}^{\Omega_h}))$.

Proposition 4.2.8. *The induced functor, $\rho_h : \mathcal{D}_D \to \mathcal{D}^b(Rep_k(\vec{D}^{\Omega_h}))$ is an equivalence of stable $(\infty, 1)$-categories.*

Proof. First, I claim ρ_h is essentially surjective.

Let $v \in \vec{D}^{\Omega_h} \subset \hat{D}_{cyc}$ a source, and $X \in \mathcal{D}^b(Rep_k(\vec{D}^{\Omega_h}))$. Define $X_{\tau v} := cofib(X_v \to mX_v)$. Note mX_v only depends on X, because v is a source.

Furthermore, we have maps to $X_{\tau v}$ from all v' entering, by the restriction of the natural map $X_{v'} \subset mX_v \to cofib(X_v \to mX_v)$. Now we have defined a derived representation on $\vec{D}^{\Omega_h} \cup \{v\} \subset \hat{D}_{cyc}$. For every Dynkin subquiver and every source, we can repeat, to get a derived representation X' of $\hat{D}_{cyc}$. By the universal property of cofibres, it is the only representation of $\hat{D}_{cyc}$ such that $\rho_h X' = X$ and $X' \to mX' \to \tilde{\tau}X'$ is a cofibre sequence.

Second, I claim ρ_h is full and faithful. Let $X, Y \in \mathcal{D}_D$. There is a clear map $Hom_{\mathcal{D}_D}(X, Y) \to Hom_{\mathcal{D}^-(Rep_k(\vec{D}^{\Omega_h}))}(\rho_h X, \rho_h Y)$, given by restriction.

In the other direction, let $f \in Hom_{D^-(Rep_k(\vec{D}_h^{\Omega}))}(\rho_h X, \rho_h Y)$. Define the extension $f : cofib(X_v \to mX_v) \to cofib(Y_v \to mY_v)$ by functoriality of mapping cones. Now repeat the above procedure to explicitly construct $f' \in Hom_{\mathcal{D}_D}(X, Y)$.

□

4.3 Universal root category via the quantum projective line.

The next idea to construct a choiceless root category, originally carried out in [KT4], is by the quantum McKay correspondence.

Let $G \subset SU(2)$ a finite subgroup. The normal McKay correspondence can be understood as a correspondence between G-equivariant coherent sheaves on $\mathbb{P}^1$ and affine ADE Dynkin diagrams.

The quantum McKay correspondence provides a correspondence between 'finite quantum subgroups' of quantum $\mathfrak{sl}_2$ and regular ADE Dynkin diagrams. So the idea is to find a quantum analogue of the category of 'G-equivariant coherent sheaves on $\mathbb{P}^1$', and hope that by the quantum McKay correspondence this is the root category we wanted.

How will we construct such a quantum analogue? We will attempt to understand the projective line in terms of $SU(2)$. Then we'll construct a 'quantum projective line' analogously in terms of quantum $SU(2)$.

The definition of a 'finite quantum subgroup', or a 'quantum irreducible representation', are complicated. We are going to sketch the construction and not be particularly careful about the specifics here. For our purposes, it suffices to consider them just like subgroups are irreducible representations. See [KO] for careful definitions.

Now, the key idea. We can consider $\mathbb{P}^1 = \mathbb{P}V^\star$, where V is the fundamental representation of $SU(2)$.

We have a well-known equivalence $Coh(\mathbb{P}^1) \simeq Sym(V){-}mod_{gr}$, the category of finitely generated graded $Sym(V)$-modules.

We can notice that

Fact 4.3.1. *We can write $Sym(V) = \oplus_n V_n$, where V_n is the irreducible representation of $SU(2)$ of highest weight n.*

Hence, we can consider 'coherent sheaves on $\mathbb{P}^1$' to be finitely generated graded modules over $\oplus_n V_n$.

This immediately suggests a quantum analogue: we are just going to *define* a coherent sheaf on quantum $\mathbb{P}^1$ to be a finitely generated graded module over the sum of all quantum irreducible representations of quantum $SU(2)$.

We have

Fact 4.3.2. *Let $q = e^{\frac{i\pi}{h}}$ a root of unity. The semisimple quotient of the category of representations of $U_q\mathfrak{sl}_2$, $\mathcal{C}_q$ is a fusion category with finitely many simple objects, $1 = V_0, \ldots, V_{h-2}$.*

The fusion rule is

$$V_n \otimes V_m = \oplus_{k=|n-m|,k+n+m \text{ is even}}^{min(n+m,2(h-2)-(n+m))} V_k$$

We consider simple objects of this category to be 'quantum irreducible representations'.

We could try to consider a 'quantum symmetric algebra' $S_q = \oplus_{i=0}^{h-2} V_i$. To do so and define a category of graded modules on it, we would need to define a grading on S_q. But there are *two* natural ways to define a grading.

1. We could define $V_n V_m$ to be the image of the projection map $V_n \otimes V_m \to V_{n+m}$, the highest module in their fusion. This is seemingly the most natural fusion structure.

2. But $V_{h-2} \otimes V_n = V_{h-2-n}$. So tensoring with V_{h-2} is a nontrivial automorphism. One can check that $S_q(h) \otimes V_{h-2}$ is also graded, where h denotes a grading shift by h.

To avoid making an arbitrary choice, we should therefore define some sort of symmetric algebra where we 'make both choices at once'.

Definition 4.3.3. The **quantum structure sheaf**, $\mathcal{O}_q$, is the two-periodic complex with trivial differential

$$\begin{tikzcd} S_q \arrow[r, bend left, "0"] & S_q(h) \otimes V_{h-2} \arrow[l, bend left, "0"] \end{tikzcd}$$

So $\mathcal{O}_q$ is an algebra. We are now going to work with the category $\mathcal{D}(Rep(\mathcal{O}_q))$, the derived category of representations of $\mathcal{O}_q$.

We will not attempt to state what a 'finite quantum subgroup' G is. But the upshot is that given such a G, we can define a category, $\mathcal{D}_G(Rep(\mathcal{O}_q))$, of G-equivariant derived representations of $\mathcal{O}_q$. Then the statement of the quantum McKay correspondence in [KO] is equivalently

Theorem 4.3.4. *Let G correspond to a Dynkin diagram D. There is an exact functor*

$$\mathcal{D}_G(Rep(\mathcal{O}_q)) \to \mathcal{D}(Rep(D \times \mathbb{Z}_{2h})$$

Of course, by theorem 3.7.2, either connected component of $D \times \mathbb{Z}_{2h}$ is the two-periodic Auslander-Reiten quiver $\hat{D}_{cyc}$.

From here on out the proof strategy is straightforward: we define a subcategory of $\mathcal{D}_G(Rep(\mathcal{O}_q))$ by pulling back the subcategory defined by a connected component of $D \times \mathbb{Z}_{2h}$, then pulling back the relation of def. 4.2.6 we used in the case of constructing the root category from the Auslander-Reiten qiver.

We call this subcategory $\mathcal{D}_G(Coh(\mathbb{P}^1_q))$. By how we defined it, it's equivalent to the subcategory $\mathcal{D}_D$ (also of def. 4.2.6) which we defined in the process of constructing the root category from the Auslander-Reiten quiver. So by prop. 4.2.8, it is equivalent to the category $\mathcal{D}(Rep_k(\vec{D}^\Omega))$, as desired.

Therefore,

Theorem 4.3.5. $\mathcal{D}_G(Coh(\mathbb{P}^1_q)) \simeq \mathcal{D}^b(Rep_k(\vec{D}^\Omega))$.

4.4 Bridgeland stability

Studying t-structures in our derived categories of Dynkin quiver representations has proven interesting; for instance, we showed that t-structures were generally induced by a choice of simple roots. How the choice of simple roots can be different is a very interesting problem; for instance, it leads us to the notion of the Weyl group.

While the space of t-structures is not rigid enough to be interesting to study, we can enhance the notion of a t-structure to a more rigid 'stability condition'. We could hope that the study of stability conditions on $\mathcal{D}^b(Rep_k(\vec{D}^\Omega))$ is interesting.

The notion of a stability condition is motivated from physics. The 'category of D-branes' (i.e. boundary conditions for B-model string theories) on some space X is often taken to be derived category of coherent sheaves on X, $D^b(Coh(X))$. But not all the boundary conditions are *physical*. That is, some choice of boundary conditions/D-branes might not admit a physical theory.

There should be some natural subcategory of physical, or 'stable', D-branes. What condition should cut out such a subcategory? Bridgeland [B1], based on physical work of Douglas [D1], defined a 'Bridgeland stability condition' as a guess at what such a condition should be. Bridgeland's notion can be defined for any triangulated category, not just derived categories of coherent sheaves.

Even in this more general setting, Bridgeland stability shares many of the properties that one would expect from the physics. For instance, the moduli space of stability conditions has a complex structure.

This section and the next are a brief introduction to stability for quiver representation categories. They lay slightly outside the main development of this thesis, and can be skipped. Much appeal is made to physics intuition to compensate for the lack of rigorous proofs, which if included would have extended this section by several dozen pages.

