

HARVARD UNIVERSITY

SENIOR THESIS

**The K -Theoretic Classification
of Topological Materials**

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Recommended Background

It is assumed that the reader is familiar with algebraic topology at the level of [14] Ch.1-3. Introductory quantum mechanics, around the level of [35] Chs. 1-7 and 11-12 is also assumed, and familiarity with classical phase transitions and other condensed matter techniques is helpful.

Acknowledgements

Parts of this thesis were inspired by previous work at the University of Chicago REU and for Math 231BR at Harvard. Personal acknowledgements may be found at the end.

List of Symbols

| symbol | meaning in this thesis | alternate notations |
|--------------------------------|--|--------------------------------|
| a_j^\dagger | fermionic creation operator at site j | c_j^\dagger, f_j^\dagger |
| a_j | fermionic annihilation operator at site j | c_j, f_j |
| c_j | j th Majorana operator | χ_j, γ_j |
| T | time reversal symmetry | |
| Q | particle number conservation | N |
| $\sigma_x, \sigma_y, \sigma_z$ | Pauli spin matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | τ_x, τ_y, τ_z |
| $K^*(X)$ | complex K -theory ring of X | $K_{\mathbb{C}}^*(X), KU^*(X)$ |
| $KO^*(X)$ | real KO -theory ring of X | $K_{\mathbb{R}}^*(X)$ |
| Cl_k | k th real Clifford algebra | C_k |
| $Cl_k^{\mathbb{C}}$ | k th complex Clifford algebra | |
| $Cl^{p,q}$ | bigraded real Clifford algebra | |
| $[X, Y]$ | homotopy classes of continuous maps $X \rightarrow Y$ | $\pi(X, Y)$ |
| Σ | reduced suspension functor | |
| Ω | based loop space functor | |
| $\mathcal{G}(\mathcal{H})$ | space of bounded operators defined on \mathcal{H} | |
| $\mathcal{F}(\mathcal{H})$ | space of Fredholm operators defined on \mathcal{H} | |

List of Tables

The main tables in [18] are reproduced here for reference.

The following two tables keep track of the groups of invariants determined by $\pi_0(C_q)$ for a given symmetry class that relies periodically on an integer q , and indicate which systems in a particular dimension have the symmetries T and Q , which stand for time-reversal invariance and particle number conservation, respectively. Table 1 includes systems with only Q -symmetry, while Table 2 includes systems with no symmetry, T -symmetry only, and both T and Q -symmetry.

Table 1

| q | $\pi_0(C_q)$ | $d = 1$ | $d = 2$ | $d = 3$ |
|-----|--------------|---------|---------|---------|
| 0 | \mathbb{Z} | | (IQHE) | |
| 1 | 0 | | | |

Classification of Topological Insulators without Time-Reversal Symmetry, from the left side of [18] Table 1

Table 2

| q | $\pi_0(R_q)$ | $d = 1$ | $d = 2$ | $d = 3$ |
|-----|----------------|------------------|-------------|-------------|
| 0 | \mathbb{Z} | | no symmetry | T only |
| 1 | \mathbb{Z}_2 | no symmetry (MC) | T only | T and Q |
| 2 | \mathbb{Z}_2 | T only | T and Q | |
| 3 | 0 | T and Q | | |
| 4 | \mathbb{Z} | | | |
| 5 | 0 | | | |
| 6 | 0 | | | |
| 7 | 0 | | | no symmetry |

Classification of Topological Superconductors and Time-Reversal Invariant Insulators, from the right side of [18] Table 1

Note that IQHE stands for integer quantum Hall effect and MC stands for Majorana chain.

The following two tables list the classifying spaces C_q and R_q of operators corresponding to each symmetry condition, which are referenced in Tables 1 and 2. Each entry is actually meant to denote the space obtained by taking the direct limit as $k, m, n \rightarrow \infty$.

Table 3

| $q \bmod 2$ | Classifying Space C_q | $\pi_0(C_q)$ |
|-------------|---|--------------|
| 0 | $(U(k+m)/U(k) \times U(m)) \times \mathbb{Z}$ | \mathbb{Z} |
| 1 | $U(n)$ | 0 |

Complex Classifying Spaces, from the left side of [18] Table 2

Table 4

| $q \bmod 8$ | Classifying Space R_q | $\pi_0(R_q)$ |
|-------------|--|----------------|
| 0 | $(O(k+m)/(O(k) \times O(m))) \times \mathbb{Z}$ | \mathbb{Z} |
| 1 | $O(n)$ | \mathbb{Z}_2 |
| 2 | $O(2n)/U(n)$ | \mathbb{Z}_2 |
| 3 | $U(2n)/Sp(n)$ | 0 |
| 4 | $(Sp(k+m)/(Sp(k) \times Sp(m))) \times \mathbb{Z}$ | \mathbb{Z} |
| 5 | $Sp(n)$ | 0 |
| 6 | $Sp(n)/U(n)$ | 0 |
| 7 | $U(n)/O(n)$ | 0 |

Real Classifying Spaces, from the right side of [18] Table 2

Chapter 1

Mathematical Framework: K -Theory, Clifford Algebras, and Index Theory

1.1 Introduction

Topological insulators and superconductors are materials with remarkable properties. They have different electrical properties in their bulk, or interior, than on their surfaces, and they can occupy symmetry-protected phases of matter that potentially host quasiparticles that could be the key to topological quantum computing. Their properties are robust to small imperfections and deformities, but the sense in which they are *topological* is much more profound than this. The tools for understanding these materials are topological invariants, which were first introduced in the study of the quantum Hall effect and which allow for a rich interaction of theoretical condensed matter physics with algebraic topology. These topological invariants provide information about the phase of a particular model, and, even better, help understand the space of all possible models. In particular, a main goal is to classify the possible kinds of topological insulators and superconductors in each spatial dimension and to determine what kinds of phases these classes can host.

1.1.1 The Periodic Table

Several classification schemes exist, but this thesis will focus on one proposed in 2009 known as the “Periodic Table,” which uses K -theory to assign invariants and explore the possible phases in each dimension. This approach is interesting mathematically because it employs a case of the Atiyah-Singer Index Theorem, which relates the topological index of a manifold to the analytical index. In the course of the classification, mathematical results including the index theorem for skew-adjoint Fredholm operators, Bott periodicity, and the loop-space suspension adjunction appear and contribute to understanding physical phenomena.

Through an exposition of Kitaev’s 2009 paper on the periodic table [18], this thesis aims to give some useful mathematical background that clarifies its use in condensed matter physics, as well as offer physical intuition and interesting examples of the classification. It will not cover the entirety of Kitaev’s paper, but instead will elaborate on low-dimensional examples.

1.1.2 Outline

The idea of the classification is to consider the possible spaces corresponding to Hamiltonian operators that possess certain symmetry properties. These spaces, R_q and C_q , are listed in Tables 3 and 4 on the previous page. Using index theory, a given Hamiltonian corresponding to one of

those classes can be assigned a K -theory index, which indicates which phase of matter it occupies. For example, the space of 1D Hamiltonians with neither of the relevant symmetries is R_1 , and there are exactly two phases that a Hamiltonian of this kind can occupy, according to the K -theory invariant, which lives in $\pi_0(R_1) = \mathbb{Z}_2$, as indicated in Table 2 above. More generally, the periodic table classification rests upon the following proposal.

Kitaev's Proposal: *The possible phases of gapped, free-fermion models in d dimensions and with p negative symmetries are classified by*

$$\widetilde{KO}^{-p+d+2}(\text{pt}) = \pi_0(R_{p-d-2 \bmod 8}) \quad \text{or by} \quad \widetilde{K}^{-p+d+1}(\text{pt}) = \pi_0(C_{p-d-1 \bmod 2})$$

where R_q and C_q denote a spaces of operators. The choice of R_q versus C_q is also determined by the symmetry properties of the system.

Rather than provide a proof of this proposal, this thesis will give the mathematical and physical motivation for why it works, with the most thorough argument occurring for the case $d = 0$. It will also explore in detail more exciting examples in dimensions 1 and 2, in which K -theory invariants can be calculated and related to measurable phenomena. References are offered where proofs and details are omitted.

This thesis begins with mathematical background, defining K -theory and Clifford algebras and giving an indication of how the assignment of an index to an operator is useful for classifying free fermion models. It then reviews some physical definitions and tools, before discussing examples of the classification in dimensions 0, 1, and 2. Several relevant symmetry cases are discussed in the $d = 0$ chapter and shown to correspond to the appropriate classifying spaces predicted by the proposal above. The subsequent chapters on models in $d = 1$ and $d = 2$ each focus primarily on one example: the Majorana chain in 1D, and the integer quantum Hall effect in 2D.

1.2 K -Theory

Topological K -theory is a generalized cohomology theory that was formalized in the 1960s by Hirzebruch and Atiyah soon after Bott's proof of the periodicity theorem. It associates a ring $K(X)$ to a topological space X , providing an invariant that can be used to distinguish phases of matter. K -theory has a geometric formulation in terms of vector bundles and can be extended to a cohomology theory using the suspension isomorphism and Bott periodicity. The following introduction to K -theory is based on [15] Chs. 1-2, [6] Ch. 1, and [23] Lecture 1. It focuses on intuition needed for physical concepts.

1.2.1 Geometric Definition

Topological K -theory is defined using vector bundles. In some cases of the classification of topological materials, vector bundles track the ground state of the Hamiltonian as a parameter continuously varies. For example, this parameter could be momentum, which in a d -dimensional lattice system lives on a torus \mathbb{T}^d . From now on, all spaces are assumed to be compact Hausdorff.

Vector Bundles

Definition 1.2.1. An n -dimensional complex vector bundle over a space X is a total space E together with a projection map $p: E \rightarrow X$ such that the following conditions hold.

- Each fiber $E_x := p^{-1}(x)$ for $x \in X$ has the structure of a complex vector space.
- Around each point x , there is a neighborhood U for which there is a local trivialization $p^{-1}(U) \cong U \times \mathbb{C}^n$.

According to this definition, a vector bundle locally resembles a product of the base space X with \mathbb{C}^n . The interesting properties of vector bundles arise from their potential failure to be written as products globally.

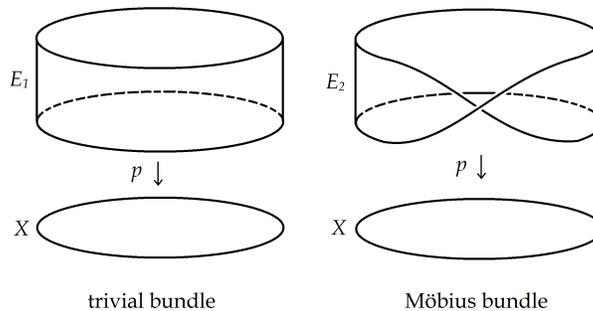
Example 1.2.2. Any product $X \times \mathbb{C}^n$ is a vector bundle. A vector bundle that can be written as a product globally is called a **trivial bundle**.

Example 1.2.3. On a complex projective space $\mathbb{C}P^n$, one can define a **tautological line bundle**, denoted H^* , whose fiber over a point $x \in \mathbb{C}P^n$ is the complex line spanned by x . This bundle is referred to as $\mathcal{O}(-1)$ in algebraic geometry, and the tautological construction works for Grassmannians in general to produce similar examples.

K-theory is usually first defined using complex vector bundles, but a version of K-theory exists for real vector bundles as well. A real vector bundle is defined as in Def. 1.2.1, but with real vector spaces instead of complex vector spaces.

Definition 1.2.4. An n -dimensional real vector bundle over X is a space E together with a projection map whose fibers have the structure of a real vector space and that satisfies the local triviality condition.

Example 1.2.5. Over $X = S^1$, there are exactly two possible one-dimensional vector bundles. One is the trivial bundle, which resembles a cylinder, and the other resembles a Möbius strip.



Example 1.2.6. Let X be a manifold. Then the tangent bundle TX is a vector bundle, with fiber the tangent space at a point. If the manifold has a complex structure, then TX is a complex vector bundle.

Definition 1.2.7. Let $p: E \rightarrow X$ and $q: F \rightarrow X$ be two vector bundles over X . They are **isomorphic** if there exists a continuous map $\phi: E \rightarrow F$ such that $\phi \circ p = q$ and ϕ maps each fiber $p^{-1}(x) \subset E$ to the fiber $q^{-1}(x) \subset F$ by a linear isomorphism.

Operations on Vector Bundles

Two operations can be defined on vector bundles that will correspond to ring operations in $K(X)$.

Definition 1.2.8. Let $E \rightarrow X$ be an n -dimensional vector bundle, and $F \rightarrow X$ an m -dimensional vector bundle. Then their **direct sum** $E \oplus F$ is a vector bundle of dimension $m + n$ over X . It can be defined at each fiber by $(E \oplus F)_x = E_x \oplus F_x$. Equivalently, it can be defined as the pullback bundle $E \times_X F$, which is defined so that the following diagram commutes.

$$\begin{array}{ccc} E \times_X F & \longrightarrow & F \\ \downarrow & & \downarrow \\ E & \longrightarrow & X \end{array}$$

Definition 1.2.9. The **tensor product** of E and F is a vector bundle $E \otimes F$ of dimension $m \cdot n$ whose fiber is $(E \otimes F)_x = E_x \otimes F_x$.

A more explicit definition of the tensor product can be given using transition functions but is not essential here.

Grothendieck Completion

Direct sum and tensor products of isomorphism classes of vector bundles give a semiring, which is almost the desired algebraic object. However, to form a ring, there must be additive inverses. The way to fix this is to perform a general construction called Grothendieck group completion.

Definition 1.2.10. Let M be a commutative monoid. The **Grothendieck group** is a quotient of the free group on M that creates formal inverses. Let $+$ denote the addition in M and $+'$ denote the addition in $\text{Gr}(M)$. Then the Grothendieck group on M is

$$\text{Gr}(M) := \mathbb{Z}(M) / ((x +' y) -' (x + y) \sim 0).$$

The idea is that the group completion identifies the addition in the monoid with the addition in the free group, creating formal inverses. The elements of the Grothendieck group are formal differences of pairs of elements of the monoid.

Example 1.2.11. The natural numbers can be completed into a group by including negative numbers, so $\text{Gr}(\mathbb{N}) = \mathbb{Z}$. Elements in the group completion are isomorphism classes of differences. For example, -2 in \mathbb{Z} corresponds to the isomorphism class $[1 - 3]$, which is the same class as $[2 - 4]$.

When this process is performed on the monoid of vector bundles under direct sum, the resulting object is a group of **virtual bundles**, which are formal differences $[E] - [F]$ of isomorphism classes of vector bundles E and F . This is the definition of the zeroth group in the cohomology theory.

Definition 1.2.12. Let X be a compact Hausdorff space, and let $\text{Vect}_{\mathbb{C}}(X)$ denote the monoid of isomorphism classes of complex vector bundles with respect to direct sum. The **zeroth K-theory group of X** is

$$K^0(X) := \text{Gr}(\text{Vect}_{\mathbb{C}}(X)).$$

With tensor product of vector bundles acting as multiplication, this in fact forms a ring.

Definition 1.2.13. Let $\text{Vect}_{\mathbb{R}}(X)$ denote the monoid of isomorphism classes of real vector bundles with respect to direct sum. The **zeroth KO-theory group of X** is

$$KO^0(X) := \text{Gr}(\text{Vect}_{\mathbb{R}}(X)).$$

1.2.2 Extension to a Generalized Cohomology Theory

Vector bundles were used to define the zeroth group $K^0(X)$ of K -theory, but K -theory actually forms a generalized cohomology theory once all of the groups $K^n(X)$ are defined. This gives it a richer algebraic structure. A quick, intuitive definition of generalized cohomology theories will be given as a reminder to the reader familiar with cohomology, but the most important attributes to understand for the physical classification are Bott periodicity and the suspension isomorphism.

Definition 1.2.14. A **generalized cohomology theory** $E^*(-)$ is a contravariant functor from the category of topological spaces to the category of graded abelian groups such that the Eilenberg-Steenrod axioms hold: homotopy invariance, a suspension isomorphism, the long exact sequence of a pair, additivity, and excision.

This definition means that for an appropriate space X , $E^*(X)$ gives a graded abelian group with levels $E^n(X)$ for $n \in \mathbb{Z}$, and if $f: X \rightarrow Y$ is a continuous map, then there is an induced homomorphism $f^*: E^*(Y) \rightarrow E^*(X)$. It will not be proved in this thesis that K -theory satisfies the axioms of a cohomology theory, but the suspension isomorphism will be discussed because it pertains to a periodic structure in the groups $K^n(X)$. To discuss suspension, one must define a reduced version of K -theory.

Reduced K -Theory and the Suspension Isomorphism

Definition 1.2.15. Let X be a based space with base point x_0 , and let $\iota: x_0 \hookrightarrow X$ be the inclusion map. The **zeroth reduced K -theory group of X** is

$$\tilde{K}^0(X) := \ker \iota^* = \ker(K^0(X) \rightarrow K^0(x_0)).$$

Taking reduced K -theory can be thought of as modding out by the dimension of the vector bundle over the base point. In general, unreduced K -theory satisfies $K(X) = \tilde{K}(X) \oplus K(x_0) = \tilde{K}(X) \oplus \mathbb{Z}$.

Definition 1.2.16. The **reduced suspension** of a based space X with base point x_0 is a quotient space

$$\Sigma X := X \wedge S^1$$

where \wedge is the smash product. Equivalently,

$$\Sigma X = X \times I / (X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I).$$

Example 1.2.17. The n -fold reduced suspension of a sphere increases the dimension of the sphere. That is, $\Sigma^n S^k \simeq S^{n+k}$, and in particular, $S^n \simeq \Sigma^n S^0$.

Now $\tilde{K}^n(X)$ can begin to be defined for n other than zero.

