Tangent Lines to Curves Arising from Automorphic Distributions

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Abstract

Automorphic distributions arise in connection with boundary values of modular forms and Maass forms. In most cases, these distributions have antiderivatives that are continuous functions. We shall look at the result of graphing the real vs. imaginary parts of these functions. Because of the automorphic properties of the distributions we consider, the graphs of their antiderivatives are curves which exhibit fractal-like self-similar behavior, as is illustrated in figures 3, 7 and 10. We show that at irrational points of these curves, this behavior is wild enough to prevent the existence of tangent lines to these curves. At rational points, these curves occasionally admit tangent lines, and we shall give a complete answer as to where these tangent lines occur.

1 Introduction

According to Weierstrass, Riemann presented

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sin n^2 x$$

as an example of a function which was everywhere continuous, but nowhere differentiable. It is not known, however, whether Riemann ever gave a proof of the non-differentiability of this function, or whether he even considered it in the first place. Weierstrass attempted to prove Riemann's statement, but was unsuccessful. Instead, he offered functions of the form

$$\sum a^n \cos b^n \pi x$$

as examples of everywhere continuous but nowhere differentiable functions.

In 1916, Hardy [4] was able to show that Weierstrass's function was not differentiable at any point x where

- x is irrational;
- x is a rational number of the form $2A/(4B+1)$ where A and B are integers;
- x is a rational number of the form $(2A+1)/2(2B+1)$ where A and B are integers.

In fact, Hardy also proved that the functions

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \sin n^2 \pi x, \ \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \cos n^2 \pi x$$

are not differentiable at irrational points for $\alpha < \frac{5}{2}$ and on a dense set of rational numbers for $2 < \alpha < \frac{5}{2}$. Hardy also offers some insight as to why showing the non-differentiability of Riemann's function eluded Weierstrass: "The question is a much more difficult one than those connected with Weierstrass's function, owing to the comparatively slow increase of the sequence n^2" [4].

In 1970, Gerver [2] showed non-differentiability at points of the form $(2A+1)/2^n$, thus extending Hardy's result. But more importantly, he showed that there were in fact points where the function

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sin n^2 \pi x$$

was differentiable–contrary to the belief of Weierstrass and Riemann. Specifically, he showed that at points p/q with p and q both odd, the derivative was $-\pi/2$. In 1971, Gerver went on further to show that these were the only points at which derivatives exist. His proof was elementary, but quite long and complicated. Several shorter proofs were given by Smith, Queffelec, Mohr, Itatsu, Luther, Holschneider and Tchamitchian. References can be found in [1].

In 1991, Duistermaat [1] further analyzed the local properties of Riemann's function, finding the pointwise Hölder exponent at rational points. He also found the pointwise Hölder exponent at a large (in a measure-theoretic sense) class of irrational points by showing that the function is sufficiently wild at these points. His insight was that the function

$$\sum_{n=1}^{\infty} \frac{1}{\pi n^2} \sin 2\pi n^2 x$$

should be analyzed alongside the function

$$\sum_{n=1}^{\infty} \frac{1}{\pi n^2} \cos 2\pi n^2 x,$$

and that they were the real and imaginary parts, respectively, of the function

$$\phi(x) = \sum_{n \neq 0} \frac{1}{2\pi \imath n^2} e^{2\pi \imath n^2 x}.$$

This function can be viewed as the "antiderivative" of the classical theta-function

$$\theta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi \imath n^2 z}$$

in the following sense: the classical theta-function is a holomorphic function on the upper half plane $\mathbb{H}$, and its limit as z approaches the real line is not a function in the usual sense (the limit at some real points is infinite or undefined) but a distribution, or a generalized function; this distribution is in turn the derivative of the Weierstrass function. The theta-function is a modular form of weight 1/2, and has long been an object of study. It has particular interest for number theorists, and can be used to evaluate Gauss sums like

$$\sum_{k=0}^{q-1} e^{2\pi \imath \frac{k^2}{q}}.$$

One of the most fundamental properties of the theta-function is automorphy, which relates the theta-function its transformation under certain group actions. For example, the values of the theta-function at z and $\frac{-1}{2z}$ are related by the identity

$$\theta(z) = \theta\Big(\frac{-1}{2z}\Big)\frac{e^{\frac{\pi \imath}{4}}}{\sqrt{2z}},$$

which was known in various forms to Gauss (1808), Cauchy (1817) and Poisson (1823). This identity yields useful identities such as