Definition 4.4.1. Let $\mathcal{C}$ a triangulated category. A **stability condition** on $\mathcal{C}$ is (Z, P), where

1. $Z : K_0(\mathcal{C}) \to \mathbb{C}$ is a group homomorphism;

2. For each $\theta \in \mathbb{R}$, a full additive subcategory $P(\theta)$

such that:

1. For $0 \neq E \in P(\theta)$, $Z(E) = m(E)e^{i\pi\theta}$ for $m(E) \in \mathbb{R}^+$;

2. $P(\theta + 1) = TP(\theta)$;

3. If $\theta_1 > \theta_2$ and $A_j \in P(\theta_j)$, then $Hom_D(A_1, A_2) = 0$;

4. For all $E \in \mathcal{D}$ there is a finite sequence $\phi_1 > \cdots > \phi_n$ of real numbers and triangles

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[dl] \arrow[r] & \ldots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dotted] & & & & A_n \arrow[ul, dotted] &
\end{tikzcd}$$

So that $A_i \in P(\phi_i)$.

Given this definition,

Definition 4.4.2. 1. Each subcategory $P(\phi)$ is abelian; if $0 \neq E \in P(\phi)$, we call it **semistable of phase** ϕ. The simple objects of $P(\phi)$ are **stable of phase** ϕ.

2. The decomposition by E_i is unique up to isomorphism. Let $\phi^+(E) = \phi_1$, $\phi^-(E) = \phi_n$. Let $m(E) := \sum_i |Z(A_i)|$.

This definition makes manifest that the space of stability conditions has a metric.

Definition 4.4.3. *Stab*$(\mathcal{C})$ is the space of stability conditions on $\mathcal{C}$. It is a metric space, with the metric topology induced by

$$d((Z_1, P_1), (Z_2, P_2)) = sup_{0 \neq E \in \mathcal{C}}\{|\phi_2^-(E) - \phi_1^-(E)|, |\phi_2^+(E) - \phi_1^+(E)|, |\log \frac{m_2(E)}{m_1(E)}|\}$$

In fact, we expect that physical moduli space should generally be better than metric spaces: they should admit complex structures. Indeed, this is true. A hard theorem of Bridgeland which we will not prove, see [B1], establishes

Fact 4.4.4. *The space of stability conditions Stab*$(\mathcal{C})$ *is a complex manifold.*

However, the previous definition of a stability condition was a lot of data. There is a another way to think of a stability condition in terms of t-structures, which allows them to be more easily constructed.

Definition 4.4.5. A **stability function** on an abelian category $\mathcal{A}$ is a group homomorphism $Z : K(\mathcal{A}) \to \mathbb{C}$ such that

- $0 \neq E \in \mathcal{A} \implies Z(E) \in \mathbb{R}^+ e^{i\pi\phi(E)}$, with $0 < \phi(E) \leqslant 1$;
- An object E is **semistable** if for all subobjects $A \subset E$ then $\phi(A) \leqslant \phi(E)$;
- if $0 \neq E \in \mathcal{A}$ then there is a finite filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$, such that the E_j/E_{j-1} are semistable and $\phi(E_1/E_0) > \phi(E_2/E_1) > \cdots > \phi(E_n/E_{n-1})$.

Proposition 4.4.6. *A stability condition is the same as a bounded t-structure and a stability function on the heart.*

4.4.1 (Lack of) Bridgeland stability for two-periodic categories

Unfortunately, there seems to be as yet no good notion of Bridgeland stability on two-periodic categories. The reason is as follows: the practical study of Bridgeland stability conditions relies heavily on the 'Hom-vanishing' condition that if $\theta_1 > \theta_2$ and $A_j \in P(\theta_j)$, then $Hom_D(A_1, A_2) = 0$.

This Hom-vanishing gives Bridgeland stability structures substantial rigidity. In the two-periodic case, unfortunately, we can have no Hom-vanishing; see remark 4.1.8. Without this rigid structure, candidate spaces of stability conditions are generically much less well behaved.

What, if any, rigid structure can replace Hom-vanishing to make for a good definition of two-periodic Bridgeland stability is as yet unclear.

Physical considerations suggest that two-periodic Bridgeland stability may not be well defined. For Bridgeland stability is motivated physically as a condition cutting out a physical subcategory of D-branes from a larger mathematical space of derived coherent sheaves. The physical spaces under consideration are supersymmetric, and generically have R-symmetry group[1] at least $U(1)$. A $U(1)_R$ symmetry induces a $\mathbb{Z}$ grading.

Yet two-periodic categories do not (generally) admit a $\mathbb{Z}$-grading. This is a sign that these categories are unphysical, therefore the question of stability is not necessarily well-defined here.

Nonetheless, it is natural to ask about stability on many two-periodic categories. The approach taken in the literature is to find additional structure as a way to promote the category in question to a $\mathbb{Z}$-graded category.

4.5 Root category via matrix factorisations with special stability structure

There is another viewpoint on the root category which provides an example of the procedure by which one can add structure to allow for the study of Bridgeland stability on a two-periodic category. This example comes from the study of *matrix factorisations*, which let us study a natural kind of two-periodic category associated to an abelian group G.

By the McKay correspondence discussed in the introduction 1.1.1, the Dynkin diagram A_n can be associated with the cyclic group $\mathbb{Z}/(n+1)$. So we can just try setting $G = \mathbb{Z}/(n+1)$, and see what sort of category comes out. Happily, it gives us the A_n root category, as we would like. In fact, constructing the category this way gives us more: instead of just a t-structure, we get a distinguished Bridgeland stability condition.

The category of matrix factorisations was originally motivated as the category of boundary conditions for a topological Landau-Ginzburg model. By this interpretation, the distinguished Bridgeland stability condition is just that which cuts out the actual physical boundary conditions for the Landau-Ginzburg model in question. See [K4] for a discussion in this language.

For brevity, we're only going to sketch this construction for the type A_n Dynkin diagrams. The constructuion can be extended to the full ADE case. See [T] for the original rigorous construction of the type A case and [KST] for the full ADE picture.

[1]The R-symmetry group is the group of global symmetries transforming the supercharges.

Definition 4.5.1.
- A $\mathbb{Z}_+$ **category** charged by Φ, $(\mathcal{C}, w, \Phi)$, is the data of a category, $\mathcal{C}$, and a natural transformation $w : id_C \to \Phi$, where $\Phi : \mathcal{C} \to \mathcal{C}$ is an equivalence.

- Let $\hat{A}_2$ be the quiver

$$\hat{A}_2 = \begin{tikzcd} 0 \arrow[r, bend left] & 1 \arrow[l, bend left] \end{tikzcd}$$

  Equip $Free(\hat{A}_2)$ with the structure of a $\mathbb{Z}_+$ category, with $w : 0 \to 0$ the map induced by the cycle $0 \to 1 \to 0$ and $w : 1 \to 1$ the cycle $1 \to 0 \to 1$.

- A **loop factorisation in** $\mathcal{C}$ charged by Φ is a functor of $\mathbb{Z}^+$ categories $(\hat{A}_2, w, id_C) \to (\mathcal{C}, w, \Phi)$.

- The **category of loop factorisations** charged by Φ, $LF(\mathcal{C}, \Phi)$, is the category of loop factorisations in $\mathcal{C}$ with morphisms generated by k-linear envelopes (i.e. sums with coefficients in k) of natural transformations. It is naturally a linear stable $(\infty, 1)$-category.

- Let L an abelian group, R an L-graded k-algebra, and $w \in R$ a 0-graded central element. Let $Perf_R^L$ the category of finitely generated projective left R-modules. $(Perf_R^L, w, \Phi)$ is then a $\mathbb{Z}_+$ category.

- The **category of matrix factorisations** charged by Φ is $MF^L(R, w, \Phi) := LF(Perf_R^L, w, \Phi)$.

- Let $\mathcal{T}_\Phi^n := MF^{\mathbb{Z}/(n+1)}(k[z], z^{n+1}, \Phi)$, where $k[z]$ is a graded with z in degree one.

Less abstractly,

- An object in $MF^L(R, w, \Phi)$ is a tuple (A, B, f_A, f_B) where

  1. $A, B \in Perf_R^L$;
  2. $f_A \in Hom(A, \Phi B)$;
  3. $f_B \in Hom(B, A)$;

  subject to the consistency condition induced by the natural transformation w

$$\begin{aligned} f_B \circ f_A &= w(id_A) \\ \Phi(f_A) \circ f_B &= w(id_B) \end{aligned}$$

- Morphism sets are naturally enriched as chain complexes, so $\mathcal{T}_\Phi^n$ is a dg-category. We have

$$\begin{aligned} (n = 2\ell) &\implies Hom_n(E, F) = Hom_{\mathcal{A}}(E_A, \Phi^\ell F_A) \oplus Hom_{\mathcal{A}}(E_B, \Phi^\ell F_B) \\ (n = 2\ell + 1) &\implies Hom_n(E, F) = Hom_{\mathcal{A}}(E_A, \Phi^\ell F_A) \oplus Hom_{\mathcal{A}}(E_B, \Phi^{\ell+1} F_B) \end{aligned}$$

  Let $(u_A, u_B) \in Hom_n(E, F)$ a morphism. Then the differential acts as

$$\begin{aligned} (n = 2\ell) &\implies d(u_A, u_B) = (u_B \circ f_{E,A} - \Phi^\ell(f_{F,A}) \circ u_A, \Phi(f_A) \circ f_{E,B} - \Phi^\ell(f_{F,B}) \circ u_B) \\ (n = 2\ell + 1) &\implies d(u_A, u_B) = (u_B \circ f_{E,A} + \Phi^\ell(f_{F,B}) \circ u_A, \Phi(f_A) \circ f_{E,B} + \Phi^\ell(f_{F,A}) \circ u_B) \end{aligned}$$

Example 4.5.2. If we set $\Phi = T$, $T^2 = 1$, we get a two periodic dg-category.