Definition 1.2.18. Let X be a well-based space, and let $n > 0$. Define, according to the *suspension isomorphism*, $\tilde{K}^{-n}(X) := \tilde{K}^0(\Sigma^n X)$ and $\widetilde{KO}^{-n}(X) := \widetilde{KO}^0(\Sigma^n X)$.

Bott Periodicity

As for positive K -groups, one must invoke Bott periodicity. Initially proven by Raoul Bott using Morse theory, Bott periodicity was one of the first applications of K -theory. In the statements below, $*$ stands for an arbitrary degree.

Theorem 1.2.19 (Complex Bott Periodicity). *There is a group isomorphism $K^*(X) \cong K^{*-2}(X)$.*

Theorem 1.2.20 (Real Bott Periodicity). *There is a group isomorphism $KO^*(X) \cong KO^{*-8}(X)$.*

There are many different formulations and proofs of Bott periodicity. For example, proof using a product theorem in K -theory can be found in [6] §2.2 or [15] Ch. 2, and a proof using the index theory and Clifford algebra techniques discussed later in this thesis can be found in [8] §5. This thesis will not include a proof.

With the suspension isomorphism defining $\tilde{K}^{-n}(X)$, and with Bott periodicity ensuring a periodic structure, positive K -groups can now be defined.

Definition 1.2.21. Define the positive complex K -groups by

$$\tilde{K}^n(X) := \begin{cases} \tilde{K}^0(X) & \text{for } n \text{ even} \\ \tilde{K}^0(\Sigma X) = \tilde{K}^{-1}(X) & \text{for } n \text{ odd} \end{cases}$$

Definition 1.2.22. Similarly, define the positive real KO -groups by

$$\widetilde{KO}^n(X) := \widetilde{KO}^0(\Sigma^j X) = \widetilde{KO}^{-j}(X) \text{ for } n \equiv -j \pmod{8}.$$

Once negative groups are defined, K -theory forms a cohomology theory. This fact will not be proven here, but will be invoked in later calculations.

Bott periodicity has wider significance in algebraic topology, but the main result of Bott periodicity for the classification of topological insulators and superconductors is that the classification is periodic in dimension. Specifically, when the symmetry classes of the classification are written in the correct order, increasing the dimension of the material by one will shift the symmetry class by one; for example, similar phases exist in the 1D Majorana chain and the 2D time-reversal invariant insulator. A few facts will be asserted next that aid in calculations.

Fact 1.2.23. Let X be compact. As long as there is a metric on the vector bundle, it is possible to find a complementary vector bundle F over X such that $E \oplus F = X \times \mathbb{C}^n$, the n -dimensional trivial bundle.

For proof, see Cor. 1.4.14 in the appendix to [6].

Fact 1.2.24. Any element of $K^0(X)$ can be represented as a virtual bundle difference in which one of the bundles is trivial. That is, any element of $K^0(X)$ can be written as $[E] - n$ for n the n -dimensional trivial bundle $X \times \mathbb{C}^n$.

Proof. As in [6] §2.1, let $[G] - [F]$ be an arbitrary element of $K^0(X)$. Using the previous fact, let H be the bundle such that $F \oplus H \simeq X \times \mathbb{C}^n$. Then $[G] - [F] = [G] + [H] - ([F] + [H]) = [G \oplus H] - n$. \square

1.2.3 Examples

Example 1.2.25. The K-theory of a point or of any contractible space is easily calculated. Assuming that the homotopy invariance axiom holds, the K-theory of a point and of a contractible space is the same. Then, a vector bundle over a point is just a single virtual vector space, meaning that the bundle cannot possibly be nontrivial. This virtual vector space is a formal difference of two isomorphism classes of vector spaces, each of which is determined by the nonnegative integer representing the vector space dimension. The difference in their dimensions determines the virtual dimension of the bundle, which can be any integer. Since the virtual bundles over a point correspond to the integers, $K^0(\text{pt}) = \mathbb{Z}$. The same reasoning holds for real vector bundles, so $KO^0(\text{pt}) = \mathbb{Z}$.

Example 1.2.26. Consider a based contractible space. The reduced K-theory mods out by the dimension over the base point, so if there is no other structure then the reduced group is trivial. Hence $\tilde{K}^0(\text{pt}) = 0$.

Example 1.2.27. Now consider the complex K-theory of the sphere. For S^0 , the two disjoint points each contribute a virtual bundle of integral dimension, so $K^0(S^0) = \mathbb{Z}^2$. Meanwhile, all complex vector bundles over S^1 are trivial, so $K^0(S^1) = \mathbb{Z}$. Invoking Bott periodicity and the suspension isomorphism, one can calculate

$$\begin{aligned}\tilde{K}^0(S^{2n}) &= \tilde{K}^{-2n}(S^0) = \mathbb{Z} \\ \tilde{K}^0(S^{2n+1}) &= \tilde{K}^{-2n-1}(S^0) = 0.\end{aligned}$$

Example 1.2.28. The real KO-theory of the sphere is more complicated, since there is an 8-fold periodicity instead of only 2-fold. The same argument as for the previous example shows that $KO^0(S^0) = \mathbb{Z}^2$ and $\widetilde{KO}^0(S^0) = \mathbb{Z}$, while the claim in Ex. 1.2.5 that there are only two real one-dimensional vector bundles over the circle gives that $\widetilde{KO}^0(S^1) = \mathbb{Z}_2$. The rest of the groups will not be calculated, but listed here for reference.

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-------------------------|--------------|----------------|----------------|---|--------------|---|---|---|
| $\widetilde{KO}^0(S^n)$ | \mathbb{Z} | \mathbb{Z}_2 | \mathbb{Z}_2 | 0 | \mathbb{Z} | 0 | 0 | 0 |

Example 1.2.29. This calculation will use cohomological techniques. Consider \mathbb{T}^2 , the two-dimensional torus. The torus can be written as a product $\mathbb{T}^2 \simeq S^1 \times S^1$. Recall that the long exact sequence of the pair $X \times Y, X \wedge Y$ gives an isomorphism $\tilde{K}(X \times Y) \cong \tilde{K}(X \wedge Y) \oplus \tilde{K}(X) \oplus \tilde{K}(Y)$, so

$$\begin{aligned}\tilde{K}(\mathbb{T}^2) &\cong \tilde{K}(S^1 \times S^1) \\ &\cong \tilde{K}(S^1 \wedge S^1) \oplus \tilde{K}(S^1) \oplus \tilde{K}(S^1) \\ &\cong \tilde{K}(S^2) \oplus 0 \oplus 0 \\ &\cong \mathbb{Z}.\end{aligned}$$

This result will be important later when discussing the quantum Hall effect in Ch. 4.

Summary

This concludes a very quick introduction to K-theory. The main takeaways for the rest of this thesis are the definition of real and complex vector bundles, the examples of K-groups and KO-groups calculated for spheres and the torus, and the fact that K-theory forms a cohomology theory.

The last fact will be important in developing index theory for skew-adjoint Fredholm operators. Elements in K -theory and KO -theory will distinguish phases in the classification of matter, but the next important ingredient for this is symmetry. The mathematical objects necessary for this are Clifford algebras.

1.3 Clifford Algebras

Clifford algebras have uses in many areas of mathematics and physics, but in this thesis they will serve to keep track of the symmetries of condensed matter systems. Specifically, it will be shown that the algebra generated by Majorana operators acting on the Fock space forms a Clifford algebra, while the symmetry operators for time-reversal symmetry and particle number conservation act as representations of Clifford generators. These will be discussed in sections 2.2 and 2.5. In section 1.4.4, commutations with Clifford representations will define more specific subspaces of Fredholm operators, whose index theory is the mathematical foundation for the K -theoretic classification. The following introduction to Clifford algebras closely follows [7] Ch. 1.

1.3.1 Definition

A Clifford algebra is a unital, associative algebra generated by a vector space or module that is equipped with a quadratic form. Let \mathbb{F} be a field, let E be a \mathbb{F} -module, and let $Q : E \rightarrow \mathbb{F}$ be a quadratic form on E . Recall that the **tensor algebra** of E is

$$T(E) = \bigoplus_{i=0}^{\infty} T^i E = \bigoplus_{i=0}^{\infty} E^{\otimes i} = \mathbb{F} \oplus E \oplus (E \otimes E) \oplus (E \otimes E \otimes E) \oplus \dots$$

with multiplication given by the tensor product and using the identification $T^i E \otimes T^j E \cong T^{i+j} E$. The Clifford algebra is a quotient of the tensor algebra that uses the quadratic form to assign a signed notion of length to vectors.

Definition 1.3.1. Let $I(Q)$ be the two-sided ideal in $T(E)$ generated by elements of the form $x \otimes x - Q(x) \cdot 1$ for $x \in E$. The **Clifford algebra** is

$$Cl(E, Q) := T(E)/I(Q),$$

the quotient of the tensor algebra by the ideal $I(Q)$.

In this thesis, the Clifford algebras of interest are real Clifford algebras $Cl^{p,q}$, which are more specifically defined.

Definition 1.3.2. The **real Clifford algebra** $Cl^{p,q}$ with $p + q = k$ is the Clifford algebra defined for the vector space \mathbb{R}^k according to the quadratic form

$$Q(x) = - \sum_{i=1}^p x_i^2 + \sum_{j=p+1}^{p+q} x_j^2$$

for $x = (x_1, \dots, x_k)^T \in \mathbb{R}^k$.

In the case that $q = 0$, the quadratic form is negative definite and the Clifford algebra is denoted Cl_p . This is the algebra most commonly referred to when one discusses Clifford algebras. Complexified versions, $Cl_p^{\mathbb{C}} = Cl_p \otimes_{\mathbb{R}} \mathbb{C}$, are also of interest.

1.3.2 Presentation

One can write down generators and relations for Clifford algebras using an injection of the module E into the Clifford algebra. If $\{e_i\}$ is a basis for the copy of E living in $Cl(E, Q)$, then the set of products with increasing indices $e_{i_1} \cdot e_{i_2} \cdots e_{i_k}$ with $i_1 < i_2 < \dots < i_k$ along with 1, form an additive basis for $Cl(E, Q)$.

For the real Clifford algebra $Cl^{p,q}$, consider the images of the standard basis vectors of \mathbb{R}^{p+q} , denoted $\{e_1, \dots, e_{p+q}\}$. The Clifford algebra is generated by these elements subject to the constraints from the quadratic form, so

$$Cl^{p,q} = \langle e_1, \dots, e_{p+q} \mid e_i e_j = -e_j e_i, e_i^2 = -1, e_j^2 = 1, \text{ for } 1 \leq i \leq p < j \leq p+q \rangle.$$

1.3.3 Grading

Whereas tensor algebras are \mathbb{Z} -graded, Clifford algebras are \mathbb{Z}_2 -graded, meaning that they decompose into direct sums of odd and even parts. Specifically, let $Cl^0(E, Q)$ be the image of $\sum_{i=0}^{\infty} T^{2i} E$ in $Cl(E, Q)$ and let $Cl^1(E, Q)$ be the image of $\sum_{i=0}^{\infty} T^{2i+1} E$ in $Cl(E, Q)$. Then the algebra decomposes as follows.

(i) $Cl(E, Q) = Cl^0(E, Q) \oplus Cl^1(E, Q)$

(ii) If $x_i \in Cl^i(E, Q)$ and $y_j \in Cl^j(E, Q)$, then $x_i y_j \in Cl^k(E, Q)$ for $k \equiv i + j \pmod{2}$.

Grading is visible for real Clifford algebras using the generators and relations above, and for an arbitrary element $e_{i_1} \cdots e_{i_n}$ is determined by $n \pmod{2}$. So, $(Cl^{p,q})^0$ consists of products of even numbers of basis elements, while $(Cl^{p,q})^1$ consists of products of odd length.

Fact 1.3.3. *The even part $(Cl^{p+1,q})^0$ of $Cl^{p+1,q}$ is isomorphic to $Cl^{p,q}$.*

Proof. Let $\{e_1, \dots, e_{p+q+1}\}$ be the standard generators of $Cl^{p+1,q}$, and define a map

$$\varphi: \text{span}\{e_i \mid i \neq p+1\} \rightarrow (Cl^{p+1,q})^0 \quad \text{by} \quad \varphi(e_i) = e_{p+1} e_i.$$

The elements $e_{p+1} e_i$ form a basis of $(Cl^{p+1,q})^0$ because any even combination of arbitrary elements $e_j e_k$ in $(Cl^{p+1,q})^0$ can be achieved by

$$(e_{p+1} e_j)(e_{p+1} e_k) = -e_{p+1}^2 e_j e_k = (-1)^2 e_j e_k.$$

So, the map is a bijection because it maps basis elements to basis elements. It respects the Clifford algebra structure because for any vector $x = \sum_{i \neq p+1} x_i e_i$,

$$\begin{aligned} \varphi(x) \cdot \varphi(x) &= \sum_{i,j} x_i x_j e_{p+1} e_i e_{p+1} e_j \\ &= (-1)^2 \sum_{i,j} x_i x_j e_i e_j \\ &= x \cdot x, \end{aligned}$$

indicating that φ is an algebra homomorphism. □

In particular, the even part $(Cl_{k+1})^0$ of the $(k+1)$ st Clifford algebra is isomorphic to the Clifford algebra Cl_k one lower. This fact is useful for calculating Clifford algebras as well as for more complicated constructions with Clifford modules.

1.3.4 Examples

Low-dimensional Clifford algebras Cl_k are isomorphic to familiar algebras.

Example 1.3.4. *The trivial Clifford algebra is just the ground field, $Cl_0 \cong \mathbb{R}$, which is generated by the element 1.*

Example 1.3.5. *The first Clifford algebra is $Cl_1 \cong \mathbb{C}$, the complex numbers, whose nonidentity element i satisfies $i^2 = -1$ and is identified with e_1 .*

Example 1.3.6. *The second Clifford algebra is $Cl_2 \cong \mathbb{H}$, the quaternions. Identify the quaternion elements as $i = e_1, j = e_2, \text{ and } k = e_1 e_2$.*

Example 1.3.7. *The third Clifford algebra, $Cl_3 \cong \mathbb{H} \oplus \mathbb{H}$ is the first to be non-simple. Using the \mathbb{Z}_2 -grading, one can define a map on the generators of the even part to \mathbb{H} . In terms of the usual notation for the elements of \mathbb{H} , the correspondence is as follows.*

| Cl_2 | Cl_3^0 |
|--------|-----------|
| 1 | 1 |
| i | $e_1 e_2$ |
| j | $e_1 e_3$ |
| k | $e_2 e_3$ |

One can check that the map defined on these basis elements extends linearly to an algebra isomorphism by computing the products.

1.3.5 Periodicity

A critical property of Clifford algebras is that their representations are periodic. This means that there are a finite number of isomorphism classes of Clifford algebra representations, which will lead to a finite number of symmetry classes in the classification of Hamiltonians. There are eight types of algebras Cl_k and two types of complexified algebras $Cl_k^{\mathbb{C}}$, as shown in the following table, where the notation $\mathbb{F}(n)$ indicates the algebra of $n \times n$ matrices over \mathbb{F} .

| k | Cl_k | $Cl_k^{\mathbb{C}}$ |
|-----|--------------------------------------|--------------------------------|
| 1 | \mathbb{C} | $\mathbb{C} \oplus \mathbb{C}$ |
| 2 | \mathbb{H} | $\mathbb{C}(2)$ |
| 3 | $\mathbb{H} \oplus \mathbb{H}$ | |
| 4 | $\mathbb{H}(2)$ | |
| 5 | $\mathbb{C}(4)$ | |
| 6 | $\mathbb{R}(8)$ | |
| 7 | $\mathbb{R}(8) \oplus \mathbb{R}(8)$ | |
| 8 | $\mathbb{R}(16)$ | |

Periodicity will be discussed and briefly proven for the algebras Cl_k and not for the bigraded algebras $Cl^{p,q}$ or complexified algebras $Cl_k^{\mathbb{C}}$, but the approach is similar in those cases. The proof will follow the approach of [7].

Theorem 1.3.8. *The algebras Cl_k are 8-periodic, with $Cl_{k+8} \cong Cl_k \otimes_{\mathbb{R}} Cl_8$.*

Lemma 1.3.9. *There are algebra isomorphisms $Cl_k \cong Cl_2 \otimes Cl^{0,k-2}$ and $Cl^{0,k} \cong Cl^{0,2} \otimes Cl_{k-2}$.*

Proof. Each isomorphism can be written using the same formula. Let e_1, \dots, e_k be a basis of Cl_k , or of $Cl^{0,k}$, respectively. Define a map on these basis elements by

$$e_i \mapsto \begin{cases} e_1 \otimes 1 & \text{if } i = 1, \\ e_2 \otimes 1 & \text{if } i = 2, \\ e_1 e_2 \otimes e_{i-2} & \text{if } i \geq 3, \end{cases}$$

and extend linearly. One can check that this process yields an algebra isomorphism in each case. \square

Proof of 1.3.8. By the isomorphisms above, $Cl_4 \cong Cl_2 \otimes Cl^{0,2} \cong Cl^{0,4}$. Then

$$Cl_{k+4} \cong Cl^{0,k+2} \otimes Cl_2 \cong Cl_k \otimes Cl^{0,2} \otimes Cl_2 \cong Cl_k \otimes Cl_4,$$

so

$$Cl_{k+8} \cong Cl_{k+4} \otimes Cl_4 \cong Cl_k \otimes Cl_4 \otimes Cl_4 \cong Cl_k \otimes Cl_8.$$

From the table, $Cl_8 \cong \mathbb{R}(16)$. \square

This periodicity has important implications for the representation theory of Clifford algebras because the algebra $Cl_8 \cong \mathbb{R}(16)$ is simple. Recall that an algebra is simple if it has no proper two-sided ideals.