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 z} = \frac{1}{\sqrt{z}}\sum_{n\in\mathbb{Z}} e^{-\pi n^2/z}.$$

Automorphy also describes the self-similar fractal behavior of the function $\phi(x)$, which can be seen in figures 8-10 of section 6.

In a paper from 2004 [7], Miller and Schmid examine the local behavior of not only the function

$$\sum_{n=1}^{\infty} \frac{1}{n^2}\sin n^2\pi x,$$

but of antiderivatives of general automorphic distributions coming from the group $SL(2,\mathbb{R})$. This is a much larger class of functions that includes the antiderivatives of boundary distributions of Maass forms and modular forms of weight 1 and 1/2 associated to subgroups of $SL(2,\mathbb{Z})$ of finite order. An earlier paper by Schmid [8] gave the Hölder exponent of these functions, in particular, showing that both the real and imaginary parts of these functions are continuous. Miller and Schmid examine the question of where these continuous functions are differentiable, and obtain a complete answer as to where derivatives of the real and imaginary parts of these functions exist: they are non-differentiable almost everywhere. The result is to exhibit a large class of functions that are everywhere continuous, but (almost) nowhere differentiable. There are of course many such functions known, like Weierstrass's family

$$\sum a^n \cos b^n \pi x.$$

However, most known examples are fairly contrived, while the antiderivatives of Maass forms and modular forms arise naturally, and are of great interest to number theorists.

As Duistermaat showed, it is natural to look at the real and imaginary parts of these functions together, not only separately. With this in mind, we shall look at the graph parameterized by the real and imaginary parts of these functions. More precisely, given an automorphic distribution τ with continuous antiderivative ϕ_τ, we shall look at the graph of the curve parameterized by

$$x(t) = \mathrm{Re}\ \phi_\tau(t)$$
$$y(t) = \mathrm{Im}\ \phi_\tau(t).$$

If we view ϕ_τ as a map $\mathbb{R}\to\mathbb{C}$, then ϕ_τ traces out a path in the complex plane, and the graph we obtain by the above parameterization is just the image of $\mathbb{R}$ under ϕ_τ. Note that automorphic properties force ϕ_τ to be periodic, so that the graph is a closed curve in the plane.

We shall examine the question of where tangents to these graphs exist. Wilfried Schmid tells me that Curt McMullen first raised this question to him in relation to the Weierstrass function.