The main theorem of of [T] implies that

Fact 4.5.3. *$D^b(Rep_k(\vec{A}_n))/\Sigma^2$ is derived equivalent to $\mathcal{T}_T^n$.*

The physical interpretation here is that an object in the category of matrix factorisations represents a brane-antibrane pair (A, B), plus a field configuration (f_A, f_B) that condenses them together. Φ, roughly, represents the charge of these branes under $U(1)_R$.

When we set $\Phi = T$, this corresponds to assigning trivial $U(1)_R$ charge. This is unphysical, and as expected, the resulting category is only two-periodic so admits no Bridgeland stable subcategory of physical D-branes.

The suggestion of [T] is to identify the Auslander-Reiten translation functor as our candidate charge.

Indeed,

Fact 4.5.4. *1. There is a Serre functor, $\mathcal{S} : Perf_R^L \to Perf_R^L$ when $L = \mathbb{Z}/(n+1)\mathbb{Z}$, $R = k[z]$. Then let $\tau := \mathcal{S}^{-1}T$.*

2. The category $\mathcal{T}_\tau^n$ is $\mathbb{Z}$-graded.

3. $\mathcal{T}_\tau^n$ is derived equivalent to $D^b(Rep_k(\vec{A}_n^\Omega))$.

Given a graded matrix factorisation (A, B, f_A, f_B), we can equivalently represent it as a pair (Q, S) by:

- Choosing a basis of $A \oplus B$ and its image under w, a basis of $A \oplus \Phi(B)$, and writing the maps f_A, f_B as a matrix, which we define to be Q;
- S is the diagonal matrix with entries the charge under $U(1)_R$ of each basis element, where for basis elements of A the charge is just the degree and for basis elements of B the charge is just the degree minus one.

The $\mathbb{Z}$-grading on $\mathcal{T}_\tau^n$ defines a t-structure, hence to define a stability condition, it suffices to define a stability function by prop. 4.4.6.

Fact 4.5.5. *There is a stability condition defined on $\mathcal{T}_\tau^n$, with t-structure coming from the chain complex structure on morphism sets and stability function $Z(Q, S) := Tr(e^{i\pi S})$.*

Chapter 5

Constructing the full Lie algebra as a Hall algebra

We have now shown that combinatorial Lie theory is entirely encoded in the two-periodic derived category of quiver representations, the 'root category'. We have constructed this category in a bunch of different ways. So can we use this category to recover the full Lie algebra in a natural way?

The answer is 'almost'. We will have to move from the root category to the category of two-periodic projective $k\vec{D}^{\Omega}$-modules. Then we can use a 'twisted Hall algebra' to recover the full quantum group associated to the Lie algebra. Taking a limit as our deformation goes to 1 then recovers the Lie algebra.

The proof that this is so is not difficult; with the proper definitions it is a straightforward computation, no hard work needed.

However, the proper definition of the twisted Hall algebra is complicated, the definition of the quantum group is complicated, and the 'straightforward computation' in question is lengthy. So this chapter, which attempts to motivate the twisted Hall algebra's construction and do the computation carefully, is a lot of definitions and computations.

First, informally: what is a Hall algebra? It is an algebra associated to a category with an $\vec{A}_3$ quiver structure. By this I mean: say you have (the k-linear envelope of) a category $\mathcal{C}$, and (the k-linear enevelope of) another category $\mathcal{C}^3$, where the objects of $\mathcal{C}^3$ are of the form $A_1 \to A_2 \to A_3$, where the A_i are objects in $\mathcal{C}$. There are distinguished maps $f_j : (A_1 \to A_2 \to A_3) \to A_j$. The Hall algebra multiplication is induced by pulling back (A, B) along (f_1, f_3) and pushing forward along f_2. In less abstract terms, the multiplication $Ob(\mathcal{C}) \times Ob(\mathcal{C}) \to Ob(\mathcal{C})$ is given by sending (A, C) to the sum of all B such that $A \to B \to C$ is in $Ob(\mathcal{C}^3)$.

Here is a sketch, subsection-by-subsection, to keep track of the bigger picture of what's going on in this section.

1. First, we'll define what a (finite field) Hall algebra is. We'll do a really simple computation to help understand what these things look like.

2. Then, we'll specialise to the case of the two-periodic derived categories of quiver representations we're interested in. We'll show that the naive Hall algebra is too commutative to admit the structure of a semisimple Lie algebra.

3. We'll define a twisting and do some computations to show how it introduces non-commutativity.

4. We'll define quantum groups associated to a Dynkin diagram in terms of a special generating basis $\{E_i, F_i, K_i\}$. We'll then define a corresponding special generating

basis of our two-periodic twisted Hall algebra, which we'll also call $\{E_i, F_i, K_i\}$. We will hope to show that these two bases can be identified. We will show that both the quantum group and the two-periodic twisted Hall algebra admit the structure of a Hopf algebra.

5. Finally, we will show that the two-periodic twisted Hall algebra $\{E_i, F_i, K_i\}$ satisfy the commutation relations of the quantum group generators, so equip the Hall algebra with the structure of the desired quantum group, by just computing the commutation relations with classical facts about Ext^1.

5.1 An introduction to Hall algebras

Definition 5.1.1. 1. Let $\mathcal{A}$ an essentially small abelian category. We say that $\mathcal{A}$ **admits a finite field Hall algebra** if :

   (a) Its Hom-spaces are finite as sets;

   (b) The category is $\mathbb{F}_q$-linear, for some finite field;

   (c) $\mathcal{A}$ has enough projectives.

2. For such $\mathcal{A}$, let $Ext^1(A, C)_B \subset Ext^1(A, C)$ be the subset of extensions with middle term isomorphic to B. To be very specific, this is the set of short exact sequences $0 \to A \to B' \to C \to 0$, where B' is isomorphic to B, under the equivalence relation that $0 \to A \to B' \to C \to 0$ is the same extension as $0 \to A \to B'' \to C \to 0$ if there is a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f_1"] \arrow[d, "id"] & B' \arrow[r, "f_2"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & A \arrow[r, "f_1'"] & B'' \arrow[r, "f_2'"] & C \arrow[r] & 0
\end{tikzcd}$$

in which case two-out-of-three implies g is an isomorphism.

3. Then the (finite field Ringel–) **Hall algebra**, $\mathcal{H}(\mathcal{A})$, is the associative algebra

   (a) With underlying set $Iso(\mathcal{A})$, the isomorphism classes of objects in $\mathcal{A}$;

   (b) With multiplication $[A][C] := \sum_{B \in Iso(\mathcal{A})} \frac{|Ext^1(A,C)_B|}{|Hom(A,C)|}[B]$.

   (c) We call the numbers $s^B_{AC} := \frac{|Ext^1(A,C)_B|}{|Hom(A,C)|}$ the **structure constants of the Hall algebra.**

Note that we include the zero map in these sets. Therefore $|Hom(A, C)| = 1$ even if there are no nontrivial maps $A \to C$, so we never divide by zero.

Example 5.1.2. The simplest example will be $\mathcal{A} = Vect_{\mathbb{F}_q}$, the category of $\mathbb{F}_q-$vector spaces.

- Isomorphism classes of objects are represented by $[\mathbb{F}_q^n], n \in \mathbb{N}$.

- Because short exact sequences conserve dimension,

$$[\mathbb{F}_q^n][\mathbb{F}_q^m] = \frac{|Ext^1(\mathbb{F}_q^n, \mathbb{F}_q^m)|}{|Hom(\mathbb{F}_q^n, \mathbb{F}_q^m)|}[\mathbb{F}_q^{n+m}]$$

- We can calculate $\frac{|Ext^1(\mathbb{F}_q^n,\mathbb{F}_q^m)|}{|Hom(\mathbb{F}_q^n,\mathbb{F}_q^m)|}$ as the number of points in the Grassmannian $Gr_{\mathbb{F}_q}(m, n+m)$ times $\frac{|Aut(\mathbb{F}_q^n)||Aut(\mathbb{F}_q^m)|}{|Aut(\mathbb{F}_q^{n+m})|}$. The upside is that

$$([n]_q!\frac{[\mathbb{F}_q^n]}{|Aut(\mathbb{F}_q^n)|})([m]_q!\frac{[\mathbb{F}_q^m]}{|Aut(\mathbb{F}_q^m)|}) = [n+m]_q!\frac{[\mathbb{F}_q^{n+m}]}{|Aut(\mathbb{F}_q^{n+m})|}$$

where

$$[n]_q := \sum_{0\leqslant i<n} q^i$$
$$[n]_q! := \prod_{1\leqslant j\leqslant n} [j]_q$$

The point is that the normalised generators $\frac{[\mathbb{F}_q^n]}{|Aut(\mathbb{F}_q^n)|}$ induce an isomorphism

$$\mathcal{H}(Vect_{\mathbb{F}_q}) \simeq \mathbb{Z}[x, \frac{x^2}{[2]_q!}, \frac{x^3}{[3]_q!}, \dots]$$

This algebra is commutative and associative.

5.2 Hall algebras of two-periodic complexes

The two-periodic derived category, $D(Rep_k(\vec{D}^\Omega))/T^2$ does not admit a Hall algebra in the way we have defined it. Any construction of the Lie algebra by preferred basis elements will correspond to a choice of simple roots, hence the data of an orientation on the Dynkin diagram D. This suggests that to construct the full Lie algebra, we will need the data of a t-structure, or, per section 4.1.1, a $\mathbb{Z}/2\mathbb{Z}$ grading.