Fact 1.3.10. *All matrix algebras are simple.*

A proof can be found in an algebra textbook, such as in [11] Chapter IX.

From this fact, one can see that the algebras Cl_k are simple for all $k \not\equiv 3 \pmod{4}$, while the algebras $Cl_k^{\mathbb{C}}$ are simple for odd k . The consequence of the algebras Cl_8 and $Cl_2^{\mathbb{C}}$ being simple is precisely periodicity up to Morita equivalence.

Definition 1.3.11. *Two rings R and S are **Morita equivalent** if the categories of right (resp. left) modules over R is equivalent to the category of right (resp. left) modules over S .*

Morita equivalence means that two rings have the same representation theory, and it is a fact that the representation theory of an algebra is preserved upon tensoring with a simple algebra; see a reference like [4]. So, the periodicity relationship

$$Cl_{k+8} \cong Cl_k \otimes_{\mathbb{R}} Cl_8 \cong Cl_k \otimes_{\mathbb{R}} \mathbb{R}(16),$$

given the fact that $\mathbb{R}(16)$ is simple, ensures that the algebras Cl_{k+8} and Cl_k are Morita equivalent. A similar explanation holds for the complex case. Morita equivalence is sufficient for the purposes of classification because Clifford representations, not Clifford generators, are what will track the symmetries of Hamiltonian operators.

1.4 Fredholm Operators and the Index Map

Index theory lies at the intersection of algebraic topology and analysis and can be used to demonstrate important topological results including Bott periodicity. It connects two calculations that can be done for a space—a topological index and an analytical index. The version of the index theorem required for the classification of topological materials is index theory for skew-adjoint Fredholm operators, which associates a particular K -theory class to a family of operators parameterized by some topological space. This section will introduce Fredholm operators, define the index map for complex K -theory, and indicate how the space of Fredholm operators classifies K -theory. Clifford algebras will enter into the picture in the next section.

1.4.1 Fredholm Operators

Recall that a Hilbert space is like a vector space with an inner product, but could be infinite-dimensional.

Definition 1.4.1. A *Hilbert space* is a real or complex inner product space that is a complete metric space with respect to the norm induced by its inner product.

Familiar examples are \mathbb{R}^k or \mathbb{C}^k with their standard inner products, or the space of periodic L^2 functions. These spaces can hold the states of particles in quantum mechanics. Fredholm operators are bounded linear operators on Hilbert space that are “almost invertible.”

Definition 1.4.2. An bounded linear operator T is *Fredholm* if $\ker T$ and $\operatorname{coker} T$ are finite-dimensional.

For a Fredholm operator T defined on a Hilbert space \mathcal{H} , the kernel of T is a subset of \mathcal{H} , and the cokernel $\operatorname{coker} T = \mathcal{H}/\operatorname{im} T$ can be identified with $(\operatorname{im} T)^\perp \subset \mathcal{H}$. It can be shown that these finite-dimensionality constraints keep the operator invertible modulo compact operators. See [6] Prop. A8.

Definition 1.4.3. The *index* of a Fredholm operator T is $\operatorname{index} T = \dim \ker T - \dim \operatorname{coker} T$.

Example 1.4.4 (Finite-dimensional Spaces). The index of a linear operator between finite-dimensional vector spaces depends only on the dimensions of the spaces and not on the choice of map. If $T : V \rightarrow W$ for finite-dimensional vector spaces V and W , then the Rank-Nullity Theorem ensures that $\operatorname{index} T = \dim V - \dim W$.

Example 1.4.5 (Shift Operator). Consider an infinite-dimensional Hilbert space \mathcal{H} with basis $\{h_1, h_2, \dots\}$. Define the right shift operator $T_k: \mathcal{H} \rightarrow \mathcal{H}$ by

$$T_k : h_i \mapsto \begin{cases} h_{i-k} & i - k > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\ker T_k = \text{Span}\{h_1, \dots, h_k\}$, while $\text{coker } T_k = \mathcal{H}/\mathcal{H} \cong 0$, so $\text{index } T_k = k$.

1.4.2 Spaces of Operators

Fredholm operators are an analytical concept, but the set of Fredholm operators defined on a particular Hilbert space also has topological and algebraic structure. In fact, Fredholm operators on a Hilbert space form a Banach algebra, which is a complete, normed vector space with an algebraic structure such that for all a, b in the algebra, $\|a\| \cdot \|b\| \geq \|a \cdot b\|$.

The norm defined on bounded operators is the usual operator norm

$$\|T\| := \sup_{h \in \mathcal{H}, h \neq 0} \frac{\|Th\|}{\|h\|},$$

while the vector space structure is given by addition of operators and scaling by elements of the field of the Hilbert space. The multiplication in the algebra is given by composition of operators. Proof that the vector space of bounded operators is complete with respect to this norm can be found in a functional analysis textbook, such as in [21] §2.10.

Definition 1.4.6. Let $\mathcal{G}(\mathcal{H})$ be the *space of bounded operators* defined on \mathcal{H} .

It can be shown that the Fredholm operators form an open subset of $\mathcal{G}(\mathcal{H})$. It is clear that linear combinations of Fredholm operators are Fredholm, and one can check that composition of Fredholm operators is Fredholm. For details, see the appendix to [6].

Definition 1.4.7. Let $\mathcal{F}(\mathcal{H})$ denote the *space of Fredholm operators* defined on \mathcal{H} .

The index as defined above is a map $\mathcal{F}(\mathcal{H}) \rightarrow \mathbb{Z}$. It is continuous on $\mathcal{F}(\mathcal{H})$ with the norm topology, so since it maps to the integers, a disconnected space, it must be constant on connected components of $\mathcal{F}(\mathcal{H})$. In fact, it determines a bijective map $\pi_0(\mathcal{F}(\mathcal{H})) \rightarrow \mathbb{Z}$. [8]

1.4.3 Extension to K-Theory

Let \mathcal{H} be a complex Hilbert space. Consider the case of an operator defined on \mathcal{H} that takes in a parameter from a space X and varies continuously as this parameter changes. For a particular $x \in X$, the operator T_x is an element of $\mathcal{F}(\mathcal{H})$. The family of operators $\{T_x\}_{x \in X}$ can be described by a continuous map $T: X \rightarrow \mathcal{F}(\mathcal{H})$ taking $x \mapsto T_x$. The definition of the index map may be extended to such a family of operators, but in this general case the target is no longer \mathbb{Z} , but $K(X)$. The reference for this section is the appendix to [6].

Definition 1.4.8. Let $A: X \rightarrow \mathcal{F}(\mathcal{H})$ define a continuous family of operators, and assume that $\ker A$ and $\text{coker } A$ are vector bundles. Then, the index of A is

$$\text{index } A = [\ker A] - [\text{coker } A].$$

In this case, $\ker A$ is the vector bundle with fiber $\ker A_x$ over x and $\operatorname{coker} A$ is the vector bundle with fiber $\operatorname{coker} A_x$ over x . It can be shown that this definition of index depends only on the homotopy class of A . The formal difference of $[\ker A]$ and $[\operatorname{coker} A]$ lies in the Grothendieck ring of isomorphism classes of vector bundles over X , which is precisely $K^0(X)$. Hence the index defines a map $[X, \mathcal{F}(\mathcal{H})] \rightarrow K^0(X)$.

Remark 1.4.9. *The formula above does not hold in general. For general A , the dimensions of $\ker A_x$ and $\operatorname{coker} A_x$ are not necessarily locally constant, meaning that $\ker A$ and $\operatorname{coker} A$ may fail to be vector bundles. In this case, the map A may be deformed to yield a homotopic admissible operator, for which $\ker A$ and $\operatorname{coker} A$ are vector bundles. Alternately, a definition of index using a finite-codimension subspace $\mathcal{V} \subset \mathcal{H}$ such that $\ker A_x \subset \mathcal{V}^\perp$ for all x works for any operator A , admissible or not. See Prop. A5 in the appendix to [6]. Other methods also exist to ensure that the index gives a virtual bundle.*

Example 1.4.10 (Trivial Case). *In the case that X is a point, or is contractible, any operator $A: X \rightarrow \mathcal{F}(\mathcal{H})$ is homotopic to an operator constant with respect to X . Hence this reduces to calculating the index of a single operator, which lies in \mathbb{Z} as discussed earlier. Indeed, $K(\operatorname{pt}) = \mathbb{Z}$, and in this case the index map gives the dimension of the trivial virtual vector bundle $[\ker A] - [\operatorname{coker} A]$.*

Nontrivial examples with physical motivation will be discussed in later chapters.

It has been argued how the target of the index map lies in K -theory, but the real power of the index map is that it is an isomorphism.

Theorem 1.4.11. *Let X be a compact Hausdorff space, and let \mathcal{H} be a complex Hilbert space. Then the map*

$$\operatorname{index}: [X, \mathcal{F}(\mathcal{H})] \rightarrow K^0(X)$$

is a natural isomorphism.

Proof Sketch. This result appears as [6] Prop. A6, and the following sketches the proof given there. After establishing rigorously that the index map is well-defined up to homotopy, is functorial, and is a homomorphism, the idea of the proof is to produce an exact sequence of semigroups

$$[X, \mathcal{G}(\mathcal{H})^*] \rightarrow [X, \mathcal{F}(\mathcal{H})] \xrightarrow{\operatorname{index}} K^0(X),$$

where $\mathcal{G}(\mathcal{H})^*$ denotes the invertible bounded operators, which are the units of the algebra of bounded operators. Kuiper's Theorem [22] establishes that this group of units is actually contractible, which renders the semigroup $[X, \mathcal{G}(\mathcal{H})^*]$ trivial and forces the second map in the exact sequence to be injective.

Surjectivity of the index can be shown by constructing a virtual bundle for every index using a projection operator. By 1.2.24, it suffices to find the preimage of an arbitrary element $n - [E]$. \square

In the case that the Hilbert space is defined over \mathbb{R} , the kernel and cokernel of the map define real vector bundles. A similar result holds for the index map to real KO -theory, and will be generalized further in the next section.

Theorem 1.4.12. *Let X be a compact Hausdorff space, and let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space. Then the map*

$$\operatorname{index}: [X, \mathcal{F}(\mathcal{H}_{\mathbb{R}})] \rightarrow KO^0(X)$$

is a natural isomorphism.

Example 1.4.13. When $X = pt$, these theorems reduce to an identification between $[pt, \mathcal{F}]$ and $K(pt)$ or $KO(pt)$, respectively. Maps $pt \rightarrow \mathcal{F}$ just detect a connected component of \mathcal{F} , which lives in $\pi_0(\mathcal{F})$.

Classifying Spaces and the Loop Space-Suspension Adjunction

This thesis will not discuss very much homotopy theory, but a short discussion of classifying spaces and loop spaces is necessary to explain how the different spaces of Hamiltonians relate to each other. Specifically, this section is meant to give context for Kitaev's comment about the classifying spaces in [18] Table 2 and to differentiate this relationship between classifying spaces from Bott periodicity.

Because that the index map is a natural isomorphism, $\mathcal{F}(\mathcal{H})$ is a **classifying space** for K -theory. This means that the functors $K^0(-)$ and $[-, \mathcal{F}(\mathcal{H})]$, which stands for homotopy classes of maps into the space of Fredholm operators, are interchangeable; one specifies the same data to calculate the K^0 -group of a space and to calculate the maps from that space into the space of Fredholm operators. In a similar way, $\mathcal{F}(H_{\mathbb{R}})$ classifies KO^0 .

Higher K -groups are defined using the suspension operation, with $\tilde{K}^{-n}(X) = \tilde{K}^0(\Sigma^n X)$. There is a related operation called taking the loop space.

Definition 1.4.14. The *loop space* of a space X is

$$\Omega X := [S^1, X]_+,$$

the space of based maps from the circle into X .

Fact 1.4.15. For two spaces X and Y , there is an isomorphism $[\Sigma X, Y] \cong [X, \Omega Y]$.

The categorical language for this fact is that suspension and looping are adjoint functors. Using this, one defines classifying spaces for the functors K^{-n} . Starting from a negative K -group,

$$K^{-n}(X) \cong [\Sigma^n X, \mathcal{F}(\mathcal{H})] \cong [X, \Omega^n(\mathcal{F}(\mathcal{H}))]$$

so one can say that $\Omega^n(\mathcal{F}(\mathcal{H}))$ classifies the functor K^{-n} . After positive K -groups are defined using Bott periodicity, these classifying space results mean that spaces of Fredholm operators define an Ω -prespectrum for K -theory, as discussed in [10] Lecture 13.

However, the takeaway for the physical application is that the spaces of Hamiltonians discussed in [18] and listed in Table 2 are related by loopings. Specifically, the classifying spaces satisfy $C_{k+1} \simeq \Omega C_k$ and $R_{k+1} \simeq \Omega R_k$. As a result, $C_q \simeq \Omega^q C_0$ and $R_q \simeq \Omega^q R_0$. It has not been explained why consecutive symmetry classes should be related by loopings or similar functors, but this will be clarified in the next section.

Remark 1.4.16. Care should be taken not to attribute these loop space relationships entirely to Bott periodicity. Bott periodicity leads to an 8-periodic structure in the real case and a 2-periodic structure in the complex case, which is why there are only ten spaces of interest. However, the relationship between consecutive spaces C_k and C_{k+1} or R_k and R_{k+1} really arises from the loop space-suspension adjunction, and only requires Bott periodicity to relate the spaces $C_k \simeq C_{k+2}$ or $R_k \simeq R_{k+7}$.

1.4.4 Spaces of Skew-Adjoint, Clifford-Equivariant Fredholm Operators

To get to the theorems that relate to the physical classification, the discussions of Clifford algebras and index theory must be combined. Instead of considering Fredholm operators in general, one must consider separate classes of Fredholm operators that possess different symmetries. These symmetries will be detected by anticommutation relations with operators J_i that serve as representations of a Clifford algebra. Specifically, an operator that anticommutes with k of these generators will be Cl_k -equivariant.

Once index theory for these operators is developed, a Hamiltonian belonging to a specific symmetry class can be associated with an invariant living in an appropriate K - or KO -group of the periodic table. One should think of connected components of a particular classifying space as determining a phase of the corresponding particular class of Hamiltonian.

Skew-Adjoint Fredholm Operators

Skew-adjoint operators will arise naturally in the discussion of Hamiltonians later.

Definition 1.4.17. A *skew-adjoint* or *skew-Hermitian* operator is an operator A such that $A^\dagger = -A$.

Definition 1.4.18. Let $\hat{\mathcal{F}}(\mathcal{H})$ denote the *space of skew-adjoint Fredholm operators* on \mathcal{H} .

Classifying Spaces of Skew-Adjoint Fredholm Operators

The space of skew-adjoint Fredholm operators also classifies a K -theory functor. However, it is no longer at the zeroth level of the cohomology theory, and there are two cases depending on whether the underlying Hilbert space is real or complex. The theorem statements and letter labels in the following discussion are from [8], and $\mathbb{1}$ denotes the identity operator on $\mathcal{H}_{\mathbb{R}}$ or \mathcal{H} , respectively.

Theorem 1.4.19 (A). Let $\alpha : \hat{\mathcal{F}}(\mathcal{H}_{\mathbb{R}}) \rightarrow \Omega\mathcal{F}(\mathcal{H}_{\mathbb{R}})$ be the map that takes an operator $A \in \hat{\mathcal{F}}(\mathcal{H}_{\mathbb{R}})$ to the path from $-\mathbb{1}$ to $\mathbb{1}$ in $\mathcal{F}(\mathcal{H}_{\mathbb{R}})$ defined by

$$\cos \pi t + A \sin \pi t, \quad 0 \leq t \leq 1.$$

Then α is a homotopy equivalence, and so $\hat{\mathcal{F}}(\mathcal{H}_{\mathbb{R}})$ is a classifying space for the functor KO^{-1} .

Remark 1.4.20. The notation $\Omega\mathcal{F}$ stands for the space of paths in \mathcal{F} from $-\mathbb{1}$ to $\mathbb{1}$. That the space $\Omega\mathcal{F}(\mathcal{H}_{\mathbb{R}})$ classifies KO^{-1} (and later, that $\Omega^k\mathcal{F}(\mathcal{H}_{\mathbb{R}})$ classifies KO^{-k}) follows from the suspension axiom and from the fact that $\mathcal{F}(\mathcal{H}_{\mathbb{R}})$ classifies KO^0 .

In the complex case, the space of operators breaks up into three components, two of which are contractible and one of which is a classifying space for the complex K -theory functor K^{-1} .

Theorem 1.4.21 (B). The space $\hat{\mathcal{F}}(\mathcal{H})$ decomposes into three components, denoted $\hat{\mathcal{F}}_+(\mathcal{H})$, $\hat{\mathcal{F}}_-(\mathcal{H})$, and $\hat{\mathcal{F}}_*(\mathcal{H})$, characterized by

$$\begin{aligned} A \in \hat{\mathcal{F}}_+(\mathcal{H}) &\iff i^{-1}A \text{ is essentially positive} \\ A \in \hat{\mathcal{F}}_-(\mathcal{H}) &\iff i^{-1}A \text{ is essentially negative} \\ A \in \hat{\mathcal{F}}_*(\mathcal{H}) &\iff A \notin \hat{\mathcal{F}}_{\pm}(\mathcal{H}). \end{aligned}$$

The two components $\hat{\mathcal{F}}_+(\mathcal{H})$ and $\hat{\mathcal{F}}_-(\mathcal{H})$ are contractible, while the component $\hat{\mathcal{F}}_*(\mathcal{H})$ is homotopy equivalent to $\Omega\mathcal{F}(\mathcal{H})$ by the map α as defined in Thm. 1.4.19. Thus $\hat{\mathcal{F}}_*(\mathcal{H})$ is a classifying space for K^{-1} .