Therefore instead of $D(Rep_k(\vec{D}^\Omega))/T^2$ we consider a naturally $\mathbb{Z}/2\mathbb{Z}$-graded relative, namely the $\mathbb{Z}_2$-graded complexes of projective objects in $\mathcal{A}$: we write this as $Kom_{\mathbb{Z}_2}(Proj(\mathcal{A}))$. This was first done by Bridgeland [B2].

We will also need to twist our product. To motivate the need for twisting, I will explain how the isomorphism fails when we do not have it.

Definition 5.2.1. A two-periodic complex is **acyclic** if it is quasi-isomorphic to the 0 object.

Definition 5.2.2. The **localised Hall algebra**, $\mathcal{DH}(\mathcal{A})$, is the Hall algebra of two-periodic projective complexes localised at the set of acyclic complexes:

$$\mathcal{DH}(\mathcal{A}) := \mathcal{H}(Kom_{\mathbb{Z}_2}(Proj(\mathcal{A})))/[[M_\bullet]^{-1}; H_\star(M_\bullet) = 0]$$

The **reduced localised Hall algebra**, $\mathcal{DH}_{red}(\mathcal{A})$ is the algebra where we set acyclic objects invariant under the translation functor T to 1;

$$\mathcal{DH}_{red}(\mathcal{A}) := \mathcal{DH}(\mathcal{A})/([M_\bullet] - 1; [TM_\bullet] = [M_\bullet])$$

This reduced localised Hall algebra is our candidate Lie algebra; after twisting, it will be our Lie algebra. To show this, we are going to explicitly construct generators K_i, E_i, F_i and show how they satisfy the Serre relations. Let's start that process.

Proposition 5.2.3. *There is a homomorphism $K : K_0(\mathcal{A}) \to \mathcal{DH}(\mathcal{A})$, sending*

$$[A] \to \left[\begin{tikzcd} A \arrow[r, bend left, "id"] & A \arrow[l, bend left, "0"] \end{tikzcd}\right] =: K_A$$

for projectives, and for classes $\alpha := [P] - [Q]$ then $K_\alpha := K_P \star K_Q^{-1}$.

To prove this, we will establish a nice lemma:

Lemma 5.2.4. *For arbitrary $M_\bullet$,*

$$K_P \star [M_\bullet] = [K_P \oplus M_\bullet] = [M_\bullet] \star K_P$$

Proof. There are no nontrivial extensions by acyclic objects, for they are quasi-isomorphic to zero. Therefore $K_P \star [M_\bullet] = [K_P \oplus M_\bullet]$ by definition. □

Proof. (Of Prop. 5.2.3) We have

$$K_A \star K_B = [K_A \oplus K_B] = K_{A \oplus B}$$

□

But the lemma 5.2.4 also implies that an untwisted Hall algebra cannot give rise to the semisimple Lie algebras we want to correspond to our Dynkin diagrams. For the lemma implies $[K_P, \mathcal{O}] = 0$ for all $\mathcal{O}$. Hence we have a huge abelian ideal generated by the $\{K_P\}$ in our candidate Lie algebra. Even if it's a Lie algebra, it is definitely not semisimple, so it isn't one of the ADE Lie algebras we want to find.

So we'll need to twist in a way that destroys the commutatitivity of this bracket.

5.3 Twisting the Hall algebra.

Because $\mathcal{A} = Rep_k(\vec{D}^\Omega)$ is hereditary, it admits a nonsymmetric **Euler form**,

$$\langle M, N \rangle := \dim RHom(M, N)$$

This is a natural nonsymmetric form on our category; hence, it seems like a good candidate for twisting. Indeed, we define (for a general hereditary $\mathcal{A}$ admitting a Hall algebra, but really we only care about $\mathcal{A} = Rep_k(\vec{D}^\Omega)$)

Definition 5.3.1. Let the **twisted Hall algebra** $\mathcal{H}_{tw}(Kom_{\mathbb{Z}_2}(Proj(\mathcal{A})))$ be the algebra with

- Underlying set $\mathcal{H}(Kom_{\mathbb{Z}_2}(Proj(\mathcal{A})))$;
- Multiplication $[M_\bullet] \star_{tw} [N_\bullet] := t^{\langle M_0,N_0\rangle + \langle M_1,N_1\rangle}[M_\bullet] \star [N_\bullet]$;
- where t is a choice of square root of q, assuming $\mathcal{A}$ is linear over some finite field $\mathbb{F}_q$.

Likewise, let the **twisted localised Hall algebra** be the same as it was before, except with twisting:

$$\mathcal{DH}_{tw}(\mathcal{A}) := \mathcal{H}_{tw}(Kom_{\mathbb{Z}_2}(Proj(\mathcal{A})))/[[M_\bullet]^{-1}; H_\star(M_\bullet) = 0]$$

and the same for the **twisted reduced localised Hall algebra** :

$$\mathcal{DH}^{red}_{tw}(\mathcal{A}) := \mathcal{DH}(\mathcal{A})/([M_\bullet] - 1; [TM_\bullet] = [M_\bullet])$$

We still have the same nice acyclic complexes K_P, but it commutes less well than before:

Lemma 5.3.2. *For arbitrary $M_\bullet$,*

$$t^{-\langle P,M_\bullet\rangle} K_P \star_{tw} [M_\bullet] = [K_P \oplus M_\bullet] = t^{\langle M_\bullet,P\rangle}[M_\bullet] \star_{tw} K_P$$

Proof. By definition; the sign swaps when we commute objects because $RHom$ swaps sign. □

So now

$$[K_P, M_\bullet] = (t^{\langle P,M_\bullet\rangle} - t^{-\langle M_\bullet,P\rangle})[K_P \oplus M_\bullet]$$

which is generically noncommutative, as desired.

Remark 5.3.3. We have so far denoted the twisted product by $\star_{tw}$. In what follows, all products are twisted so we use $\star_{tw}$ and $\star$ interchangeably.

What kind of objects do we need deal with in our categry of complexes? First, recall that in a hereditary category, every complex is quasi-isomorphic to one of the form $A \oplus TB$, where A and B are objects in $\mathcal{A}$. Choosing minimimal projective resolutions, we have the following

Proposition 5.3.4. *Every object in $Kom_{\mathbb{Z}_2}(Proj(\mathcal{A}))$ has a direct sum decomposition*

$$Res_A \oplus TRes_B \oplus K_{P_1} \oplus TK_{P_2}$$

where the K_{P_i} are acyclic projectives, and given a minimal projective resolution

$$0 \longrightarrow P_A \xrightarrow{f_A} Q_A \longrightarrow A \longrightarrow 0$$

we let

$$Res_A := \left[\begin{tikzcd} P_A \arrow[r, bend left, "f_A"] & Q_A \arrow[l, bend left, "0"] \end{tikzcd}\right]$$

Example 5.3.5. Consider the Hall algebra $\mathcal{H}_{tw}(Kom_{\mathbb{Z}_2}(Proj(Vect_{\mathbb{F}_q})))$. By the above lemma, isomorphism classes of objects are generated by

- $K_{[\mathbb{F}_q^n]} = \begin{tikzcd} \mathbb{F}_q^n \arrow[r, bend left, "id"] & \mathbb{F}_q^n \arrow[l, bend left, "0"] \end{tikzcd}$, $TK_{[\mathbb{F}_q^n]} = \begin{tikzcd} \mathbb{F}_q^n \arrow[r, bend left, "0"] & \mathbb{F}_q^n \arrow[l, bend left, "id"] \end{tikzcd}$

- $Res_{[\mathbb{F}_q^n]} = \begin{tikzcd} 0 \arrow[r, bend left, "0"] & \mathbb{F}_q^n \arrow[l, bend left, "0"] \end{tikzcd}$, $TRes_{[\mathbb{F}_q^n]} = \begin{tikzcd} \mathbb{F}_q^n \arrow[r, bend left, "0"] & 0 \arrow[l, bend left, "0"] \end{tikzcd}$

We'll prove that the Lie bracket on this algebra will end up endowing it with the structure of quantum $SU(2)$; hence, the relations to define this algebra are going to be relatively complicated. So we won't try to exhaustively describe this algebra. But here are some observations:

- The subalgebra spanned by all $Res_{[\mathbb{F}_q^n]}$ is a twisted copy of the Hall algebra we analysed in our previous example. Because $\dim Hom(\mathbb{F}_Q^n, \mathbb{F}_q^m) = nm$ and all extensions are trivial,

$$([n]_q!\frac{[\mathbb{F}_q^n]}{|Aut(\mathbb{F}_q^n)|}) \star_{tw} ([m]_q!\frac{[\mathbb{F}_q^m]}{|Aut(\mathbb{F}_q^m)|}) = t^{nm}[n+m]_q!\frac{[\mathbb{F}_q^{n+m}]}{|Aut(\mathbb{F}_q^{n+m})|}$$

  This is a commutative subalgebra, and of course the $\{TRes_{[\mathbb{F}_q^n]}\}$ generate an isomorphic subalgebra.