Note that i is the imaginary unit, and to say that $i^{-1}(A)$ is essentially positive means $i^{-1}(A)$ is positive on some invariant subspace of \mathcal{H} of finite codimension.

Remark 1.4.22. *The notion of index for these operators is more complicated than the previous case. It requires an elaboration on the theory of Clifford modules and enters into the realm of noncommutative geometry. Since this thesis will only include lattice models, which are easier to solve than the general case, a discussion of noncommutative geometry will not be included, but it can be found in sources like [31].*

Clifford Representations

Using Clifford representations, the preceding theorems can each be generalized to subspaces of the spaces of Fredholm operators to inductively provide classifying spaces for KO^{-k} and K^{-k} .

Let $\mathcal{H}_{\mathbb{R}}$ be a Hilbert space equipped with an action of Cl_{k-1} and having the property that there exists an operator J_k on $\mathcal{H}_{\mathbb{R}}$ such that $J_k^* = -J_k$. This means that there is a $*$ -representation ρ from the Clifford algebra Cl_{k-1} to the space of bounded operators on $\mathcal{H}_{\mathbb{R}}$. If $\rho(e_i) = J_i$ for e_i the Clifford generator, then the operators J_i must satisfy

$$J_i^2 = -1, J_i^* = -J_i, \text{ and } J_i J_j = -J_j J_i \text{ for } i, j = 1, \dots, k-1, \text{ and } J_k^* = -J_k.$$

Clifford-Equivariant Operators

Subspaces of the skew-adjoint operators defined on $\mathcal{H}_{\mathbb{R}}$ can interact with this Clifford algebra action.

Definition 1.4.23. *Let $\mathcal{F}^k(\mathcal{H}_{\mathbb{R}}) = \{A \in \hat{\mathcal{F}}(\mathcal{H}_{\mathbb{R}}) \mid AJ_i = -J_i A \text{ for all } i = 1, \dots, k-1\}$. This is the subspace of real Cl_k -equivariant skew-adjoint Fredholm operators.*

Definition 1.4.24. *Similarly, let $\mathcal{F}^k(\mathcal{H}) = \{A \in \hat{\mathcal{F}}(\mathcal{H}) \mid AJ_i = -J_i A \text{ for all } i = 1, \dots, k-1\}$. This is the subspace of complex Cl_k -equivariant skew-adjoint Fredholm operators.*

For an analog of Thm. 1.4.19, there are two cases. When the Clifford algebra is simple (when $k \not\equiv 3 \pmod{4}$), take $\mathcal{F}^k(\mathcal{H}_{\mathbb{R}})_* = \mathcal{F}^k(\mathcal{H}_{\mathbb{R}})$, but otherwise the space of operators A decomposes according to whether the operator $J_1 \cdots J_{k-1} A$ is essentially positive, essentially negative, or neither, to give subspaces $\mathcal{F}^k(\mathcal{H}_{\mathbb{R}})_{\pm}$ and $\mathcal{F}^k(\mathcal{H}_{\mathbb{R}})_*$ as above. It can be shown that each of these spaces is nonempty whenever they are defined.

Theorem 1.4.25 (A(k)). *The spaces $\mathcal{F}_{\pm}^k(\mathcal{H}_{\mathbb{R}})$ —which are defined only for $k \equiv 3 \pmod{4}$ —are contractible. For all $k \geq 1$, the space $\mathcal{F}_*^k(\mathcal{H}_{\mathbb{R}})$ is homotopy equivalent to $\Omega(\mathcal{F}^{k-1}(\mathcal{H}_{\mathbb{R}}))$ by the map α , which takes each $A \in \mathcal{F}_*^k(\mathcal{H}_{\mathbb{R}})$ to the path in $\mathcal{F}^{k-1}(\mathcal{H}_{\mathbb{R}})$ from J_{k-1} to $-J_{k-1}$ defined by*

$$J_{k-1} \cos \pi t + A \sin \pi t, \quad 0 \leq t \leq 1.$$

Thus $\mathcal{F}_^k(\mathcal{H}_{\mathbb{R}})$ is a classifying space for the functor KO^{-k} .*

Remark 1.4.26. *The path $J_{k-1} \cos \pi t + A \sin \pi t$ lies within $\mathcal{F}^{k-1}(\mathcal{H}_{\mathbb{R}})$ because at any value of t it is a linear combination of the operators J_{k-1} and A , which lie in $\mathcal{F}^{k-1}(\mathcal{H}_{\mathbb{R}})$. This is because J_{k-1} anticommutes with all J_i for $i = 1, \dots, k-2$ by definition, while $A \in \mathcal{F}^k$ implies that A anticommutes with all J_i for $i = 1, \dots, k-1$, so in particular for $i = 1, \dots, k-2$.*

In the complex case, Clifford algebras are simple for even k .

Theorem 1.4.27 ($B(k)$). *The spaces $\mathcal{F}_\pm(\mathcal{H})$ —defined only for odd k —are contractible. For all $k \geq 1$, the space \mathcal{F}_*^k is homotopy equivalent to $\Omega(\mathcal{F}_*^{k-1}(\mathcal{H}))$ by the map α defined as in Thm. 1.4.25. Thus \mathcal{F}_*^k is a classifying space for the functor K^{-k} .*

Theorems $A(k)$ and $B(k)$ reduce to Theorems A and B, respectively, when $k = 1$. And combined with the periodicity of the Clifford algebras Cl_k and complexified Clifford algebras $Cl_k \otimes \mathbf{C}$, they grant Bott periodicity statements as corollaries.

This thesis will not go into detail of the proof in [8] of the two theorems above because it is fairly involved. There are more readable discussions, including in [10] Lectures 12-14. The proof uses spectral theory for normal operators, facts about fiber bundles, and Kuiper's theorem to prove a series of homotopy equivalences that yield the desired statements.

The index maps for these spaces of operators are more complicated than those for the non- Cl_k -equivariant case, but the classifying space results above are enough to understand the structure of the spaces of operators in the periodic table.

Periodicity Theorems

As mentioned earlier, the two theorems above can also lead to proofs of Bott periodicity. Along the way to that result, one arrives at an important periodic relationship among the spaces of Fredholm operators. Namely, if one considers Fredholm operators acting on a \mathbb{Z}_2 -graded Hilbert space, then the following two facts hold.

Fact 1.4.28. *In the real case, $\Omega^8 \mathcal{F}_*^k \simeq \mathcal{F}_*^k$, so $\mathcal{F}_*^{k+8} \simeq \mathcal{F}_*^k$.*

Fact 1.4.29. *In the complex case, $\Omega^2 \mathcal{F}_*^k \simeq \mathcal{F}_*^k$, so $\mathcal{F}_*^{k+2} \simeq \mathcal{F}_*^k$.*

The full proof can be found in [8], where these statements are included in Thm. (5.1). One main consequence of these statements is that now only those $8 + 2 = 10$ spaces of operators are distinct, and these correspond to the ten spaces of fermionic Hamiltonians in [18]. Specifically, the real spaces satisfy $\mathcal{F}_*^q \simeq R_q$ and the complex spaces satisfy $\mathcal{F}_*^q \simeq C_q$ for R_q and C_q as listed in Table 2 of [18] and in Tables 3 and 4 on page viii. Another important takeaway is that these spaces of fermionic Hamiltonians can now be interpreted as classifying spaces for K -theory functors. Critically, this allows a K -theory invariant to be associated to a given Hamiltonian.

1.5 Summary

This chapter attempted to build up the mathematical framework for the classification of fermionic Hamiltonians. First, the definition and some examples of K -theory classes were given, using vector bundles and some properties of generalized cohomology theories. Next, Clifford algebras were introduced to track symmetries, and it was found that they have a periodic structure that underlies Bott periodicity and periodicity relationships in spaces of operators. Index theory was discussed to show how Fredholm operators can be associated K -theory classes, and finally Clifford representations were incorporated to divide up fermionic Hamiltonians into ten classes with different symmetry properties. It was suggested how the Fredholm index can be used to assign invariants in general cases, though the noncommutative geometry required to perform these calculations was not developed. The rest of this thesis will expand upon the physical meaning of these results and offer a few examples of invariants in lattice models.

Chapter 2

Physical Framework: Hamiltonians and Symmetries

This chapter seeks to put the results about classifying spaces of Fredholm operators into context and to develop the physical definitions and techniques that will be required to understand the examples in the next three chapters. After some background has been built up, the end of this chapter revisits the Kitaev's proposal.

2.1 Topological Materials

2.1.1 Topological Insulators and Superconductors

Topological materials are materials with special electronic properties that are particularly robust to surface imperfections and fluctuations in temperature. Topological insulators are insulating in the bulk, or interior, of the material, but conducting on the surface. Meanwhile, topological superconductors exhibit very low electrical resistance protected by symmetry. Technically, a material is topological if it can host a symmetry-protected topological phase. The phases exhibited in topological materials are particularly exciting for theorists and materials scientists alike for their potential to encode quantum information.

2.1.2 Symmetry-Protected Topological Phases

Recall that a Hamiltonian operator acts on a space of quantum states to determine how a system evolves in time. The eigenvalues of the Hamiltonian correspond to energy levels of different states in the system.

A quantum phase is a phase of matter at zero temperature determined by some order parameter such that if this order parameter varies within a given range, the ground state energy varies analytically. While classical order parameters might be macroscopically measurable things like temperature and pressure, quantum order parameters can be more subtle and phase transitions happen at zero temperature. Phase transitions occur when a change in order parameter causes the ground state energy to make a non-analytic transition, usually causing some qualitative properties of the system to change. The kind of transitions of interest in this thesis are a kind of second-order phase transitions, in which at the phase transition the first excitation energy vanishes. That is, at a second-order phase transition, the lowest nonzero eigenvalue of the Hamiltonian goes to zero.

[26]

Definition 2.1.1. A Hamiltonian is said to be *gapped* if its energy spectrum is discrete around zero. In particular, there is a finite gap Δ between zero and the energy of the first excited state.

A Hamiltonian that is not gapped is said to be gapless. In that case, arbitrarily low energy excitations exist. In a gapped system, when the first excited energy goes to zero, one says that the “gap closes,” and when an excited energy becomes nonzero, one says that a “gap opens.” Both changes in order parameter and changes in symmetry properties can lead the gap to open or close.

To understand what is special about symmetry-protected topological phases, one must first define trivial phases. A system is in a **trivial** quantum phase if its ground state can be adiabatically varied until it may be written as a product state. That is, the ground state of the system is adiabatically connected to a state that features no quantum entanglement. A nontrivial phase, then, is capable of hosting a ground state with entanglement that prevents it from globally being written as a product state, even after being deformed slightly. What defines symmetry-protected topological phases is that without symmetry, they become trivial.

Definition 2.1.2. A *symmetry-protected topological (SPT) phase* is a nontrivial gapped quantum phase of matter possessing a combination of symmetries such that if these symmetries are broken, the phase becomes trivial.

In the language of the previous section, an SPT phase is a connected component of a space of Fredholm operators with certain symmetries. The trivial phase is the connected component containing the identity. Two Hamiltonians are in the same SPT phase if they are homotopic through gapped Hamiltonians with the same symmetries, meaning that they can be connected in the space of operators by a path consisting only of gapped Hamiltonians that all have the same symmetry properties. Physically, these homotopies are achieved by adiabatically varying parameters. The definition of SPT phase used in this thesis will be the following.

Definition 2.1.3. A *symmetry-protected topological phase* is an element of π_0 of a classifying space of gapped Hamiltonian operators with certain symmetries.

SPT phases exhibit interesting properties. Examples will be discussed later, including the topological phase of the Majorana chain in Ch. 3 and the integer quantum Hall effect in Ch. 4.

2.2 Second Quantization

Second quantization provides a way to express a generic fermionic Hamiltonian. Using this framework, Hamiltonians are written as matrix combinations of operators acting on the space of electron states. Analyzing the symmetry properties of the matrix allows Hamiltonians to be divided into different symmetry classes, and the matrices in each class form the classifying space associated to the K -theory invariant. This section introduces second quantization with examples.

2.2.1 Fock Space

A single particle has a probability amplitude of occupying a certain location that is represented by some vector in a Hilbert space \mathcal{H} , usually a complex number in \mathbb{C} . The Hamiltonians considered in the classification are all single-particle Hamiltonians, meaning that interactions will not be taken into account, but even the single-particle Hamiltonian must track whether sites are already

occupied or not because fermions, due to Pauli exclusion, cannot have multiple occupancies at one site. So it is necessary to track the locations of multiple particles, and to do this one uses the **Fock space**. The Fock space is a tensor algebra of identical Hilbert spaces, symmetrized appropriately for the kind of particle. For fermions, it is antisymmetrized to be compatible with their exchange statistics, so it becomes the exterior algebra.

Definition 2.2.1. *The fermionic Fock space of a Hilbert space \mathcal{H} over \mathbb{C} is the exterior algebra on \mathcal{H} .*

$$F(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\wedge n} = \mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H} \wedge \mathcal{H}) \oplus (\mathcal{H} \wedge \mathcal{H} \wedge \mathcal{H}) \cdots$$

A useful basis for the Fock space is the **occupancy number basis**, which consists of states $|n_1 n_2 \dots\rangle$ where n_i tracks the number of particles in state i . This notation is short for $|n_1\rangle \otimes |n_2\rangle \otimes \dots$. The state i might be occupancy at a specific site; in the following examples n_i will encode the number of particles at site i , through for fermions, this will only ever be 0 or 1. Thus if the original Hilbert space is of dimension N , the fermionic Fock space will be of dimension 2^N .

2.2.2 Creation and Annihilation Operators

In second quantization, one uses operators to change the number of particles in a given state. Consider the site j . The corresponding **creation operator**, denoted a_j^\dagger , acts on the Fock space to create a particle at the site j . If the site is unoccupied, a_j^\dagger takes $|\dots 0 \dots\rangle$ to $|\dots 1 \dots\rangle$. If the site is already occupied, it cannot host an additional fermion due to Pauli exclusion, so the action of the creation operator is defined to be $a_j^\dagger |\dots 1 \dots\rangle = 0$, the zero vector of \mathcal{H} . The adjoint operator a_j is called the **annihilation operator**; it removes the particle from an occupied site. Hence if the site j is occupied, the annihilation operator acts as $a_j |\dots 1 \dots\rangle = |\dots 0 \dots\rangle$, where the digit shown is in the j th position. It is defined to act on unoccupied sites so that $a_j |\dots 0 \dots\rangle = 0$.

Fermionic exchange statistics require that if two fermions are exchanged, the overall wavefunction negates. This condition requires that the creation and annihilation operators have the following anticommutation relations:

$$a_j a_k + a_k a_j = 0, \quad a_j^\dagger a_k^\dagger + a_k^\dagger a_j^\dagger = 0, \quad a_k a_j^\dagger + a_j^\dagger a_k = \delta_{jk}.$$

That is, the operators anticommute except for adjoint operators. A more thorough introduction to these operators can be found in a quantum mechanics textbook, but some examples will be given next that demonstrate their algebraic properties.

Remark 2.2.2. *Some sources will denote creation and annihilation operators by c^\dagger and c instead of a^\dagger and a , but in this thesis the former notation is reserved for Majorana operators. This thesis will only discuss fermionic creation and annihilation operators, which in other sources might also be denoted f^\dagger and f , in contrast to bosonic creation and annihilation operators, which might be denoted b^\dagger and b .*

2.2.3 Examples

Example 2.2.3 (One Site). *A single site has $N = 1$, so the dimension of the Fock space is $2^1 = 2$. The two states correspond to the site holding zero or one electron and can be written as $|0\rangle$ and $|1\rangle$, respectively. The creation operator a^\dagger adds an electron to the empty site, so $a^\dagger |0\rangle = |1\rangle$. By Pauli exclusion, the creation*

operator sends the occupied state to zero: $a^\dagger|1\rangle = 0$. Note that the resulting state is 0, not the vacuum state $|0\rangle$. Meanwhile, the annihilation operator takes the occupied state to the vacuum state, so $a|1\rangle = |0\rangle$, and $a|0\rangle = 0$ because one cannot remove an additional electron from an empty site.

These operators can be written as matrices. Let the states $|0\rangle$ and $|1\rangle$ be identified as standard basis vectors, with

$$|0\rangle \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then, according to their action on the basis vectors as explained above, the creation and annihilation operators correspond to the matrices

$$a^\dagger \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad a \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is straightforward to check that these matrices each square to the zero matrix and that they are Hermitian adjoints, as suggested by the notation. One can also check that they satisfy the fermionic anticommutation relation $\{a_\alpha, a_\beta^\dagger\} = \delta_{\alpha\beta}$. Here subscript notation is suppressed because there is only one site; the equation to check is $aa^\dagger + a^\dagger a = 1$, the identity matrix.

As an algebra over \mathbb{C} , the 2×2 matrices representing a^\dagger and a generate the matrix algebra $M_2(\mathbb{C})$. That is, any 2×2 complex matrix can be written as an additive or multiplicative combination of a^\dagger , a , and the identity matrix with complex scalar multiplication.

Note that this algebra is isomorphic to the second complex Clifford algebra $Cl_2^{\mathbb{C}}$ discussed in §1.3. However, the matrices representing a^\dagger and a are not the usual anticommuting generators for this algebra. Instead, new generators called Majorana operators can be formed from combinations of the creation and annihilation operators. These operators make the isomorphism to the Clifford algebra more apparent, as well as encode important physical information. For one site, the Majorana operators c_1 and c_2 are defined as

$$c_1 = a + a^\dagger \quad \text{and} \quad c_2 = \frac{a - a^\dagger}{i}.$$

Their corresponding matrices are

$$c_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad c_2 \mapsto \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

These operators satisfy the anticommutation relation $\{c_j, c_k\} = 2\delta_{jk}$, meaning that they anticommute and square to the identity. They correspond to the standard generators of the second complex Clifford algebra when defined with a positive-definite quadratic form, and they are Hermitian, with $c_j^\dagger = c_j$.

Remark 2.2.4. Some sources denote Majorana operators by χ^\dagger and χ , or γ^\dagger and γ , instead of c^\dagger and c .

Example 2.2.5 (Two Sites). A two-site system has $N = 2$ and a Fock space of dimension $2^2 = 4$ with basis states $|00\rangle$, $|10\rangle$, $|01\rangle$, and $|11\rangle$, which represent the occupancy of the two sites. In this case, there are a pair of creation operators a_1^\dagger, a_2^\dagger and a pair of annihilation operators a_1, a_2 . Each operator acts on its corresponding site and leaves the other site's occupancy unchanged. For example, $a_1^\dagger|00\rangle = |10\rangle$ and $a_2|11\rangle = |10\rangle$, while $a_1|01\rangle = 0$.

However, now that two sites are involved, fermionic exchange statistics must be taken into account. The state produced by first placing an electron at site 1 and then placing an electron at site 2 should be the negative of first placing an electron at site 2. In terms of operators, the requirement is

$$a_2^\dagger a_1^\dagger |00\rangle = -a_1^\dagger a_2^\dagger |00\rangle.$$

This is why the operators must anticommute, as claimed above. Then defining

$$a_2^\dagger a_1^\dagger |00\rangle = a_2^\dagger |10\rangle = |11\rangle \quad \text{necessitates that} \quad a_1^\dagger a_2^\dagger |00\rangle = a_1^\dagger |01\rangle \equiv -|11\rangle.$$

Fermionic statistics also require that $a_1 |11\rangle = -|01\rangle$. This property can be derived in a similar way as above, or seen from the fact that a_1^\dagger and a_1 are Hermitian adjoints.

Otherwise, the operators act on their corresponding sites in the same way as in the single site example. Making the identification of $|00\rangle$, $|10\rangle$, $|01\rangle$, and $|11\rangle$ with the standard basis vectors of \mathbb{R}^4 in that order, the operators can be represented by 4×4 matrices acting on the Fock space as follows.

$$\begin{aligned} a_1^\dagger &\mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} & a_1 &\mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ a_2^\dagger &\mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} & a_2 &\mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Again, the pairs a_1, a_1^\dagger and a_2, a_2^\dagger are adjoint operators, and each squares to zero. These operators satisfy the fermionic anticommutation relation $\{a_j, a_k^\dagger\} = \delta_{jk}$. One can check that these matrices generate the algebra $M_4(\mathbb{C})$, which is isomorphic to $Cl_4^{\mathbb{C}}$.

As in the single site example, one can recombine these creation and annihilation operators to form Majorana operators that square to the identity and anticommute, corresponding to the positive generators of the Clifford algebra $Cl_4^{\mathbb{C}}$. In this case, the Majorana operators are

$$\begin{aligned} c_1 &= a_1 + a_1^\dagger & c_2 &= \frac{a_1 - a_1^\dagger}{i} \\ c_3 &= a_2 + a_2^\dagger & c_4 &= \frac{a_2 - a_2^\dagger}{i} \end{aligned}$$

and their matrix representations are

$$\begin{aligned}
 c_1 &\mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} & c_2 &\mapsto \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} \\
 c_3 &\mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} & c_4 &\mapsto \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}
 \end{aligned}$$

One can check using this representation that the Majorana operators are Hermitian and satisfy the anticommutation relation $\{c_j, c_k\} = 2\delta_{jk}$.

2.3 Majorana Fermions

Majorana operators have been introduced as the canonical generators of the Clifford algebra that corresponds to the action of creation and annihilation operators on the Fock space of electrons on a finite collection of sites. They allow Hamiltonians to be rewritten in more convenient ways; in particular, all parameters in a Hamiltonian written in the Majorana representation will be real numbers. This representation allows for the application of results in index theory to lead to a K -theory invariant for a given Hamiltonian. However, the significance of Majorana operators goes beyond easing calculations.

Physically, they represent creation and annihilation operators for particles called Majorana fermions. Since each Majorana operator is Hermitian, with $c_j^\dagger = c_j$, the operators can act as both creation and annihilation operators. The physical condition associated with this is that Majorana fermions are their own antiparticles, a property posited by Ettore Majorana in 1937. These particles, while not yet definitively detected in experiment, are of interest for the development of protected qubits in topological quantum computing.

2.3.1 Definition

A one-dimensional chain with N sites can host at most N electrons and at most $2N$ Majorana fermions, since each site can host a pair of Majorana fermions.

Definition 2.3.1. *At site j , the associated Majorana operators are*

$$c_{2j-1} = a_j + a_j^\dagger \quad \text{and} \quad c_{2j} = \frac{a_j - a_j^\dagger}{i}.$$

One can check using these formulas and the properties of a_j and a_j^\dagger that the Majorana operators are Hermitian and satisfy the anticommutation relation $\{c_j, c_k\} = 2\delta_{jk}$ for all $j, k \leq N$. These relations are familiar.

Claim 2.3.2. *The algebra generated by the representations of the Majorana operators acting on the Fock space of an N site system is isomorphic to the $2N$ th complex Clifford algebra.*

Proof. Let M denote the algebra of Majorana operators. Define a map $\varphi: M \rightarrow \text{Cl}_{2N}^{\mathbb{C}}$ on basis elements by $\varphi: c_j \mapsto e_j$ and extend linearly. This map is a bijection because it is a bijection on the generators of each algebra, and it is an algebra homomorphism because the generators in each case satisfy the same relations. \square

2.3.2 Topological Quantum Computing

One reason that physicists are excited about Majorana fermions is that they have potential for applications in quantum computing due to their nonabelian statistics and due to the ability of topological materials to host Majorana modes that are localized far away from each other.

Exchange statistics, as discussed above for fermions, dictate how the wavefunction corresponding to a state changes if two particles are switched. For bosons and fermions, the wavefunction is preserved or negated, respectively, so that a wavefunction $|\psi\rangle$ goes to $\pm|\psi\rangle$ and always returns to the original state when the particles are switched back. However, topological materials have the potential to host Majoranas and other quasiparticles whose wavefunctions are not preserved by two switches; particles with these more complicated statistics are known as **anyons**. This allows for information to be encoded by moving particles relative to each other, a process known as **braiding**. [20]

The other draw of Majoranas is that if the pairs of Majoranas can be spatially separated, then it is very difficult for any local perturbation to affect their state, meaning that they could offer a robust quantum memory [19]. Specifically, setups have been proposed that encode a qubit using four Majoranas. Exactly how a mode—a linear combination of Majoranas—could separate pairs of Majoranas while still being distributed across the sample will become more clear when discussing the Majorana chain in §4.1.

2.3.3 Experimental Signatures

Majorana particles have not conclusively been detected in experiment, but there is evidence for their existence. There are several experimental signatures that Majorana particles should exhibit, though these signatures are difficult to detect experimentally with current instrument precision. Namely, Majorana particles should exhibit a 4π -periodic Josephson effect, as opposed to a 2π -periodic one, as well as Andreev conductance quantization [9]. The fact that Majoranas are their own antiparticles precludes Majoranas from carrying charge, but they can conduct heat. Recent experiments in quantized Hall heat transport have found evidence for Majorana edge modes [28].

2.4 Quadratic Hamiltonians

The matrix representations in the previous section were discussed so that the action of the Majorana operators on the physical system was more concrete. However, the Hamiltonian describing a physical system must contain higher than linear terms in Majorana operators to encode couplings between sites, and may even track arbitrarily many sites, so it will become too difficult to write down the action of the Hamiltonian on the Fock space as was done in the previous section for individual Majorana operators. Instead, a Hamiltonian will be written in terms of Majorana operators and will be classified on this level. For simplicity, no terms of higher than quadratic degree will be included.

A generic quadratic Hamiltonian, containing linear combinations of terms $c_j c_k$, can be written

$$H = \frac{i}{4} \sum_{j,k} A_{jk} c_j c_k \quad (2.1)$$

where A is a matrix. On a discrete system of N sites, this matrix is $2N \times 2N$ and the Hamiltonian acts on the Fock space of dimension 2^N . A priori, A has entries in \mathbb{C} , but the benefit of writing Hamiltonians in terms of Majoranas instead of standard creation and annihilation operators is that the matrices A will necessarily be real. This is because the Hermitian property of the Hamiltonian requires

$$H^\dagger = H \implies (iA_{jk}c_jc_k)^\dagger = iA_{jk}c_jc_k \iff A_{jk}c_jc_k = -A_{kj}^*c_k^\dagger c_j^\dagger = -A_{kj}^*c_kc_j \implies A^\dagger = -A,$$

so the matrix A must be skew adjoint, while anticommutation relations of the Majoranas require

$$A_{jk}c_jc_k = -A_{jk}c_kc_j = -A_{kj}c_jc_k \implies A^T = -A$$

so the matrix is actually skew-symmetric. Then since $A^\dagger = A^T$, $A^* = A$. Hence these two conditions together require A to be real skew-symmetric, which is exactly the form of matrices in the index theorem.

Example 2.4.1 (The Trivial Hamiltonian). *Perhaps the simplest quadratic Hamiltonian is the “trivial” Hamiltonian, which only has terms localized to one site. That is, the trivial Hamiltonian is*

$$H = \frac{i}{2} \sum_j c_{2j-1} c_{2j}$$

where each pair of Majorana operators c_{2j-1} and c_{2j} belong to the same site, j . Use the anticommutation relations to rewrite the Hamiltonian as

$$H = \frac{i}{4} \sum_j (c_{2j-1} c_{2j} - c_{2j} c_{2j-1}).$$

Then, the real, skew-symmetric matrix for this Hamiltonian is the block-diagonal matrix

$$A_{jk} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ -1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & -1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This Hamiltonian will be revisited later.

2.5 Physical Symmetries

Hamiltonians will be separated into symmetry classes according to their dimension and symmetry properties. The two main symmetries, which generate the relevant Clifford algebra, are time-reversal symmetry and particle number conservation, but other symmetries will be relevant to discuss for the purposes of solving example Hamiltonians. Here, their physical significance and relevance to the classification are discussed.

2.5.1 Time-Reversal Symmetry

Time-reversal symmetry (T) is possessed by systems that are unchanged when time is reversed by a transformation $t \mapsto -t$.

Example 2.5.1 (Projectile Motion). *Consider a ball thrown along a parabolic trajectory. If time is reversed, the velocity of the ball is also reversed, so velocity is not a time-reversal invariant property. Acceleration, however, is. The gravitational acceleration of the ball along its trajectory is the same whether the ball goes forward or backward.*

Example 2.5.2 (Electromagnetism). *Electric fields are time-reversal invariant, but magnetic fields are not. Since magnetic forces on a charge depend on the velocity of the charge, the negation of velocity under time-reversal causes the magnetic force to negate as well. Chiral edge modes, which occur in the quantum Hall effect and other systems, can only exist in the absence of time-reversal symmetry. This is because chiral modes have a preferred direction, which reverses under time reversal. In 5.2.2, a T-symmetry-breaking perturbation will model the effects of a magnetic field for the quantum Hall effect.*

Example 2.5.3 (Kramers Degeneracy). *The **Kramers Degeneracy Thm.** states that in a system with time-reversal symmetry and half-integer spin, every eigenstate is doubly degenerate. In particular, electron systems with time-reversal symmetry feature **Kramers pairs** of electrons, which are time-reversed copies of one another. [5]*

A Hamiltonian of the form in eqn. (2.1) has time-reversal symmetry when it *anticommutes* with the block-diagonal matrix

$$T = \begin{pmatrix} 0 & 0 & -1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & -1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

written in the Majorana basis from eqn. (15) of [18].

2.5.2 Particle Number Conservation

Fermionic systems possess **particle number conservation**, also sometimes referred to as charge conservation symmetry, when the net number of fermions is conserved. Mathematically, this means that a quadratic Hamiltonian describing such a system can only have terms of the form $a_j^\dagger a_k$, and none of the form $a_j^\dagger a_k^\dagger$ or $a_j a_k$. This symmetry is also called $U(1)$ **symmetry** because a substitution $a_j \mapsto e^{i\phi} a_j$, which changes the creation and annihilation operators by a phase $\phi \in [0, 2\pi)$,

will not change a Hamiltonian with only terms of the form $a_j^\dagger a_k$. See that

$$(e^{i\phi} a_j)^\dagger e^{i\phi} a_k = e^{-i\phi} a_j^\dagger e^{i\phi} a_k = a_j^\dagger a_k.$$

Terms of the form $a_j^\dagger a_k^\dagger$ or $a_j a_k$ break this symmetry.

Example 2.5.4. *Insulators possess particle number conservation, while superconductors do not. This is because in addition to having electrons hop, superconductors can have electrons be added and removed from the system in pairs. According to Bardeen–Cooper–Schrieffer (BCS) theory, superconductors can be viewed as a sort of Bose-Einstein condensate of particles, in which particles all try to occupy the lowest energy state. However, as fermions, no two electrons can occupy the same state. Instead, electrons can bypass Pauli exclusion by condensing in pairs known as **Cooper pairs** [16]. The term $a_j^\dagger a_{j+1}^\dagger$ creates a Cooper pair, while its conjugate $a_{j+1} a_j$ annihilates the pair.*

A Hamiltonian of the form in eqn. (2.1) has particle number conservation when it *commutes* with the block-diagonal matrix

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ -1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & -1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

from eqn. (14) of [18]. This is actually the matrix for the trivial Hamiltonian when written in the Majorana basis, which measures the occupation of each site in the lattice. Hence commuting with this matrix means that the number of particles is the same whether it is measured before or after the Hamiltonian acts.

2.5.3 Particle-Hole Symmetry

A system has **particle-hole symmetry**, also called **charge-conjugation symmetry**, when exchanging particles with their antiparticles leaves the system unchanged. Note that a “hole” describes a site on a lattice of particles that could potentially hold a particle, usually an electron, but does not. In semiconductors and superconductors, electron holes are areas of relative positive charge that can move and behave like particles themselves. Systems with particle-hole symmetry have symmetric spectrums around the Fermi level, since the exchange of particles with antiparticles negates energy eigenvalues. This symmetry is used for the Cartan classification discussed in [3], but is not used directly for classification in [18]. It is introduced here because it will aid calculations for the Majorana chain in §4.1.

Example 2.5.5 (Electromagnetism). *Since particle-hole symmetry exchanges particles and antiparticles, it negates the charge of individual particles and thus reverses the direction of electric and magnetic fields. However, these two changes cancel out, and the laws of electromagnetism are unchanged.*

Example 2.5.6 (Superconductors). *Superconducting systems possess particle-hole symmetry. This allows their Hamiltonians to be rewritten in a more compact form called the **Bogoliubov-de Gennes** or BdG form.*

Specifically, the Bogoliubov-de Gennes Hamiltonian H_{BdG} for a Hamiltonian

$$H = \sum_{j,k=1}^N h_{jk} a_j^\dagger a_k + \frac{1}{2} (\Delta_{jk} a_j^\dagger a_k^\dagger + \Delta_{kj}^* a_k^\dagger a_j^\dagger)$$

with h Hermitian and Δ antisymmetric, satisfies

$$H = \frac{1}{2} \sum_{j,k=1}^N \begin{pmatrix} a_j^\dagger & a_j \end{pmatrix} H_{\text{BdG}} \begin{pmatrix} a_k \\ a_k^\dagger \end{pmatrix} + \frac{1}{2} \text{Tr}(h) \quad \text{with} \quad H_{\text{BdG}} = \begin{pmatrix} h_{jk} & \Delta_{jk} \\ \Delta_{kj}^* & -h_{kj} \end{pmatrix}.$$

When particle-hole symmetry holds, the trace of h is zero, allowing the analysis of the overall Hamiltonian to reduce to studying H_{BdG} .

2.5.4 Lattice Translation Symmetry

Periodic systems possess lattice translation symmetry. This symmetry is generated by lattice translation operators, which translate unit cells onto unit cells.

In crystallography, **Bloch's theorem** states that the wavefunctions of a single-particle Hamiltonian in a perfectly periodic potential can be written as $\psi_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} u_{\mathbf{k}}(\mathbf{r})$, for \mathbf{r} a position vector, \mathbf{k} the "wave vector," and $u_{\mathbf{k}}$ spatially periodic with respect to the unit cell of the system. This approach to solving Hamiltonians on a lattice basically amounts to Fourier transforming, and will be useful in solving the Majorana chain in 4.1 and honeycomb lattice model in 5.2. Generally, the coordinates k_i of the wave vector \mathbf{k} are interpreted as momenta. The span of possible \mathbf{k} vectors determines the momentum space, or **Brillouin zone**, of the system. [5]

2.5.5 Clifford Algebra Structure

The next chapter will discuss how tuning a Hamiltonian to have certain commutation relations with the matrices T and Q turns into a question about extending Clifford algebras. For now, observe that the matrices T and QT can serve as representations of negative real Clifford generators because they each square to -1 and anticommute. That is,

- $T^2 = -1$
- $(QT)^2 = -1$
- $T(QT) + (QT)T = 0$

Each condition can be easily tested on 4×4 matrices, then generalized because the matrices are block-diagonal. Because they satisfy the appropriate relations, T and QT can be viewed as representations of the generators of the Clifford algebra $Cl_2 \cong \mathbb{H}$.

For reasons that will be explained in the next chapter, one separately considers the case of Q symmetry alone. However, Q also satisfies $Q^2 = -1$ and so corresponds to the representation of the generator in $Cl_1 \cong Cl_0^{\mathbb{C}} \cong \mathbb{C}$.

A Ten Fold Way

Whether a system possess T or Q symmetry or both determines four possible classes of Hamiltonians, but viewing these symmetries as living in real and complex Clifford algebras, one can see how the 8-fold and 2-fold periodicities arise as dimension increases. Refer to Tables 1 and 2 on page vii to see where the four symmetry cases show up in the wider classification of ten possible cases. This kind of approach to classification that leads to ten possible cases is sometimes called a “ten-fold way.”