- The subalgebra spanned by all $K_{[\mathbb{F}_q^n]}$ (or the $TK_{[\mathbb{F}_q^n]}$) is the abelian group generated by $K_{\mathbb{F}_q}$, hence a copy of $\mathbb{Z}$;

- Because $\dim Hom(\mathbb{F}_q, T\mathbb{F}_q) = 0$ and $\dim Ext^1(\mathbb{F}_q, T\mathbb{F}_q) = 1$, and the extension is clearly spanned by $K_{\mathbb{F}_q}$,

$$Res_{\mathbb{F}_q} \star_{tw} TRes_{\mathbb{F}_q} = [Res_{\mathbb{F}_q} \oplus TRes_{\mathbb{F}_q}] + (q-1)K_{\mathbb{F}_q}$$
$$TRes_{\mathbb{F}_q} \star_{tw} Res_{\mathbb{F}_q} = [Res_{\mathbb{F}_q} \oplus TRes_{\mathbb{F}_q}] + (q-1)TK_{\mathbb{F}_q}$$

  Hence, they have nonzero commutator:

$$[Res_{\mathbb{F}_q}, TRes_{\mathbb{F}_q}] = (q-1)(K_{\mathbb{F}_q} - TK_{\mathbb{F}_q})$$

5.4 Quantum groups, Hopf algebras and triangular decomposition

The quantum group is a continuous one-parameter deformation of our Lie algebra. In terms of generators, as an algebra dependent on a parameter t,

Definition 5.4.1.
- Let Γ a finite graph; let n_{ij} the number of edges connecting $i-j$. Let $\alpha_{ij}^{\Gamma} := 2\delta_{ij} - n_{ij}$, the **generalised Cartan matrix** associated to Γ.

- Let $\binom{n}{k}_t := \prod_{i=0}^{m-1} \frac{1-t^{n-1}}{1-t^{i+1}}$, the $t-$**binomial coefficient**.
- Let $U_t(\mathfrak{g})$, the **quantum enveloping algebra of the corresponding derived Kac-Moody Lie algebra** $\mathfrak{g}$, be generated by symbols E_i, F_i, K_i, K_i^{-1}, subject to relations

$$\begin{aligned}
K_iK_i^{-1} &= 1 = K_i^{-1}K_i \\
[K_i, K_j] &= 0 \\
K_iE_j &= t^{a_{ij}^\Gamma}E_jK_i \\
K_iF_j &= t^{-a_{ij}^\Gamma}F_jK_i \\
[E_i, F_j] &= \delta_{ij}\frac{K_i - K_i^{-1}}{t-t^{-1}} \\
\sum_{n=0}^{1-a_{ij}^\Gamma}(-1)^n\binom{1-a_{ij}^\Gamma}{n}_t E_i^nE_jE_i^{1-a_{ij}^\Gamma-n} &= 0, i \neq j \\
\sum_{n=0}^{1-a_{ij}^\Gamma}(-1)^n\binom{1-a_{ij}^\Gamma}{n}_t F_i^nF_jF_i^{1-a_{ij}^\Gamma-n} &= 0, i \neq j
\end{aligned} \tag{5.1}$$

If we define $H_i := \frac{K_i-K_i^{-1}}{t-t^{-1}}$ and take a limit $t \to 1$ in some suitable sense, this recovers the classical Serre relations, verifying that this algebra is somehow a deformation of the universal enveloping algebra of the underlying Lie algebra.

In what space is it a deformation? In the space of Hopf algebras: algebras which are also coalgebras, with nice consistency conditions.

Definition 5.4.2. A k-**bialgebra** $(A, m, m^\vee, \eta, \eta^\vee)$ is

- A vector space A;
- An associative multiplication $m : A \otimes A \to A$
- A unit $\eta : k \to A$ so that (A, η, m) is an associative unital algebra;
- A coassociative comultiplication $m^\vee : A \to A \otimes A$;
- A counit $\eta^\vee : A \to k$ so that $(A, \eta^\vee, m^\vee)$ is a coassociative counital coalgebra (i.e. all the diagrams an algebra satisfies are satisfied with the arrows in reverse).
- The unit and multiplication are homomorphisms of A as a coalgebra;
- The counit and comultiplication are homomorphisms of A as an algebra

Definition 5.4.3. A **Hopf algebra** is a k-bialgebra $(A, m, m^\vee, \eta, \eta^\vee)$ with an **antipode map**, $S : A \to A$, so that

$$\begin{tikzcd}
& A \arrow[dl, "m^\vee"] \arrow[dr, "m^\vee"] \arrow[dd, "\eta^\vee"] & \\
A \otimes A \arrow[dd, "id\otimes S"] & & A \otimes A \arrow[dd, "S\otimes id"] \\
& k \arrow[dd, "\eta"] & \\
A \otimes A \arrow[dr, "m"] & & A \otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

Fact 5.4.4. *The quantum group is a Hopf algebra, with*

- *Structure of an algebra given by multiplication under the relations listed;*
- *Coproduct*

$$\begin{aligned}
m^\vee(K_i) &= K_i \otimes K_i \\
m^\vee(E_i) &= E_i \otimes 1 + K_i \otimes E_i \\
m^\vee(F_i) &= 1 \otimes F_i + F_i \otimes K_i^{-1}
\end{aligned}$$

- *Antipode*

$$\begin{aligned}
S(K_i) &= K_i^{-1} \\
S(E_i) &= -K_i^{-1}E_i \\
S(F_i) &= -F_iK_i
\end{aligned}$$

- *Counit*

$$\begin{aligned}
\eta^\vee(K_i) &= 1 \\
\eta^\vee(E_i) &= 0 = \eta^\vee(F_i)
\end{aligned}$$

To prove this, we would need to check that all the given operations satisfy the relations given on generators, and that all diagrams are satisfied. This is a lot of data, so for brevity we omit checking.

Just like the Lie algebra can be split into a Cartan subalgebra and positive and negative halves, we can split the quantum group.

Fact 5.4.5. • *There is an involution*

$$\begin{aligned}
\sigma : U_t\mathfrak{g} &\to U_t\mathfrak{g} \\
\sigma(E_i) &= F_i \\
\sigma(F_i) &= E_i \\
\sigma(K_i) &= K_i^{-1}
\end{aligned}$$

- *Let $U_t(\mathfrak{n})^+$ be the subalgebra generated by the $\{E_i\}$. Let $U_t\mathfrak{h} \simeq \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the subalgebra generated by the $K_i^{\pm 1}$.*
- *Then the natural multiplication map*

$$U_t(\mathfrak{n}^+) \otimes_C U_t(\mathfrak{h}) \otimes_C \sigma(U_t(\mathfrak{n}^+)) \to U_t\mathfrak{g}$$

is an isomorphism.

The Hopf algebra structure is very rigid, and hence a powerful tool for analysing quantum groups. However, a Hopf algebra is a lot of data, and the theory of Hopf algebras takes a while to build up. Hence, in what follows we avoid the use of all but the algebra structure on the Hall algebra and the quantum group.

5.5 The quantum group is the two-periodic reduced twisted Hall algebra

Our proof strategy is as follows.

First, we will define a candidate basis of our reduced hall algebra E_A, F_A, K_A, which we want to relate to the generating basis of the quantum group of def. 5.4.1. We will show the E_A, resp. F_A generate $\mathcal{H}_{tw}(\mathcal{A})$ subalgebras and K_A a $\mathbb{C}[K_0(\mathcal{A})]$.

Lemma 5.5.1. ***(Triangular decomposition for the Hall algebra)*** *Let $\mathcal{A} = Rep_k(\vec{D}^\Omega)$, with D Dynkin. The multiplication map*

$$\begin{aligned} \mathcal{H}_{tw}(\mathcal{A}) \otimes_{\mathbb{C}} \mathbb{C}[K_0(\mathcal{A})] \otimes_{\mathbb{C}} \mathcal{H}_{tw}(\mathcal{A}) &\to \mathcal{DH}(\mathcal{A}) \\ A \otimes B \otimes C &\to E_A \star_{tw} K_B \star_{tw} F_C \end{aligned}$$

is an isomorphism.

Then we will try to show the equivalence on each part of the decomposition. That is, we want to show

Lemma 5.5.2. ***(The twisted Hall algebra is the positive half of the quantum group)*** *Let $\mathcal{A} = Rep_k(\vec{D}^\Omega)$, with D Dynkin. There is an isomorphism of Hopf algebras*

$$\iota : H_{tw}(\mathcal{A}) \simeq U_t(\mathfrak{n}^+)$$

Combining the lemmas with the obvious isomorphism

$$h : C[K_0(\mathcal{A})] \simeq U_t\mathfrak{h}$$

Proves:

Proposition 5.5.3. *There is a commutative diagram of vector spaces*

$$\begin{tikzcd} U_t(\mathfrak{n}^+) \otimes_{\mathbb{C}} U_t(\mathfrak{h}) \otimes_{\mathbb{C}} U_t(\mathfrak{n}^-) \arrow[d, "\sim"] & \mathcal{H}_{tw}(\mathcal{A}) \otimes_C \mathbb{C}[K_0(\mathcal{A})] \otimes \mathbb{C}\mathcal{H}_{tw}(\mathcal{A}) \arrow[l, "\iota\otimes h\otimes\iota"'] \arrow[d, "\sim"] \\ U_t(\mathfrak{g}) & \mathcal{DH}_{red}(\mathcal{A}) \arrow[l] \end{tikzcd}$$

which furnishes an isomorphism $U_t(\mathfrak{g}) \simeq \mathcal{DH}_{red}(\mathcal{A})$.

It then suffices to show that this morphism of vector spaces is in fact a morphism of (Hopf) algebras. That is, we need to check that we have a candidate generating basis (E_i, F_i, K_i) which satisy the quantum group relations. Once we do so, then

Theorem 5.5.4. *There is an isomorphism of Hopf algebras $U_t(\mathfrak{g}) \simeq \mathcal{DH}_{red}(\mathcal{A})$.*

5.5.1 Candidate E, F, K basis

We already defined our candidate K_A. To recall the definition,

Definition 5.5.5.
- For P projective,

$$K_P := \begin{tikzcd} P \arrow[r, bend left, "id"] & P \arrow[l, bend left, "0"] \end{tikzcd}$$

  And $K_P^{-1} := TK_P$. For α in the class $[P] - [Q]$, then $K_\alpha := K_P \star K_Q^{-1}$.

- For $0 \to P_A \to Q_A \to A \to 0$ a minimal projective resolution,

$$Res_A := \left[\begin{tikzcd} P_A \arrow[r, bend left, "f_A"] & Q_A \arrow[l, bend left, "0"] \end{tikzcd} \right]$$

We will now define our E and F. The idea is as follows. We want E_A, F_A, K_A, K_A^{-1} to generate the reduced Hall algebra. We know by 5.3.4 that every object has a decomposition $K_A^{-1} \oplus K_B \oplus [Res_C] \oplus T[Res_D]$. Hence a natural candidate for E_A, F_A would be $[Res_A], T[Res_A]$. In fact, this definition does not quite work. An intuitive reason is that we *chose* a minimal resolution of A to define $[Res_A]$; our definition of E, F, should be independent of the choice of resolution we made. Rather, we have

Definition 5.5.6. Let A an object; denote its minimal projective resolution $0 \to P_A \to Q_A \to A \to 0$. Then define

$$E_A := t^{\langle P_A, A\rangle} K_{-P_A} \star [Res_A]$$
$$F_A := TE_A$$

The reason why we have E_A, not just $[Res_A]$, is that this twisting makes the definition of E_A invariant under choice of resolution. A different resolution, nonminimal, would be of the form $[Res_A] \oplus K_R$, for some R.

Then we would get (applying the identity of lemma 5.3.2)

$$\begin{aligned} &t^{\langle P_A \oplus R, A\rangle} K_{-P_A \oplus R} \star [K_R \oplus Res_A] \\ &= t^{\langle P_A \oplus R, A\rangle - \langle R, A\rangle} K_{-P_A \oplus R} \star K_R \star [Res_A] \\ &= t^{\langle P_A, A\rangle} K_{-P_A} \star [Res_A] \end{aligned}$$

So that E_A does not depend on our choice of resolution.

Proposition 5.5.7. *There are injective morphisms of rings*

$$\begin{aligned} I_+ : \mathcal{H}_{tw}(\mathcal{A}) &\to \mathcal{DH}_{tw}(\mathcal{A}) \\ I_+[A] &= E_A \\ I_- : \mathcal{H}_{tw}(\mathcal{A}) &\to \mathcal{DH}_{tw}(\mathcal{A}) \\ I_-[A] &= F_A \end{aligned}$$

Proof. By definition,

$$E_{A_1} \star_{tw} E_{A_2} = t^{\langle P_{A_1}, A_1\rangle + \langle P_{A_2}, A_2\rangle} K_{-P_{A_1}} \star_{tw} [Res_{A_1}] \star_{tw} K_{-P_{A_2}} \star_{tw} [Res_{A_2}]$$

We use lemma 5.3.2 to commute $[Res_{A_1}]$ and $K_{-P_{A_2}}$,

$$E_{A_1} \star_{tw} E_{A_2} = t^{\langle P_{A_1}, A_1\rangle + \langle P_{A_2}, A_2\rangle + \langle P_{A_2}, A_1\rangle} K_{-(P_{A_1}+P_{A_2})} \star_{tw} [Res_{A_1}] \star_{tw} [Res_{A_2}]$$

We know that $[Res_{A_1}] \star_{tw} [Res_{A_2}] = t^{\langle Q_{A_1}, Q_{A_2}\rangle + \langle P_{A_1}, P_{A_2}\rangle} \sum s^{A_3}_{A_1 A_2} [Res_{A_3}]$. Putting all this together,

$$\begin{aligned} & E_{A_1} \star_{tw} E_{A_2} \\ = & t^{\langle P_{A_2}, A_1\rangle + \langle A_1, P_{A_2}\rangle + \langle P_{A_1}, A_1\rangle + \langle P_{A_2}, A_2\rangle - \langle P_{A_1}+P_{A_2}, A_1+A_2\rangle + \langle P_{A_1}, P_{A_2}\rangle + \langle Q_{A_1}, Q_{A_2}\rangle - 2\langle Q_{A_1}, P_{A_2}\rangle} \sum s^{A_3}_{A_1 A_2} E_{A_3} \end{aligned}$$

Now $Q_{A_i} = A_i + P_{A_i}$ in equivalence class, hence substituting this in and doing algebra we get

$$E_{A_1} \star_{tw} E_{A_2} = t^{\langle A_1, A_2\rangle} \sum s^{A_3}_{A_1 A_2} E_{A_3}$$

establishing the required homomorphism.

To establish injectivity, we will explicitly identify a left inverse; the map $[M_\bullet] \to t^{-\langle M_1, H_0(M_\bullet)\rangle}[H_0(M_\bullet)]$ provides the required inverse, by computation.

By translation symmetry, the map $I_- : A \to F_A$ is also an injective morphism of rings. ☐

5.5.2 Triangular decomposition for the two-periodic twisted Hall algebra

Now we've defined the maps; we need a little bit more information about $\mathcal{DH}(\mathcal{A})$ to establish the desired triangular decomposition. By prop 5.3.4, $\mathcal{DH}(\mathcal{A})$ has a basis of elements of the form

$$[Res_A \oplus T Res_B] \star K_\alpha \star T K_\beta$$

because the effect of multiplication with K_α is to take the direct sum, up to an invertible coefficient.

Definition 5.5.8. Let $\alpha \in K_0(\mathcal{A})$. Let $\mathcal{DH}_{\leqslant \gamma} \subset \mathcal{DH}(\mathcal{A})$ be the subspace spanned by those basis vectors for which $[A]+[B] \leqslant \gamma$ in the Grothendieck group, where γ is positive.

Lemma 5.5.9. $\mathcal{DH}_{\leqslant\gamma} \star \mathcal{DH}_{\leqslant\zeta} \subset \mathcal{DH}_{\leqslant \gamma+\zeta}$. *Consequently, these subspaces define a filtration on* $\mathcal{DH}(\mathcal{A})$.

Proof. Let $0 \to M_\bullet \to N_\bullet \to P_\bullet \to 0$ be an extension. It is a short exact sequence of complexes, therefore induces a long exact sequence in homology,

$$\begin{tikzcd} H_0(M_\bullet) \arrow[r] & H_0(N_\bullet) \arrow[r] & H_0(P_\bullet) \arrow[d] \\ H_1(P_\bullet) \arrow[u] & H_1(N_\bullet) \arrow[l] & H_1(M_\bullet) \arrow[l] \end{tikzcd}$$

Taking kernels and cokernels of the upper and lower rows, we get two exact sequences

$$0 \to \ker_0 \to H_0(M_\bullet) \to H_0(N_\bullet) \to H_0(P_\bullet) \to coker_0 \to 0$$
$$0 \to coker_0 \to H_1(M_\bullet) \to H_1(N_\bullet) \to H_1(P_\bullet) \to ker_0 \to 0$$

Where the kernels and cokernels identify as such because exactness means the map $H_0(P_\bullet) \to H_1(M_\bullet)$ factors through the cokernel, hence also applying exactness at $H_1(M_\bullet) \to H_1(N_\bullet)$, we have a diagram

$$\begin{tikzcd} H_0(N_\bullet) \arrow[r] \arrow[d] & H_0(P_\bullet) \arrow[d] & \\ 0 \arrow[r] & coker_0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ & H_1(M_\bullet) \arrow[r] & H_1(N_\bullet) \end{tikzcd}$$

where the upper square is a pushout square but the lower square is not necessarily a pullback. So by the universal property of kernels, we get a map $coker_0 \to ker_1$. We could run the analysis in reverse with ker_1 in the middle, and the lower square a pullback square, to get a map $\ker_1 \to coker_0$. Hence $coker_0 \simeq \ker_1$.

Summing the Grothendieck group relations induced by the short exact sequence,

$$[H_0(N_\bullet)] + [H_1(N_\bullet)] + 2[\ker_0] + 2[coker_0] = [H_0(M_\bullet)] + [H_1(M_\bullet)] + [H_0(P_\bullet)] + [H_1(P_\bullet)]$$

Classes of objects in the Grothendieck group are always nonnegative; hence, dropping all the kernel and cokernel terms, we get an inequality

$$[H_0(N_\bullet)] + [H_1(N_\bullet)] \leqslant [H_0(M_\bullet)] + [H_1(M_\bullet)] + [H_0(P_\bullet)] + [H_1(P_\bullet)] \tag{5.2}$$

so that the filtration works, as desired. □

Now we are ready to establish that

Lemma 5.5.10. *The map*

$$\mu := \begin{array}{c} \mathcal{H}_{tw}(\mathcal{A}) \otimes_{\mathbb{C}} \mathbb{C}[K_0(\mathcal{A})] \otimes_{\mathbb{C}} \mathcal{H}_{tw}(\mathcal{A}) \\ \big\downarrow{\scriptstyle I_+ \otimes K \otimes I_-} \\ \mathcal{DH}(\mathcal{A}) \otimes_{\mathbb{C}} \mathcal{DH}(\mathcal{A}) \otimes_{\mathbb{C}} \mathcal{DH}(\mathcal{A}) \\ \big\downarrow{\scriptstyle \star \otimes id} \\ \mathcal{DH}(\mathcal{A}) \otimes_{\mathbb{C}} \mathcal{DH}(\mathcal{A}) \\ \big\downarrow{\scriptstyle \star} \\ \mathcal{DH}(\mathcal{A}) \end{array}$$

is an isomorphism.