However, not all “ten-fold ways” are the same. The symmetry classes in Kitaev’s periodic table are not exactly the same as those in the earlier Altland-Zirnbauer [3] and Ryu et al. [25] classifications. The latter classifications track time-reversal symmetry and *charge conjugation symmetry*, while Kitaev’s classification tracks time-reversal symmetry and *particle number conservation*. The latter classifications also focus on first-quantized Hamiltonians when testing symmetries, and specify ten classes by allowing symmetry generators \mathcal{T} and \mathcal{C} to square to either ± 1 .

Example 2.5.7 (Different Symmetry Classes). *The Majorana chain, to be discussed in §4.1, has “no symmetry” according to Kitaev’s classification, but in the Altland-Zirnbauer classification the charge-conjugation symmetry it possesses places it in the Cartan class D.*

For clarification, the following table compares the Cartan classes of the Altland-Zirnbauer classification with Kitaev’s labels for the classifying spaces of operators. Note that they are not in the same order. For a discussion of these Cartan classes and alternate classifications, see [3] or [25].

| | | | | | | | | | | |
|-------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| Classifying Space | C_0 | C_1 | R_0 | R_1 | R_2 | R_3 | R_4 | R_5 | R_6 | R_7 |
| Cartan Label | A | AIII | DIII | D | AII | CII | C | CI | AI | BDI |

2.5.6 Review of Kitaev’s Proposal

Recall the proposal from §1.1.2:

Kitaev’s Proposal: *The possible phases of gapped, free-fermion models in d dimensions and with p negative symmetries are classified by*

$$\widetilde{KO}^{-p+d+2}(\text{pt}) = \pi_0(R_{p-d-2 \bmod 8}) \quad \text{or by} \quad \widetilde{K}^{-p+d+1}(\text{pt}) = \pi_0(C_{p-d-1 \bmod 2})$$

where R_q and C_q denote a spaces of operators. The choice of R_q versus C_q is also determined by the symmetry properties of the system.

Parts of this result have been argued. After an identification is made of R_q and C_q with the appropriate spaces \mathcal{F}_*^q , Thms. 1.4.25 and 1.4.27 give the equivalences with the KO -theory and K -theory groups, respectively, in the proposal. That is, the equalities in the proposal have been explicated. That the connected components of the spaces of possible Hamiltonians determine the possible phases of a particular model follows from the definition of a SPT phase as a space of Hamiltonians that are homotopic through gapped Hamiltonians.

It remains to see why the spaces R_q and C_q of operators with certain symmetries characterize the possible Hamiltonians in each dimension. This part of the proposal is essentially taken as an assumption in this thesis, but it will be motivated and explained in more detail in the $d = 0$ classification, in particular when discussing the Clifford extension problem in §3.3.

Chapter 3

Zero-Dimensional Systems

Even in zero dimensions, a number of topological phases can exist. In this model, fermions are added and removed from states in the system using creation and annihilation operators, but the situation has a slightly different physical interpretation from the other examples in this thesis. Specifically, the indices of the creation and annihilation operators no longer stand for sites on a lattice, but for different electron orbitals on a single site. Due to Pauli exclusion, two fermions cannot occupy the same state, so if only one electron state were allowed on the single site in the zero-dimensional system, the classification would only have two possibilities: occupied or unoccupied. Instead, a Hamiltonian is written down that adds, removes, and pair fermions in different orbitals.

There are two different cases to discuss, corresponding to Tables 3 and 4 on page [viii](#). Morally, the reason that two tables are necessary is that each set of loop spaces of Fredholm operators \mathcal{F}^k tracks operators with k specific symmetries, so each table only contains one space of operators $\mathcal{F}^1 = \hat{\mathcal{F}}$ with a single symmetry. However, there are two cases that the classification needs to consider that possess only a single symmetry—the case with only T -symmetry and the case with only Q -symmetry. The two cases one takes are the cases with Q -symmetry only, which are classified using complex K -theory, and the cases with T and Q -symmetry, only T -symmetry, and no symmetry, which are classified with real KO -theory. This zero-dimensional classification clarifies how the spaces of Fredholm operators naturally arise from the constraints imposed on Hamiltonians. This chapter follows the sections “Classification Principles” and “Symmetries and Clifford Algebras” in [18].

3.1 Cases With Q -Symmetry Only

As discussed previously, Hamiltonians that conserve particle number cannot contain superconducting terms like $a_j^\dagger a_k^\dagger$ or $a_j a_k$, so it is easiest to write the Hamiltonian for such a system using electron creation and annihilation operators. A general free-fermion quadratic Hamiltonian with particle number conservation takes the form

$$H = \sum_{j,k} X_{jk} a_j^\dagger a_k \quad (3.1)$$

for X_{jk} a Hermitian matrix. The Hamiltonian acts on the orbitals of the zero-dimensional system, so if there are N available orbitals, X_{jk} will be $N \times N$. Analysis of this Hamiltonian will proceed using spectral theory.

3.1.1 Spectral Flattening

Recall the following fact from linear algebra.

Fact 3.1.1 (Spectral Theorem for Hermitian Matrices). *Hermitian matrices are diagonalizable and have only real eigenvalues.*

So, to understand the space of possible matrices, it suffices to consider diagonal matrices with real eigenvalues. However, not all such matrices are possible. In order to correspond to an SPT phase, the Hamiltonian must be gapped, so one can assume that the eigenvalues of X_{jk} are bounded away from zero by some positive Δ . Meanwhile, an infinite system with eigenvalues tending to infinity is unphysical, so one can also assume that the eigenvalues of X_{jk} are bounded above. For ε_j an eigenvalue, write $\Delta \leq |\varepsilon_j| \leq E_{\max}$.

Now, scaling the eigenvalues ε_j by real, positive constants will yield a homotopic matrix without closing the gap, resulting in a Hamiltonian in the same SPT phase. Specifically, a homotopy f_t can be defined that acts on the eigenvalues of X_{jk} such that $f_0(\varepsilon_j) = \varepsilon_j$ and $f_1(\varepsilon_j) = \text{sgn}(\varepsilon_j)$; for example, take

$$f_t(\varepsilon_j) = \frac{\varepsilon_j}{|\varepsilon_j|^{1-t}}.$$

Applying this function to each eigenvalue in the diagonalized form of X_{jk} defines the **spectral flattening** transformation.

3.1.2 Classifying Space of Hamiltonians

Now, to study all possible Hamiltonians in this class, it suffices to study Hamiltonians whose eigenvalues have been flattened to $+1$ and -1 . The set of possible matrices is

$$C_0 := \bigcup_{0 \leq k \leq N} U(N)/(U(k) \times U(N-k)),$$

where N is the size of the system and k is the number of negative eigenvalues. The connected components of this space consist of matrices with the same number of negative eigenvalues, corresponding to an integer invariant. That is, $\pi_0(C_0) = \mathbb{Z}$. Referring to Tables 3 and 4 on page [viii](#) and invoking index theory for Fredholm operators, one can see that this group corresponds to $\tilde{K}^0(\text{pt})$. For a Hamiltonian that depends on some parameter in a space X , the group of possible invariants generalizes to $[X, C_0] \cong \tilde{K}^0(X)$. Physically, this integer parameter corresponds to the number of filled negative energy orbitals in the system.

3.2 Other Cases

3.2.1 No Symmetry

First consider systems with no symmetry. Since particle number is not conserved, superconducting terms may exist and it is thus convenient to write the general quadratic Hamiltonian in terms of Majorana operators. As in eqn. (2.1), the general Hamiltonian is

$$H = \frac{i}{4} \sum_{j,k} A_{jk} c_j c_k$$

with A real skew-symmetric. For a system with N states, this matrix is $2N \times 2N$ since there are twice as many Majorana operators as states. To characterize the space of possible matrices A , first recall the spectral theorem.

Fact 3.2.1 (Spectral Theorem for Skew-Adjoint Operators). *Bounded, skew-adjoint operators defined on complex Hilbert spaces have only imaginary eigenvalues $i\varepsilon_j$, and every such operator A can be brought into a block-diagonal form in the following way. Let*

$$S = \begin{pmatrix} 0 & i\varepsilon_1 & 0 & 0 & \dots \\ -i\varepsilon_1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & i\varepsilon_2 & \dots \\ 0 & 0 & -i\varepsilon_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then there is some orthogonal matrix R such that $A = RSR^{-1}$.

Using this theorem, one can generalize the spectral flattening transformation from above to take the matrix A to a flattened matrix $\tilde{A} = -i\text{sgn}(iA)$ that consists of blocks of positive or negative ones off of the diagonal. However, this representation is not unique, because the matrix R can be multiplied by any orthogonal matrix M that commutes with the matrix Q and still yield $(MR)S(MR)^{-1} = A$. The representation will become unique after the quotient is taken by the space of matrices M .

The trick is that the space of these $2N \times 2N$ matrices M that commute with Q can be identified with $N \times N$ complex matrices.. This is implicitly why the matrices with only Q -symmetry can be taken to be Hermitian matrices and were classified with complex K -theory in the previous section. Hence the classifying space of matrices with T -symmetry only is

$$R_2 := \bigcup_N O(2N)/U(N) = \lim_{N \rightarrow \infty} O(2N)/U(N).$$

In this case, the invariant lives in $\pi_0(R_2) = \widetilde{KO}^{-2}(\text{pt}) \cong \mathbb{Z}_2$. The physical interpretation of this invariant is the number of filled orbitals modulo 2.

Full arguments are not given for the next two cases, but they are included for completeness.

3.2.2 T -Symmetry Only

The space of matrices corresponding to Hamiltonians with T -symmetry alone is

$$R_3 := \bigcup_N U(2N)/Sp(N).$$

There is no meaningful invariant because $\pi_0(R_3) = \widetilde{K}^{-3}(\text{pt}) = 0$.

3.2.3 T and Q -Symmetry

The space of matrices corresponding to Hamiltonians with T and Q symmetry is

$$R_4 := \bigcup_{k+m=N} Sp(k+m)/(Sp(k) \times Sp(n)) \times \mathbb{Z}.$$

The invariant lives in $\pi_0(R_4) = \tilde{K}^{-4}(\text{pt}) \cong \mathbb{Z}$ and represents the number of Kramers pairs of electrons in the system.

3.3 Clifford Extension Problem

In the real cases above, spaces of Hamiltonians were considered that commuted with Q and anti-commuted with T , and it was discussed in the previous chapter how T and QT could be viewed as representations of Clifford generators. Now, the Clifford algebra problem can be more explicitly defined. If \tilde{A} represents the spectrally flattened matrix $-i\text{sgn}(iA)$, then each of the real cases can be recast as the problem of extending a given Clifford algebra by adding \tilde{A} as the representation of another symmetry generator.

- In the case of no symmetry, \tilde{A} is free, but can be imagined as the representation of a negative generator in a Clifford algebra because $\tilde{A}^2 = -1$.
- In the case of T -symmetry only, T acts as a Clifford generator representation, corresponding to, say, e_1 . Then specifying possible matrices \tilde{A} is the same as specifying the representation of a second generator e_2 .
- In the case of T and Q symmetry, take T as the representation corresponding to a generator e_1 and take QT as the representation corresponding to a generator e_2 . Then the possible matrices \tilde{A} correspond to extensions of the Clifford algebra by another element e_3 . Note that this case illustrates the asymmetry in the treatment of T and Q .

In general, the process of finding representations of an additional Clifford generator that anti-commutes with the representations of given generators is called the **Clifford extension problem**.

Claim 3.3.1. *The classification of free-fermion Hamiltonians in zero dimensions with p negative symmetries is equivalent to the Clifford extension problem with p negative generators.*

This claim has been argued through examples for $p = 0, 1, 2$. But how does this classification relate back to Kitaev's proposal? In particular, how do KO -theory classes arise? As in the previous discussion, the possible Clifford representations that extend in each case form the classifying space R_{p-2} , to which index theory allows a KO -theory invariant to be assigned. Then, the statement is the same as that in the proposal in the case $d = 0$.

Remark 3.3.2. *Calculating the index corresponding to the Clifford representation that determines a particular phase can be done without directly viewing the representation as an operator. Using the **difference bundle construction**, one can form a K -theory or KO -theory class from a difference of two Clifford representations. This process was not explained in the Clifford algebras section above, but is detailed in [7] §7-9.*

Remark 3.3.3. *One might wonder where the extra “−2” comes from in R_{p-2} , or indeed where it arises in the more general result. If one were to pose the Clifford extension problem in terms of positive Clifford generators instead of negative generators, using the isomorphism $Cl^{0,p} \cong Cl^{0,2} \otimes Cl_{p-2}$ from Lemma 1.3.8 and recalling the fact that the representation theory is unchanged by tensoring with a simple algebra like $Cl^{0,2} \cong \mathbb{R}(2)$, the 2 would disappear. Hence it arises only from the choice to take negative generators.*

Chapter 4

One-Dimensional Systems

This chapter will review a few cases of the 1D classification, focusing on the famous Majorana chain. Attention is restricted to lattice systems, since they are easier to solve exactly.

4.1 The Majorana Chain

The Majorana chain, also called the Kitaev chain or wire, is a one-dimensional quantum system designed to host Majorana fermions. In this system, there is neither particle-number conservation nor time-reversal symmetry. The invariant, the **Majorana number**, can be calculated using the Pfaffian of the Fourier-transformed Hamiltonian. This invariant lives in $\pi_0(R_1) = \mathbb{Z}_2$, distinguishing two possible phases. The nontrivial phase in this case has two unpaired Majorana modes at the edges of the wire.

4.1.1 The Hamiltonian

The Majorana chain is a **spinless p -wave superconductor**, a kind of “unconventional superconductor” that breaks time-reversal symmetry. For a wire with N sites, the Hamiltonian is

$$H = -u \sum_{j=1}^N a_j^\dagger a_j - \frac{v}{2} \sum_{j=1}^{N-1} (a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}) + \frac{v}{2} \sum_{j=1}^{N-1} (a_j^\dagger a_{j+1}^\dagger + a_{j+1} a_j). \quad (4.1)$$

The first summation in the Hamiltonian consists of trivial terms confined to single sites, which encode the onsite energy or chemical potential, the second summation contains hopping terms between adjacent sites, and the last summation in the Hamiltonian includes superconducting terms that break $U(1)$ symmetry. Breaking this symmetry is desirable because if the symmetry transformation $a_j \mapsto e^{i\phi} a_j$ held, it could mix different kinds of operators, which runs counter to the goal of localizing individual Majorana operators [19].

Remark 4.1.1. *The Hamiltonian above may not seem general because the parameter v is repeated. In some sources, the superconducting parameter is denoted Δ instead. However, this form is sufficient to demonstrate the two phases of the model and allows for a more convenient expression in the next section.*

Hamiltonian in Terms of Majorana Operators

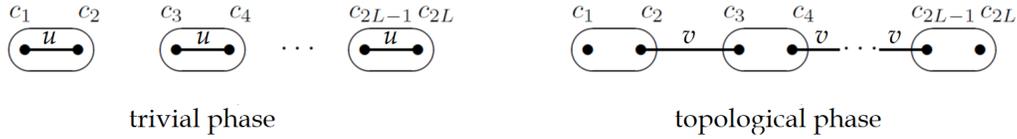
The form of the Hamiltonian in (4.1) will be useful when solving for its spectrum. However, writing the Hamiltonian in terms of Majorana operators makes it clearer how unpaired modes can

arise. Equation (4.2) distinguishes the two different kinds of Majorana pairings—either within the same site, in which the Majoranas c_{2j-1} and c_{2j} pair to form a normal fermion, or between sites, which pairs the Majoranas c_{2j} and c_{2j+1} but can leave dangling, unpaired Majoranas at the end of a finite sample.

$$H = \frac{i}{2} \left(u \sum_{j=1}^N c_{2j-1} c_{2j} + v \sum_{j=1}^{N-1} c_{2j} c_{2j+1} \right) \quad (4.2)$$

Notice that the Majoranas c_1 and c_{2N} do not show up in the second summation, corresponding to them being left unpaired. It will be shown that when $|u| > |v|$, the Hamiltonian is in the trivial phase, while when $|u| < |v|$ the system is in the topological phase.

A useful way to depict these two situations is with a “domino picture,” in which each site j is represented as an oval and the two Majorana modes associated to it are shown as two dots within the oval. Pairings are drawn as lines between the Majorana modes. Note that the Hamiltonian corresponding to the left image below has $v = 0$ and the the Hamiltonian for the right image has $u = 0$.



This domino picture, adapted from [19], shows the pairings in the two phases of the Majorana chain.

The matrix A from the form of the Hamiltonian in eqn. (2.1) can also be written out. For convenience, the two summations are separated into different matrices, and written in a standard block diagonal form. This block-diagonal form is achievable using the fact that $2c_j c_k = c_j c_k - c_k c_j$.

$$A = \begin{pmatrix} 0 & u & 0 & 0 & 0 & \dots & 0 & 0 \\ -u & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & u & 0 & \dots & 0 & 0 \\ 0 & 0 & -u & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -u & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & v & 0 & 0 & \dots & 0 & 0 \\ 0 & -v & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & v & \dots & 0 & 0 \\ 0 & 0 & 0 & -v & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

Zero Modes

Note that in the topological phase, there are two zero eigenvalues, which correspond to the existence of unpaired Majorana zero modes [29]. The eigenvectors for these two zero eigenvalues are

$$\left(1, 0, \frac{v}{w}, 0, \left(\frac{v}{w}\right)^2, 0, \dots, \left(\frac{v}{w}\right)^N, 0 \right) \quad \text{and} \quad \left(0, \left(\frac{v}{w}\right)^N, 0, \left(\frac{v}{w}\right)^{N-1}, \dots, 0, 1 \right)$$

and these correspond, respectively, to left and right boundary modes

$$b_l = c_1 + \frac{v}{w} c_3 + \dots + \left(\frac{v}{w}\right)^N c_{2N-1} \quad \text{and} \quad b_r = \left(\frac{v}{w}\right)^N c_2 + \left(\frac{v}{w}\right)^{N-1} c_4 + \dots + c_{2N}.$$

In the topological phase with $|u| < |v|$, these modes decay exponentially away from the opposite boundary of the sample, meaning that the modes are effectively spatially separated from each other even though they are distributed across the wire [20].

4.1.2 Checking Symmetries

The Majorana chain does not possess T or Q symmetry. This can be seen explicitly by checking the commutation relations with the symmetry matrices. For example, when $N = 2$,

$$AT + TA = \begin{pmatrix} 0 & v & 0 & 0 \\ v & 0 & 0 & 0 \\ 0 & 0 & 0 & -v \\ 0 & 0 & -v & 0 \end{pmatrix} \neq 0 \quad \text{and} \quad AQ - QA = \begin{pmatrix} 0 & 0 & v & 0 \\ 0 & 0 & 0 & -v \\ -v & 0 & 0 & 0 \\ 0 & v & 0 & 0 \end{pmatrix} \neq 0.$$

Note that the superconducting terms, which had the coefficient $\frac{v}{2}$, contribute to the breaking of particle number conservation.

Remark 4.1.2. While the Majorana chain does not possess the symmetries that are part of the classification, it does have a \mathbb{Z}_2 -symmetry generated by the parity operator $P = \prod_{j=1}^N (-ic_{2j-1}c_{2j})$, which is of physical interest in mathematically-equivalent models including the transverse field Ising model. This is discussed in §2.3 of [19], and a more general classification that does include this \mathbb{Z}_2 -symmetry is given in [36].

4.1.3 Calculating the Invariant

The \mathbb{Z}_2 invariant that distinguishes the trivial from the topological phase can be calculated by passing to momentum space and solving for when the spectral gap closes, since when the gap closes the phase is allowed to change. Mathematically, this involves using an approximate translation invariance of the wire to invoke Bloch's theorem and perform a Fourier transform, then examining the spectrum of the Hamiltonian under different parameter values.

Bogoliubov-de Gennes Hamiltonian

To solve for the spectrum of the Hamiltonian, it is easiest to return to the form (4.1) and take advantage of the particle-hole symmetry of the system. This approach follows [9]. Using the vector $C = (a_1, \dots, a_n, a_1^\dagger, \dots, a_n^\dagger)^T$, the Hamiltonian can be written as a $2N \times 2N$ matrix H_{BdG} with $H = C^\dagger H_{\text{BdG}} C$. In matrix form, this is

$$H = \begin{pmatrix} a_1^\dagger & a_2^\dagger & \dots & a_n^\dagger & a_1 & a_2 & \dots & a_n \end{pmatrix} \left(\begin{array}{cccc|cccc} -u & -\frac{v}{2} & & & & & & \\ -\frac{v}{2} & -u & \ddots & & \frac{v}{2} & & & \\ & \ddots & \ddots & & & \ddots & & \\ & & & -\frac{v}{2} & -u & & & \\ \hline & & & \frac{v}{2} & & & \frac{v}{2} & \\ & & & & \ddots & & & \\ & & & & & \frac{v}{2} & & \end{array} \right) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ a_1^\dagger \\ a_2^\dagger \\ \vdots \\ a_n^\dagger \end{pmatrix}.$$

The eigenvalue in this case is e^{ik} . Similarly, e^{-ikn} is also an eigenvector. The possible values of k range from 0 to 2π , and are periodic, meaning that k is picked from the circle S^1 . This circle is the Brillouin zone of the system.

Remark 4.1.3. *Formally, these eigenvectors are the characters of the group \mathbb{Z} , which is the group generated by \mathbf{t} . That is, they are maps $\mathbb{Z} \rightarrow U(1)$, by $n \mapsto e^{ikn}$. This formalism will be used again with more examples in §5.2.*

Spectrum of the Hamiltonian

Now that the eigenvectors are determined, the Hamiltonian can be applied to get its eigenvalue spectrum. Note that the hopping operators $a_{j+1}^\dagger a_j$ and $a_{j+1} a_j^\dagger$ effectively play the role of \mathbf{t} and \mathbf{t}^{-1} because they shift the electron state between adjacent sites, so their eigenvalues are e^{ik} and e^{-ik} , respectively, for the eigenvector e^{ikn} . Meanwhile, the onsite terms $a_n^\dagger a_n$ effectively annihilate and then recreate the state $|n\rangle$, so return e^{ikn} unchanged and have eigenvalue 1. A slightly different argument is required for the Fourier transform of the superconducting terms and will be omitted here. Substituting these eigenvalues allows the BdG Hamiltonian to be rewritten with respect to the Fourier basis $|k\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^N e^{-ikn} |n\rangle$ as

$$H(k) = \langle k | ((-v \cos k - u)\sigma_z + v \sin k i\sigma_y) | k \rangle,$$

which in matrix form is

$$H(k) = \begin{pmatrix} a_k^\dagger & a_k \end{pmatrix} \begin{pmatrix} -v \cos k - u & iv \sin k \\ -iv \sin k & v \cos k + u \end{pmatrix} \begin{pmatrix} a_k \\ a_k^\dagger \end{pmatrix}.$$

This form of the Hamiltonian makes it relatively easy to diagonalize, revealing the spectrum to be

$$\varepsilon(k) = \pm \sqrt{u^2 + v^2 + 2uv \cos k}.$$

The Majorana Number

The boundary between phases lies along the parameter conditions that close the gap in the Hamiltonian. This happens in only two cases:

$$\begin{aligned} k = 0, u = -v &\implies \varepsilon(0) = \sqrt{u^2 + v^2 + 2uv} = 0 \\ k = \pi, u = v &\implies \varepsilon(\pi) = \sqrt{u^2 + v^2 - 2uv} = 0. \end{aligned}$$

Now $|u| = |v|$ can be seen as a phase transition point, as was suggested earlier. The invariant that keeps track of this change involves the Pfaffian, which is defined for skew-symmetric matrices A and satisfies $\text{Pf}(A)^2 = \det(A)$.

Definition 4.1.4. *The Pfaffian of a $2N \times 2N$ skew-symmetric matrix A is*

$$\text{Pf}(A) := \frac{1}{2^N \cdot N!} \sum_{\sigma \in S_{2N}} \text{sgn}(\sigma) \left(\prod_{i=1}^N A_{\sigma(2i-1), \sigma(2i)} \right)$$

where S_{2N} is the symmetric group on $2N$ letters.

Measuring the Pfaffian at the two points where H may be gapped, namely $k = 0, \pi$, determines the phase. Specifically, the formula

$$\mathcal{M}(H) = \text{sign}(\text{Pf}[iH(0)] \text{Pf}[iH(\pi)])$$

determines $\mathcal{M}(H) = 1$ for H in the trivial phase and $\mathcal{M}(H) = -1$ for H in the topological phase. A rigorous justification for why this formula works more generally may be found in [19], but it is clear in this case because the Pfaffians evaluate to

$$\begin{aligned} \mathcal{M}(H) &= \text{sign}(\text{Pf}[iH(0)] \text{Pf}[iH(\pi)]) \\ &= \text{sign}((-v - u)(v - u)) \\ &= \text{sign}(u^2 - v^2). \end{aligned}$$

This invariant determines whether the Hamiltonian lives in the trivial or nontrivial component of $R_2 \simeq O$, the infinite orthogonal group, which is the limit of $O(n)$. Recall that the trivial component is defined to be the component containing $\mathbb{1}$.

Remark 4.1.5. *It is also justified in [19] that the invariant has the following property: if H_1 and H_2 are Hamiltonians describing two different chains, and $H_1 \oplus H_2$ is defined to be the Hamiltonian of the chains concatenated with each other, then $\mathcal{M}(H_1 \oplus H_2) = \mathcal{M}(H_1)\mathcal{M}(H_2)$. This provides a \mathbb{Z}_2 group structure for the invariant, which is technically more structure than is provided by π_0 alone.*

4.1.4 Quantum Computing and Experimental Realization

As discussed earlier, part of the allure of Majorana fermions for quantum computing comes from their nonabelian exchange statistics. However, these Majorana fermions cannot move past each other with only one degree of spatial freedom, so how can they be implemented in a wire? One proposal is to use a wire network with T-junctions, which would allow the Majorana fermions to be moved one at a time into a side chain and exchanged this way. A discussion of how this process allows for the desired exchange statistics is in [2]. Experimentally, the most promising platforms for the Majorana chain may be electron-doped InAs and InSb wires [1].

4.2 T -Invariant Superconductor

Time-reversal invariant superconductors possess time-reversal symmetry but particle number is not conserved. The invariant of the system is the parity of the number of Kramers pairs, which defines an element of $\pi_0(R_2) = \mathbb{Z}_2$. An even number of Kramers pairs corresponds to spin-singlet pairing, while an odd number corresponds to spin-triplet pairing [27].

Models of a time-reversal invariant topological superconductor, or TRITOPS, can be found in [32] and [12]. Materials promising for fabrication of TRITOPS include $(\text{TMTSF})_2\text{X}$, where TMTSF stands for tetramethyltetraselenafulvalene and X is an inorganic anion like ClO_4 or PF_6 [18].

4.3 *T*-Invariant Insulator

Time-reversal invariant insulators possess both time-reversal symmetry and particle-number conservation. Only a trivial phase exists in this case, since $\pi_0(R_3) = \pi_0(U(2n)/Sp(n)) = 0$.

Chapter 5

Two-Dimensional Systems

5.1 Integer Quantum Hall Effect

The integer quantum Hall effect is one of the most important areas of study within topological materials, and it features a more concrete example of a K -theory invariant. The effect has a very well-developed theory, but it was actually discovered experimentally and only subsequently explained.

5.1.1 The Classical Hall Effect

In the late 1870s, during his doctoral studies, Edwin Hall decided to investigate the effects of magnetism on the distribution of charge in a conducting material with a current running through it. Existing literature on the topic was contradictory on whether a magnetic field should act on the current or not, and the electron had not yet been identified as a charge carrier. To resolve the issue, Hall with the help of his advisor designed a remarkably precise experiment to measure the charge distribution in a thin strip of metal. After several adjustments, Hall observed a potential difference across one end of the metal strip, indicating that the magnetic field was exerting a force on the current after all. This is the **classical Hall effect**. [13]

The Hall effect can be seen as a result of the **Lorentz force law**, which determines the force experienced by a charge q moving at a velocity \vec{v} that experiences an electric field \vec{E} and a magnetic field \vec{B} . This charge might be an electron, a hole, or an ion. The force is

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}).$$

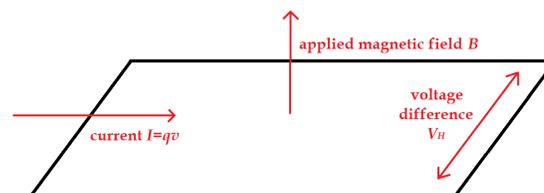


FIGURE 5.1: The Hall effect is a voltage gradient across one end of a conducting sample subjected to a transverse magnetic field.

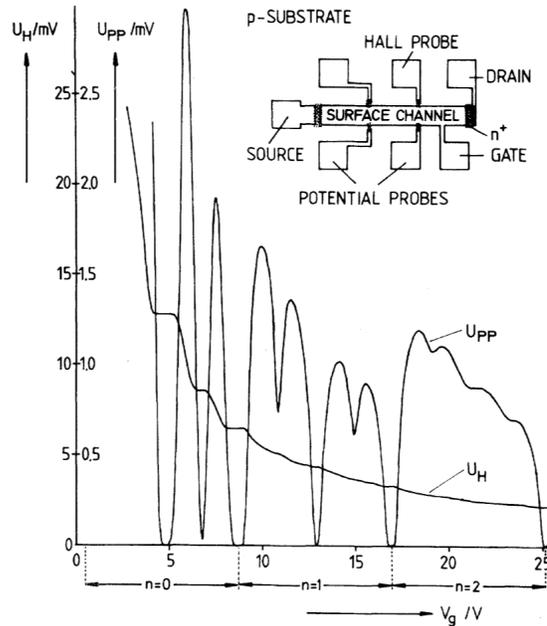


FIGURE 5.2: This data from [33] demonstrates the constant plateaus of the Hall voltage U_H at each multiple of $\frac{1}{\nu}$. It also shows the voltage drop U_{pp} between potential probes. Each is plotted as a function of gate voltage v_g .

In Hall's setup, the magnetic field was transverse to the direction of current flow, meaning that $\vec{v} \perp \vec{B}$ and the magnetic field was able to exert a force on the moving charges perpendicular to the current and to the field. Due to the application of this force, the charges were deflected to one side as the current flowed, resulting in an uneven distribution of charge at the end of the metal strip and an observed voltage difference.

5.1.2 The Quantum Hall Effect

In the regime of low temperatures and strong magnetic fields, the Hall effect in effectively two-dimensional materials exhibits an even more interesting behavior. In 1980, von Klitzing observed in silicon MOSFETs¹ prepared by Dorda and Pepper that the Hall resistivity, which is Hall voltage divided by the current strength, was quantized as $\frac{h}{e^2\nu}$. Here, h is Planck's constant, e is the electron charge, and ν , the **filling factor**, is an integer. Klitzing used these measurements to obtain a more precise value of the fine-structure constant. [33]

This phenomenon is often called the **integer quantum Hall effect** to differentiate it from the fractional quantum Hall effect, which only appears once electron interactions are taken into account. The quantum Hall effect has been observed in other MOSFETs, in GaAs heterostructures, and in bilayer graphene. The integer quantum Hall effect can be understood theoretically by solving the one-particle Hamiltonian, and an intuitive derivation can be found in [34] §1.4 or [31] §2.6.

¹MOSFET stands for "metal oxide semiconductor field effect transistor."

5.1.3 In the Periodic Table

In the periodic table classification, the integer quantum Hall effect is an example of a $d = 2$ system possessing Q symmetry but not T symmetry. The T symmetry is broken due to the applied magnetic field. The invariant produced is the Hall conductance filling factor ν , which is an integer in $\mathbb{Z} = \pi_0(C_0)$.

Since Q symmetry holds, the Hamiltonians describing quantum Hall systems are described by Hermitian matrices and classify complex K -theory. The invariant ν is calculated as the **Chern number**, or first Chern class, of the vector bundle over the Brillouin zone, whose fibers describe the ground state subspace of the Hamiltonian with a given momentum parameter. This calculation is explained in the context of an example in the next section.

5.2 The Honeycomb Lattice Model

The honeycomb lattice model for graphene is an exactly solvable model that allows for a calculation of topological invariant similar to the Chern number. This model does not quite represent the quantum Hall effect because Q -symmetry does not hold, and the edge modes that will be symbolized by the topological invariant by the end of the calculation are real fermions, not complex fermions. However, it is an instructive example of the meaning of the Chern number, so it will be discussed in the terms of the quantum Hall effect.

Remark 5.2.1. *Since it possesses neither Q nor T symmetry but has an integer invariant similar to that of the quantum Hall effect, this model actually belongs to the “no symmetry” 2D case in the periodic table.*

The model, as described in [17], constitutes a tight-binding model for graphene, which is a material made up of sheets of carbon atoms arranged in a hexagonal lattice. It originally possesses time-reversal symmetry, before this is broken by a perturbation of the Hamiltonian. This perturbation opens a spectral gap and allows for the topological invariant ν to achieve nonzero values.

Remark 5.2.2. *Graphene itself is actually notable for exhibiting the quantum Hall effect at room temperatures. However, as mentioned previously, this occurs in bilayer graphene, not in the single layer described by the honeycomb lattice model.*

5.2.1 Unperturbed Hamiltonian

Lattice Setup

In graphene, each carbon atom makes one σ bond with each of its three neighbors, and the tight-binding model for graphene assumes that only these nearest-neighbor interactions are relevant. To write down the Hamiltonian that dictates these bonds, it is useful to break up the hexagonal lattice into two triangular sublattices, which have different colors.

Now nearest-neighbor atoms alternate in lattice color, and the translational symmetries of the lattice are more apparent. Two translation operators, \mathbf{s} and \mathbf{t} , can be defined that shift the lattice into itself and that preserve each sublattice. In [17], these correspond to the vectors \mathbf{n}_1 and \mathbf{n}_2 . Since the Hamiltonian is assumed to be translation invariant, defining these translation operators is useful for finding an appropriate unit cell and basis for writing down the Hamiltonian. The unit cell should contain one site of each sublattice so that unit cells may tile the entire hexagonal lattice.

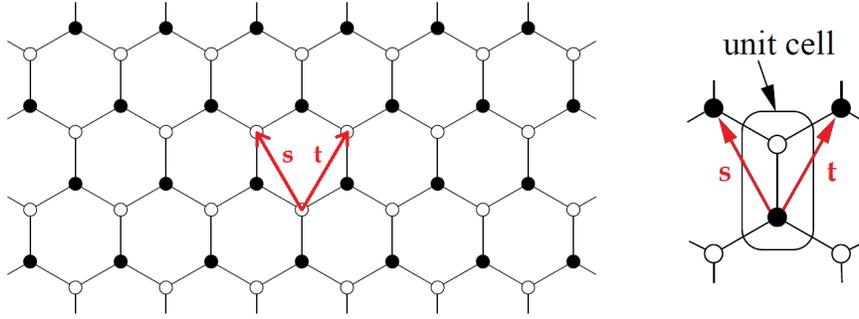


FIGURE 5.3: The honeycomb lattice has two triangular sublattices and two lattice translation vectors, \mathbf{s} and \mathbf{t} . The unit cell contains one site of each sublattice; say the black site is site 1 and the white site is site 2. This figure is adapted from [17].