Proof. In the analysis of the previous lemma, let $M_\bullet = Res_A, P_\bullet = TRes_B$. By definition of a resolution, $Res_A = \begin{tikzcd} P_A \arrow[r, bend left, "f"] & Q_A \arrow[l, bend left, "0"] \end{tikzcd}$, the map f is injective. Hence $H_0(Res_A) = coker f \simeq A$, and $H_1(Res_A) = 0$.

Hence, $H_1(M_\bullet) = 0$ and $H_0(P_\bullet) = 0$. Therefore, $coker_0 = 0$.

I claim $\ker_0 = 0$ if and only if the extension is trivial. If the extension is trivial, then $N_\bullet \simeq Res_A \oplus TRes_B$ and $\ker_0 = 0$ by computing its homology. Now suppose $\ker_0 = 0$.

Then $H_0(N_\bullet) \simeq H_0(M_\bullet) \simeq A$ and $H_1(N_\bullet) \simeq H_1(P_\bullet) \simeq B$. But because by assumption our category is hereditary, by 3.2.4 $P_\bullet \simeq Res_A \oplus TRes_B$, therefore the extension is trivial.

Therefore: if the inequality 5.2 is an equality, the extension is trivial. Hence, because $\mu([A] \otimes \alpha \otimes [B])$ attains the bound,

$$\begin{aligned} &\mu([A] \otimes \alpha \otimes [B]) \\ &= t^n K_{-P_A} \star [Res_A] \star (K_\alpha \star TK_\beta) \star TK_{-P_B} \star T[Res_B] \\ &= t^{n'} [Res_A \oplus TRes_B] \star K_{\alpha - P_A} \star TK_{\beta - P_B} \end{aligned}$$

where n' is some number.

Hence, basis elements are sent to basis elements, up to an invertible power of t. This map is surjective and injective clearly.

□

5.5.3 Isomorphism of the positive half

Here is the idea. We want a basis directly corresponding to the E_i which generate the positive half of the quantum group. What should it be? The best idea is that E_i should correspond to $E_{Simple(i)}$.

Proposition 5.5.11. *The classes* $[Simple(i)]$ *generate* $\mathcal{H}_{tw}(Rep_k(\vec{D}^\Omega))$.

Proof. Grade quiver representations in $Rep_k(\vec{D}^\Omega)$ by the number of vertices, n, of the largest connected subquiver on which the quiver representation is nontrivial.

We'll prove the result by induction on n. Let X_n an indecomposable on n vertices. It suffices to show that X_n is generated by algebra elements of lesser grading. Clearly all indecomposables nontrivial on one vertex are simple, so the base case is clear. Now let X_n an indecomposable on n vertices. Choose a vertex, v, which is a sink for the subquiver on which X_n is nontrivial. Let $X_1 := Simple(v)$. Let X_{n-1} be the quiver representation which agrees with X_n except on the vertex v, where it is trivial.

I claim $[X_{n-1}]\star[X_1] = A[X_{n-1}\oplus X_1]+B[X_n]$, where A and B are invertible. Proof: the existence of both these extensions is clear, and we can easily calculate $\dim Ext(X_{n-1}, X_1) = 1$.

Further, $[X_1] \star [X_{n-1}] = C[X_{n-1} \oplus X_1]$ by construction. So we can write $[X_n]$ in terms of these two elements.

Therefore done. □

Therefore, it remains to prove

Proposition 5.5.12. *The morphism $\iota : U_t\mathfrak{n}^+ \to \mathcal{H}_{tw}(Rep_k(\vec{D}^{\Omega}))+$ generated by*

$$\iota : E_i \to Simple(i)$$

is an isomorphism of Hopf algebras.

Proof. Because the $E_i, Simple(i)$ generate, the morphism is a surjective morphism of Hopf algebras if the $Simple(i)$ obey the quantum Serre relation obeyed by E_i,

$$\sum_{n=0}^{1-a^{\Gamma}_{ij}} (-1)^n \binom{1-a^{\Gamma}_{ij}}{n}_t E_i^n E_j E_i^{1-a^{\Gamma}_{ij}-n} = 0, i \neq j$$

It is injective if the $Simple(i)$ obey no other relation.

We will check both these conditions independently.

□

Lemma 5.5.13. *The $\{Simple(i)\}$ obey the quantum Serre relation. Hence, ι is surjective.*

Proof.

1. Suppose i and j are not connected, $i \neq j$, so $a_{ij} = 0$ and there are no nontrivial extensions. The quantum Serre relation says

$$Simple(j) \star Simple(i) - Simple(i) \star Simple(j) = 0$$

which is true because $\dim RHom(Simple(i), Simple(j)) = 0 = \dim RHom(Simple(j), Simple(i))$, so the twisting is trivial.

2. Suppose i and j are connected by an arrow $i \to j$, so $a_{ij} = -1$ and the space of extensions is spanned by

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] & k \arrow[r] \arrow[d, "id"] & \dots \\
\dots \arrow[r] & k \arrow[r, "id"] \arrow[d, "id"] & k \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & k \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

i.e. $\dim Ext(Simple(i), Simple(j)) = 1$, $\dim Ext(Simple(j), Simple(i)) = 0$.

We need to do a bunch of calculation.

$$Simple(j) \star Simple(i) = [Simple(j) \oplus Simple(i)]$$

because there are no nontrivial extensions $Simple(j) \to Simple(i)$, but there are in the other direction, so

$$Simple(i) \star Simple(j) = t^{-1}([Simple(j) \oplus Simple(i)] + (t^2 - 1)[P_{i \to j}])$$

We have so far calculated $\frac{|Ext^1(A,C)_B|}{|Hom(A,C)|}$ without comment. For brevity, we will mostly continue to do so. But to very careful just once,

- We have $|Hom(Simple(i), Simple(j))| = 1$ because there is only the zero map. And $|Ext^1(Simple(i), Simple(j))_{Simple(i) \oplus Simple(j)}| = 1$ because any extension, for invertible maps f, f'

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & k \arrow[d, "f"] \\ k \arrow[r, "0"] \arrow[d, "f'"] & k \arrow[d] \\ k \arrow[r] & 0 \end{tikzcd}$$

  there is an automorphism of $Simple(i) \oplus Simple(j)$, namely $f^{-1} \oplus (f')^{-1}$, demonstrating that this extension is equivalent to the one where all maps are the identity.

- We have $|Ext^1(Simple(i), Simple(j))_{P_{i \to j}}| = (t^2 - 1)$ because given some extension

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & k \arrow[d, "f"] \\ k \arrow[r, "id"] \arrow[d, "f'"] & k \arrow[d] \\ k \arrow[r] & 0 \end{tikzcd}$$

  An automorphism of $P_{i \to j}$ is totally determined by its value on either vertex i or j, so we can only invert one of the two maps, say for instance f, to the identity. So we have free choice of f'. There are $(q - 1) = (t^2 - 1)$ invertible elements in $\mathbb{F}_q$, so that many choices of f'.

Hence,

$$Simple(j) \star Simple(j) \star Simple(i) = (Simple(j) \star Simple(j)) \star Simple(i)$$
$$= t^2[2!]_t[Simple(j) \oplus Simple(j)] \star Simple(i) = t(t^2 + 1)[Simple(j) \oplus Simple(j) \oplus Simple(i)]$$

$$Simple(i) \star Simple(j) \star Simple(j) = t(t^2 + 1)Simple(i) \star [Simple(j) \oplus Simple(j)]$$
$$= t^{-1}(t^2 + 1)([Simple(j) \oplus Simple(j) \oplus Simple(i)] + (t^2 - 1)[Simple(j) \oplus P_{i \to j}])$$

And

$$\begin{aligned}Simple(j) \star Simple(i) \star Simple(j) &= [Simple(j) \oplus Simple(i)] \star [Simple(j)] \\ = (t^2+1)[Simple(j) \oplus Simple(j) \oplus Simple(i)] &+ (t^2-1)[Simple(j) \oplus P_{i \to j}]\end{aligned}$$

Which imply a linear relation

$$\begin{aligned}Simple(j) \star Simple(j) \star Simple(i) - (t+t^{-1})Simple(j) \star Simple(i) \star Simple(j) \\ + Simple(i) \star Simple(j) \star Simple(j) = 0\end{aligned}$$

which is the quantum Serre relation desired.

3. Suppose i and j are connected by an arrow $j \to i$. Then almost the same calculation as we just did above gets the quantum Serre relation.

□

Hence, ι is a morphism of Hopf algebras, and surjective. We still need to show it is injective.

Proposition 5.5.14. *ι is injective.*

Proof. Let

$$\begin{aligned}\mathcal{H}'_t &:= \mathcal{H}_{tw}(Rep_k(\vec{D}^{\Omega})) \otimes_{\mathbb{Z}[t,t^{-1}} \mathbb{Q}[t,t^{-1}] \\ U'_t &:= U_t(\mathfrak{n}^+) \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Q}[t,t^{-1}]\end{aligned}$$

Functorially, we get $\iota' : U'_t \to \mathcal{H}'_t$. It suffices to show ι' is injective. Both $U'_t.\mathcal{H}'_{tw}$ are graded by $K_0(Rep_k(\vec{D}^{\Omega}))$, and ι' respects this grading. For an arbitrary class γ in K_0, it hence suffices to show the restricted mapped ι'_γ is injective.