Remark 5.2.3. Note that shorter translation operators than \mathbf{s} and \mathbf{t} cannot be defined. In particular, shifting the lattice by the length of one bond will not return it to its original position: shifting site 1 to site 2 will shift all of the white sites into the middle of where the hexagons should be.

Group Ring

The motivation for Fourier transforming to solve this Hamiltonian is similar to that given for the Majorana chain, but the algebraic object is slightly more complicated in two dimensions. Now each unit cell contains two sites, each of which may host an electron with a probability amplitude represented by some complex number. The Hilbert space the Hamiltonian acts on at a particular unit cell is $\mathcal{H} \cong \mathbb{C}^2$, say with basis $\{h_1, h_2\}$, where each h_i corresponds to one of the sites. Then, then the Hilbert space over the unit cell adjacent via the lattice vector \mathbf{s} can be expressed as $\mathbf{s}\mathcal{H}$, with basis vectors $\{\mathbf{s}h_1, \mathbf{s}h_2\}$ for the quantum states at that site. Similarly, a unit cell that is two \mathbf{s} vectors below and one \mathbf{t} vector to the left of the central unit cell has a basis $\{\mathbf{s}^{-2}\mathbf{t}h_1, \mathbf{s}^{-2}\mathbf{t}h_2\}$ for the Hilbert space above it.

As in the previous chapter, the overall Hilbert space \mathcal{H}_{tot} is a module over the Laurent polynomial ring $\mathbb{C}[\mathbf{s}^{\pm 1}, \mathbf{t}^{\pm 1}]$. The polynomial ring is isomorphic to the ring $\mathbb{C}[\mathbb{Z}^2]$ but with chosen generators \mathbf{s} and \mathbf{t} , and the module \mathcal{H}_{tot} is the Hilbert space completion of a free module of rank 2. It can be written

$$\mathcal{H}_{\text{tot}} = \mathbb{C}[\mathbf{s}^{\pm 1}, \mathbf{t}^{\pm 1}]\{h_1, h_2\}.$$

A generic translation-invariant Hamiltonian defined over this group ring will be 2×2 , since 2 is the dimension of the Hilbert space $\mathbf{s}^a \mathbf{t}^b \mathcal{H}$ at each site. The entries in the 2×2 matrix will be Laurent polynomials in \mathbf{s}, \mathbf{t} with complex coefficients.

Example 5.2.4. Consider the Hamiltonian $\begin{pmatrix} 0 & 0 \\ \mathbf{s} & \mathbf{t} \end{pmatrix}$. This matrix maps the basis vector $\mathbf{s}^a \mathbf{t}^b h_1$ at the unit cell (a, b) to $\mathbf{s}^{a+1} \mathbf{t}^b h_2$, which is translated to the adjacent unit cell and switches sublattices. Meanwhile, $\mathbf{s}^a \mathbf{t}^b h_2$ is mapped to $\mathbf{s}^a \mathbf{t}^{b+1} h_2$.

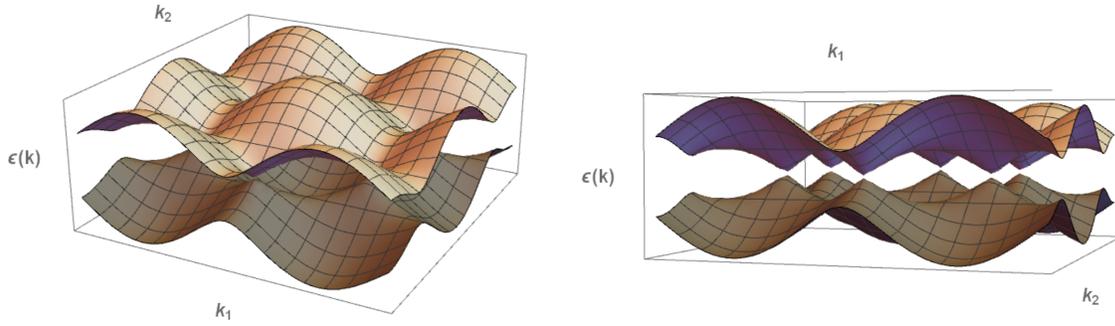


FIGURE 5.4: These plots show the spectrum $\varepsilon(k)$ as a function of \mathbf{k} . The Dirac cones are found around where the positive and negative sheets touch and close the gap.

The Hamiltonian

To write down the particular Hamiltonian for graphene, consider an electron at site 1 in the unit cell (a, b) . It can hop to any of its nearest neighbors, who live on the other sublattice in site 2 of each adjacent unit cell. One of these sites is on the same unit cell, right above, while one is on a site displaced by \mathbf{s} , and the last is on a site displaced by \mathbf{t} . Hence the hopping coefficient is $1 + \mathbf{s} + \mathbf{t}$ to account for each of these possibilities. Since the electron must switch sublattices when it hops, this term is off diagonal. Meanwhile, electrons from adjacent sites can also jump *on* to the site 1 at (a, b) , corresponding to a coefficient $1 + \mathbf{s}^{-1} + \mathbf{t}^{-1}$ since the translation vectors are now pointing in instead of out of the unit cell. This term goes in the other off-diagonal entry in the matrix. The Hamiltonian is

$$H = \begin{pmatrix} 0 & 1 + \mathbf{s}^{-1} + \mathbf{t}^{-1} \\ 1 + \mathbf{s} + \mathbf{t} & 0 \end{pmatrix}.$$

Calculating the Spectrum

It is not obvious how to find the spectrum of this Hamiltonian because its entries are not numbers, but Laurent polynomials. However, eigenspaces can be found by using the lattice translation operators because the Hamiltonian again must share eigenspaces with them.

At a given point (a, b) in the lattice, the exponential e^{ika} will be an eigenvector of \mathbf{s} because when \mathbf{s} is applied to it, the lattice position will change to $(a + 1, b)$ and the vector becomes

$$\mathbf{s}e^{ika} = e^{ik(a+1)} = e^{ik} \cdot e^{ika}.$$

The eigenvalue in this case is e^{ik} , where k is a momentum parameter. Such a vector is also an eigenvector for \mathbf{t} , with $\mathbf{t}e^{ikb} = e^{ik(b+1)} = e^{ik} \cdot e^{ikb}$. The eigenspace of each operator is two-dimensional, spanned by $\{e^{ik}, e^{-ik}\}$.

Remark 5.2.5. Formally, these eigenvectors are the characters of the group \mathbb{Z}^2 . That is, they are maps $\mathbb{Z}^2 \rightarrow U(1)$, by $(a, b) \mapsto e^{i(k_1 a + k_2 b)}$, and passing to these eigenvectors constitutes a Fourier transform. For a more explicit treatment of the Fourier transform, see the derivation in §4 of [17] assuming $J_x = J_y = J_z = 1$.

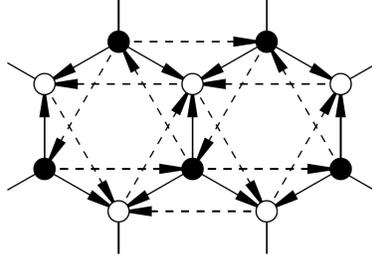


FIGURE 5.5: In this diagram from [17], the nearest-neighbor interactions are directed solid lines, while the second-neighbor interactions are dashed lines.

In general, a vector $e^{i(k_1 a + k_2 b)}$ is an eigenvector of each of these operators and hence also of H . Applied to this eigenvector, the Hamiltonian gives

$$\begin{aligned} H(e^{i(k_1 a + k_2 b)}) &= 2e^{i(k_1 a + k_2 b)} + e^{i(k_1(a+1) + k_2 b)} + e^{i(k_1 a + k_2(b+1))} + e^{i(k_1(a-1) + k_2 b)} + e^{i(k_1 a + k_2(b-1))} \\ &= (2 + e^{ik_1} + e^{ik_2} + e^{-ik_1} + e^{-ik_2}) e^{i(k_1 a + k_2 b)} \\ &= (2 + 2 \cos k_1 + 2 \cos k_2) e^{i(k_1 a + k_2 b)}. \end{aligned}$$

Physically, the parameters k_1 and k_2 are interpreted as momenta because if the cosines in the expression for energy above are expanded, $\cos k_i \approx 1 - k_i^2/2$, the quadratic term resembles a kinetic energy $\frac{1}{2}mv^2$, meaning that a linear term in k_i resembles a momentum mv . Recall that the momentum space of a lattice system is referred to as the **Brillouin zone**. In this case, the parameters are periodically constrained, with $k_1, k_2 \in [0, 2\pi)$ with endpoints identified, so (k_1, k_2) can be viewed as living on the torus \mathbb{T}^2 .

However, there is a problem with this spectrum—it is not gapped. That is, it has zero-energy eigenvalues corresponding to (k_1, k_2) for which $1 + e^{ik_1} + e^{ik_2} = 0$, as well as eigenvalues accumulating around zero. Plotting the spectrum in Fig. 5.4, reveals the existence of **Dirac cones** at the points that the spectrum hits zero, so-called because near these points, the Hamiltonian resembles a Dirac operator. While Dirac cones are physically interesting, they need to be eliminated for an invariant to be assigned.

5.2.2 T -Symmetry-Breaking Perturbation

One way to eliminate the accumulation of eigenvalues around zero is to perturb the Hamiltonian by adding extra terms that break the time-reversal symmetry and open a gap. These terms are meant to represent the action of a magnetic field. The perturbation needs to break T -symmetry because the Dirac cones are actually protected by this symmetry; any perturbation that does not break T -symmetry cannot open a gap, as shown in [17] §6.1.

However, to break time-reversal symmetry, it actually suffices to add diagonal terms to the Hamiltonian. These diagonal terms represent interactions between neighboring sites on the *same sublattice*, which ultimately constitute second-neighbor interactions, as depicted in Fig. 5.5. Physically, the reason that these new interactions break time-reversal symmetry is that they introduce a chirality to the system: these interactions have a preferred cyclical direction that will reverse

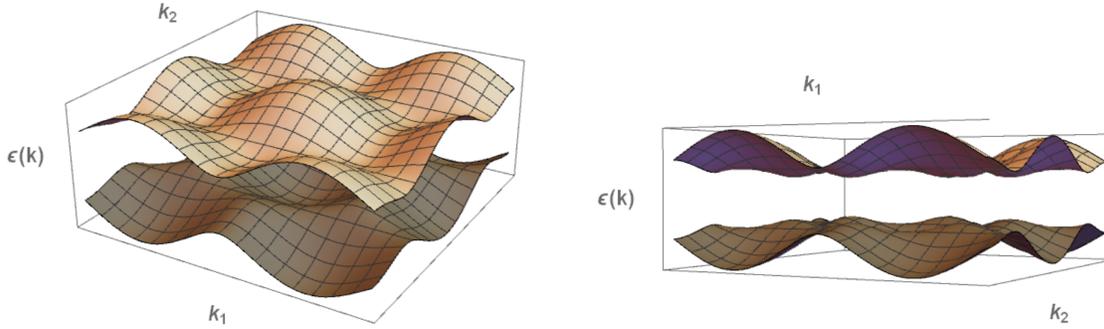


FIGURE 5.6: The extra terms added to the Hamiltonian have eliminated the Dirac cones, gapping out the system.

under time reversal. In contrast, reversing the original nearest-neighbor interactions would leave the system unchanged up to a lattice translation.

In a simple model in which the second-neighbor pairing is just a real number Δ , the perturbed Hamiltonian is

$$H' = \begin{pmatrix} \Delta & 1 + \mathbf{s}^{-1} + \mathbf{t}^{-1} \\ 1 + \mathbf{s} + \mathbf{t} & \Delta \end{pmatrix},$$

and its spectrum is

$$\varepsilon'(k) = 2 + \Delta + 2 \cos k_1 + 2 \cos k_2,$$

which is always positive, as demonstrated in Fig. 5.6. Now that the system is gapped, an invariant can be calculated.

5.2.3 The Chern Invariant

Traditionally, the **Chern number** represents the Hall conductance. It is often referred to as the **TKNN invariant**, after the authors of the famous 1982 paper that first proved that this conductance was quantized and thus offered a topological invariant. In the paper, Thouless, Kohmoto, Nightingale, and den Nijs wrote the Hamiltonian for a particle in a periodic lattice potential and used Bloch's theorem to Fourier transform, then demonstrated the quantization of the Hall conductance by rewriting the **Kubo formula** for two-dimensional conductors. [30]

There are several ways to interpret this formula. One is as the integral of a quantity called the **Berry curvature**, which tracks the change in the phase of certain complex vectors as the momentum varies over the Brillouin zone. This phase is alternately called the **Berry phase**, the **geometric phase**, or the **Pancharatnam phase**. Another is as the **first Chern class** of a vector bundle associated to a Hamiltonian, which in turn is related to a spectral projection matrix. This second view will be explained for the honeycomb lattice model, since it is more intuitive how K -theory classes might arise this way and because it corresponds to the more general case with non-lattice systems in which topological invariants are calculated as the indices of spectral projection operators.

The Vector Bundle

The vector bundle defined over the Brillouin zone has fiber at the point \mathbf{k} the ground state subspace of the Hamiltonian $H(\mathbf{k})$. That is, it encodes how the eigenvectors of the Hamiltonian that correspond to negative eigenvalues vary as the momentum of the system does. Assuming the Hamiltonian acts on \mathbb{C}^n and has an m -dimensional ground state, the vector bundle is thus determined by a map from the Brillouin zone to the space of m -dimensional subspaces in \mathbb{C}^n . This map acts as a projection of \mathbb{C}^n onto the ground state subspace [20]. So, there is a classifying map

$$P: \mathbb{T}^2 \rightarrow U(n)/(U(m) \times U(n-m)),$$

where the target should be recognized as the classifying space C_0 . The first Chern class of the vector bundle can be calculated using the pullback from the universal bundle over the classifying space, but the invariant can also be calculated using the index theorem to give the appropriate element in $\tilde{K}(\mathbb{T}^2)$, which as shown in Ex. 1.2.29 is indeed \mathbb{Z} . For the honeycomb lattice model, $n = 2$ and $m = 1$, so the classifying space is

$$U(2)/(U(1) \times U(1)) \simeq \mathbb{C}P^1 \simeq S^2.$$

The Projection Map

How does one construct the projection map? Start with the spectrally flattened Hamiltonian $\tilde{H}(\mathbf{k})$, whose eigenvalues are all ± 1 . Now the eigenspaces of positive eigenvalue and the eigenspaces of negative eigenvalue, respectively, are grouped together. The projection matrix is defined to be

$$P(\mathbf{k}) := \frac{1}{2}(\mathbb{1} - \tilde{H}(\mathbf{k})),$$

so that the eigenvalue of P on a positive eigenvector of H is zero, but the eigenvalue of P on a negative eigenvector of H is $+1$. This ensures that P projects onto the negative eigenvector subspace of H . Now, the formula for the Chern number is

$$Ch(P) = \int_{\mathbb{T}^2} \frac{d\mathbf{k}}{2\pi i} \text{Tr} \left(P(\mathbf{k}) \left(\frac{\partial}{\partial k_1} \frac{\partial}{\partial k_2} P(\mathbf{k}) - \frac{\partial}{\partial k_2} \frac{\partial}{\partial k_1} P(\mathbf{k}) \right) \right).$$

The formula that TKNN calculated was tantamount to $\sigma_H = \frac{e^2}{h} Ch(P)$ [31, 34].

A full calculation can be found in [17] §6.3, but the Chern number for the honeycomb lattice model actually reduces to $Ch(P) = \text{sgn}(\Delta)$. Physically, this makes sense because a phase change occurs when $\Delta = 0$, which is where T -symmetry re-emerges and the gap closes. In a more general system, noncommutative geometry techniques could associate a Fredholm index to the projection operator that would correspond to the Chern number in this lattice case.

In the context of the quantum Hall effect, this Chern number corresponds to the quantum Hall conductance. In the honeycomb lattice model, the interpretation of the Chern number is slightly different, because this model lacks Q -symmetry and is not exactly the quantum Hall effect. Instead, the invariant determines the edge mode chirality, meaning which direction that edge modes can travel around the sample [17]. This is, in turn, determined by the direction of the magnetic field applied: up or down.

Remark 5.2.6. *The sphere S^2 that is the target of the projection map is known as the **Bloch sphere**, and the topological invariant calculated above can also be understood as the winding number of the torus around the sphere according to P [34].*

5.2.4 Non-Abelian Anyons

Like the Majorana chain, the honeycomb lattice model is exciting for its ability to host quasiparticles with nonabelian exchange statistics. In this case, the quasiparticles take the form of vortices, which are singularities around which the spin or phase angle of particles undergoes a rotation. In the honeycomb lattice model, they can be detected using hexagonal operators as in [17].

Braiding of these quasiparticles is easier to implement than in the Majorana chain because there are now two dimensions of spatial freedom, allowing particles to move around each other. As shown in §8 of [17], these vortices have nonabelian statistics for any odd ν , and these statistics only depend on $\nu \bmod 16$.

5.3 Conclusion

This thesis explored the Kitaev's proposed K -theoretic classification of matter, developing some mathematical and physical background before reviewing some of the most significant examples of topological materials in current literature: the Majorana chain and the integer quantum Hall effect. The power of topological invariants applied to condensed matter systems stretches far beyond what this thesis could discuss and is still being developed by mathematicians and physicists. Some directions that generalize the discussion in this thesis are classifying Hamiltonians in arbitrary dimension, as in [18], or using noncommutative geometry and a more general C^* -algebra framework to treat non-lattice models [31, 24].

An important future direction for the classification of topological materials is to generalize from free-fermion models to incorporate interactions. This will ensure that models are more physically realizable, as well as help to describe topological phenomena that only arise in the existence of interactions, such as the fractional quantum Hall effect.

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