Now $U'_{t,\gamma}$ is torsion free, hence because $\mathbb{Q}[t,t^{-1}]$ is a PID it is a free module over $\mathbb{Q}[t,t^{-1}]$. The dimension of $U'_{t,\gamma}$ does not change as t varies. Now a standard result in the theory of quantum groups is that $U'_{t,\gamma}/(t-1)$ is $U(\mathfrak{n}^+)_\gamma$, the γ-graded part of the (universal enveloping algebra of) the positive part of the usual Lie algebra associated to our quiver. $\mathcal{H}_{t,\gamma}$ is also a free $\mathbb{Q}[t,t^{-1}]$-module.

Let Φ^+ a choice of set of positive roots associated to our Dynkin diagram. Both $\mathcal{H}_{t,\gamma}$ and $U(\mathfrak{n}^+)_\gamma$ have dimension

$$u(\gamma) := \text{the number of maps } m : \Phi^+ \to \mathbb{N}^+ \text{ so } \sum_\alpha m(\alpha)\alpha = \gamma$$

the prior by definition, and the latter by the Poincare-Birkhoff-Witt theorem (see [K2].)

Now ι'_γ is a surjection between free modules of the same dimension, so by rank-nullity is an injection also.

□

5.5.4 Quantum group relations

Now that we have established isomorphisms of Hopf algebras between each part in the triangular decomposition, we need to show that the linear map $U_t(\mathfrak{n}^+) \otimes_{\mathbb{C}} U_t(\mathfrak{h}) \otimes_{\mathbb{C}} U_t(\mathfrak{n}^-) \to \mathcal{H}_{tw}(\mathcal{A}) \otimes_C \mathbb{C}[K_0(\mathcal{A})] \otimes_{\mathbb{C}} \mathcal{H}_{tw}(\mathcal{A})$ is an isomorphism of Hopf algebras. To do so, we need to show all the relations of eq. 5.1 are satisfied, except for the quantum Serre relations, which we have already checked. We need not show that it is an isomorphism, which follows because it is an isomorphism on each of the tensor components.

The remainder of this section is a series of extensive, and not particularly enlightening, calculations. They are given here for completeness, but one does not lose any conceptual understanding by skipping them.

Proposition 5.5.15. $[E_{Simple(i)}, F_{Simple(i)}] = (t^2-1)(TK_{Simple(i)} - K_{Simple(i)})$.

Proof. We have $Ext^1(Simple(i), TSimple(i)) \simeq Hom(Simple(i), Simple(i))$. Clearly the nontrivial extension is by $TK_{Simple(i)}$.

Hence, as always using lemma 5.3.2 to commute the Ks,

$$\begin{aligned}
&E_{Simple(i)} \star F_{Simple(i)}\\
&= t^{2\langle P_{Simple(i)}, Simple(i)\rangle} K_{-P_{Simple(i)}} \star [Res_{Simple(i)}] \star TK_{-P_{Simple(i)}} \star [TRes_{Simple(i)}]\\
&= t^{\langle P_{Simple(i)}, Simple(i)\rangle - \langle Simple(i), P_{Simple(i)}\rangle} K_{-P_{Simple(i)}} \star TK_{-P_{Simple(i)}} \star [Res_{Simple(i)}] \star [TRes_{Simple(i)}]\\
&= t^{\langle P_{Simple(i)}, Simple(i)\rangle - \langle Simple(i), P_{Simple(i)}\rangle} [Res_{Simple(i)}] \star [TRes_{Simple(i)}]\\
&= t^{\langle P_{Simple(i)}, Simple(i)\rangle - \langle Simple(i), P_{Simple(i)}\rangle} t^{\langle Q_{Simple(i)}, P_{Simple(i)}\rangle - \langle P_{Simple(i)}, Q_{Simple(i)}\rangle}\\
&\quad \times([Res_{Simple(i)} \oplus TRes_{Simple(i)}] + (t^2-1)TK_{Simple(i)})\\
&= [Res_{Simple(i)} \oplus TRes_{Simple(i)}] + (t^2-1)TK_{Simple(i)}
\end{aligned}$$

where in the last step the coefficient of t is zero by inserting the Grothendieck group relation $[A] = [Q] - [P]$ to simplify the Euler forms.

Now taking T of both sides,

$$F_{Simple(i)} \star E_{Simple(i)} = [Res_{Simple(i)} \oplus TRes_{Simple(i)}] + (t^2-1)K_{Simple(i)}$$

Subtracting the difference gets the desired commutation relation. □

Proposition 5.5.16. *Suppose $i \neq j$. Then* $[E_{Simple(i)}, F_{Simple(j)}] = 0$.

Proof. There are no nontrivial extensions because

$$\begin{aligned}
Ext^1(Simple(i), TSimple(j)) &\simeq Hom(Simple(i), Simple(j))\\
&= 0\\
= Hom(Simple(j), Simple(i)) &\simeq Ext^1(Simple(j), TSimple(i))
\end{aligned}$$

So that

$$\begin{aligned}
&E_{Simple(i)} \star F_{Simple(j)}\\
&= t^{\langle P_{Simple(i)}, Simple(i)\rangle + \langle P_{Simple(j)}, Simple(j)\rangle - \langle Simple(i), P_{Simple(j)}\rangle - \langle Simple(j), P_{Simple(i)}\rangle}\\
&\quad \times K_{-P_{Simple(i)}} \star TK_{-P_{Simple(j)}} \star [Res_{Simple(i)}] \star [TRes_{Simple(j)}]\\
&= t^{\langle P_{Simple(i)}, Simple(i)\rangle + \langle P_{Simple(j)}, Simple(j)\rangle - \langle Simple(i), P_{Simple(j)}\rangle - \langle Simple(j), P_{Simple(i)}\rangle}\\
&\times t^{\langle Q_{Simple(i)}, P_{Simple(j)}\rangle - \langle P_{Simple(i)}, Q_{Simple(j)}\rangle} K_{-P_{Simple(i)}} \star TK_{-P_{Simple(j)}} \star [Res_{Simple(i)} \oplus TRes_{Simple(j)}]
\end{aligned}$$

Rewriting $[Q] = [A] + [P]$ as in the previous proof,

$$\begin{aligned}
= t^{\langle P_{Simple(i)}, Simple(i)\rangle + \langle P_{Simple(j)}, Simple(j)\rangle - \langle P_{Simple(i)}, Simple(j)\rangle - \langle P_{Simple(j)}, Simple(i)\rangle} \times\\
K_{-P_{Simple(i)}} \star TK_{-P_{Simple(j)}} \star [Res_{Simple(i)} \oplus TRes_{Simple(j)}]
\end{aligned}$$

Now writing the overall power of t as t^n, noting that it is invariant under interchange of i and j,

$$\begin{aligned}[E_{Simple(i)}, F_{Simple(j)}] &= E_{Simple(i)} \star F_{Simple(j)} - T(E_{Simple(j)} \star F_{Simple(i)}) \\ &= t^n\{K_{-P_{Simple(i)}} \star TK_{-P_{Simple(j)}} \star [Res_{Simple(i)} \oplus TRes_{Simple(j)}] \\ &\quad -TK_{-P_{Simple(j)}} \star K_{-P_{Simple(i)}} \star [Res_{Simple(i)} \oplus TRes_{Simple(j)}]\}\end{aligned}$$

and $TK_{-P_{Simple(j)}} \star K_{-P_{Simple(i)}} = K_{-P_{Simple(i)}} \star TK_{-P_{Simple(j)}}$, because they commute because $\langle P_{Simple(i)}, P_{Simple(j)} \rangle = 0$, so this is just zero, as desired. □

Proposition 5.5.17. *$K_{Simple(i)}E_{Simple(j)} = t^{a^\Gamma_{ij}} E_{Simple(j)}K_{Simple(i)}$. Likewise, $K_{Simple(i)}F_{Simple(j)} = t^{-a^\Gamma_{ij}} F_{Simple(j)}K_{Simple(i)}$.*

Proof. Translation $Simple(i) \to TSimple(i)$ flips the sign of the Euler form, hence it suffices to show for the K_iE_j case. Now $a^\Gamma_{ij} = \langle Simple(i), P_{Simple(j)} \rangle + \langle P_{Simple(j)}, Simple(i) \rangle$ by the symmetrised Euler form description of the inner product on roots, theorem 3.8.1. □

Now we have verified all the necessary quantum group relations. Hence, the map of proposition 5.5.3 is a map of Hopf algebras. Because it is an isomorphism on each of the triangular components by 5.5.2, the map is an isomorphism. Hence we have proved theorem 5.5.4.

5.5.5 Using the Hall algebra to understand the quantum group

Now that we have equated the two-periodic Hall algebra and the quantum group, we could use all the tools we defined for quiver representation categories to study the quantum group. For instance, BGP reflection induces an automorphism of the two-periodic twisted Hall algebra, hence on the quantum group. In the literature, this set of automorphisms carries the name "Lusztig's symmetries", see [C] for a full discussion.

Hall algebras can also be used to understand Lusztig's canonical basis, see [R3], and Kashiwara's crystal basis, see [S2].

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