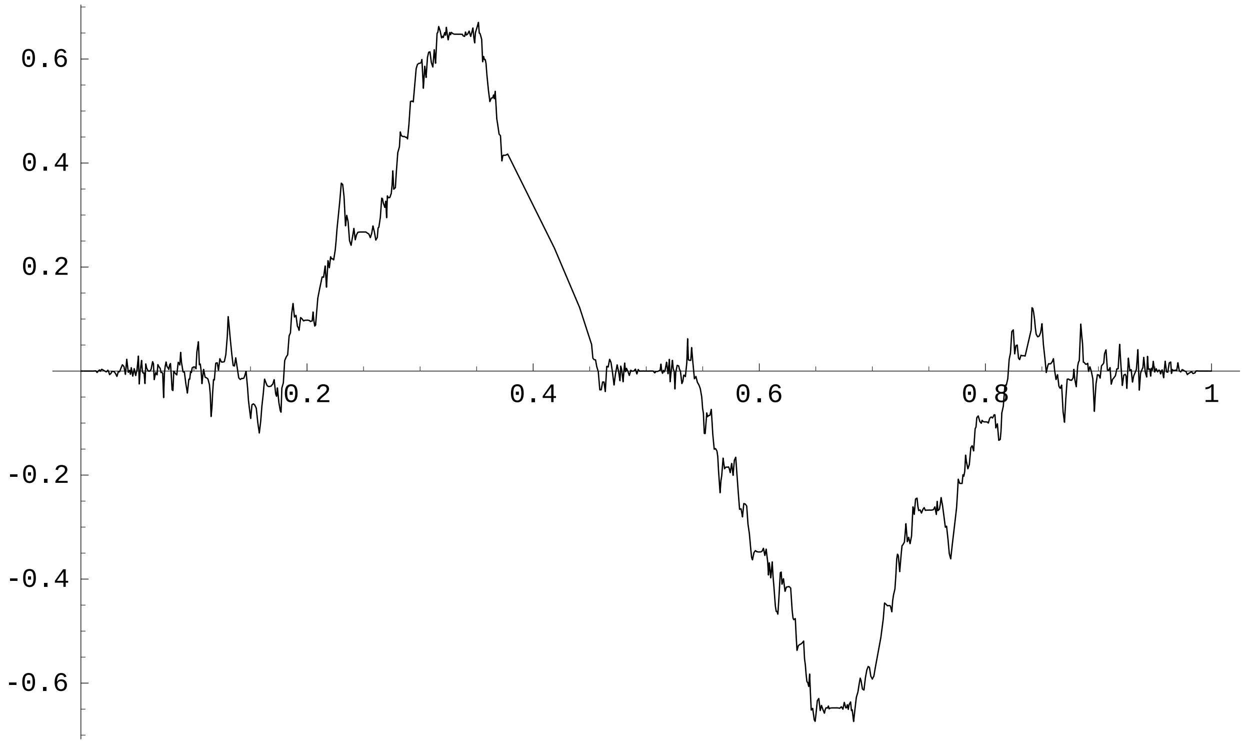

Figure 1: The real part of the antiderivative of the boundary distribution associated to the Maass form for $SL_2(\mathbb{Z})$ with $\lambda \approx 27.56\imath$

This question is in some ways analogous to the question of where derivatives exist for the real and imaginary parts of these functions. The existence of a derivative for Re $\phi_\tau(t)$ at some point is the same as the question of whether a tangent line exists to the graph of t vs. Re $\phi_\tau(t)$, whereas we are interested in the existence of tangent lines to the graph of Re $\phi_\tau(t)$ vs. Im $\phi_\tau(t)$. Figures 1, 2, and 3 are graphs of the real part, imaginary part and real part vs. imaginary part of the curve arising from the Maass form associated to $SL_2(\mathbb{Z})$ for $\lambda \approx 27.56\imath$. These graphs were created from Fourier coefficients calculated by Michael Rubinstein, and transmitted through Stephen Miller. The first of these graphs also appears in [7]. The third graph pictured shows that the curve parameterized by the real and imaginary parts of the antiderivative of the boundary of the Maass form has no tangent lines. From these graphs, we can also see that the real and imaginary parts may behave in very different ways, and the graph obtained by graphing the real part vs. imaginary part is of an entirely different nature than the graphs of either part alone.

Moreover, though the existence of derivatives for Re $\phi_\tau(t)$ and Im $\phi_\tau(t)$ is related to the existence of tangent lines, these questions are logically independent. A differentiable curve can fail to have tangents, and a curve with tangents can fail to be differentiable. For example, the curve

$$(t^2, 0) \text{ for } t > 0$$
$$(0, t^2) \text{ for } t < 0$$

has derivatives, even continuous derivatives near $t = 0$ in both its co-ordinates. However, at $t = 0$,

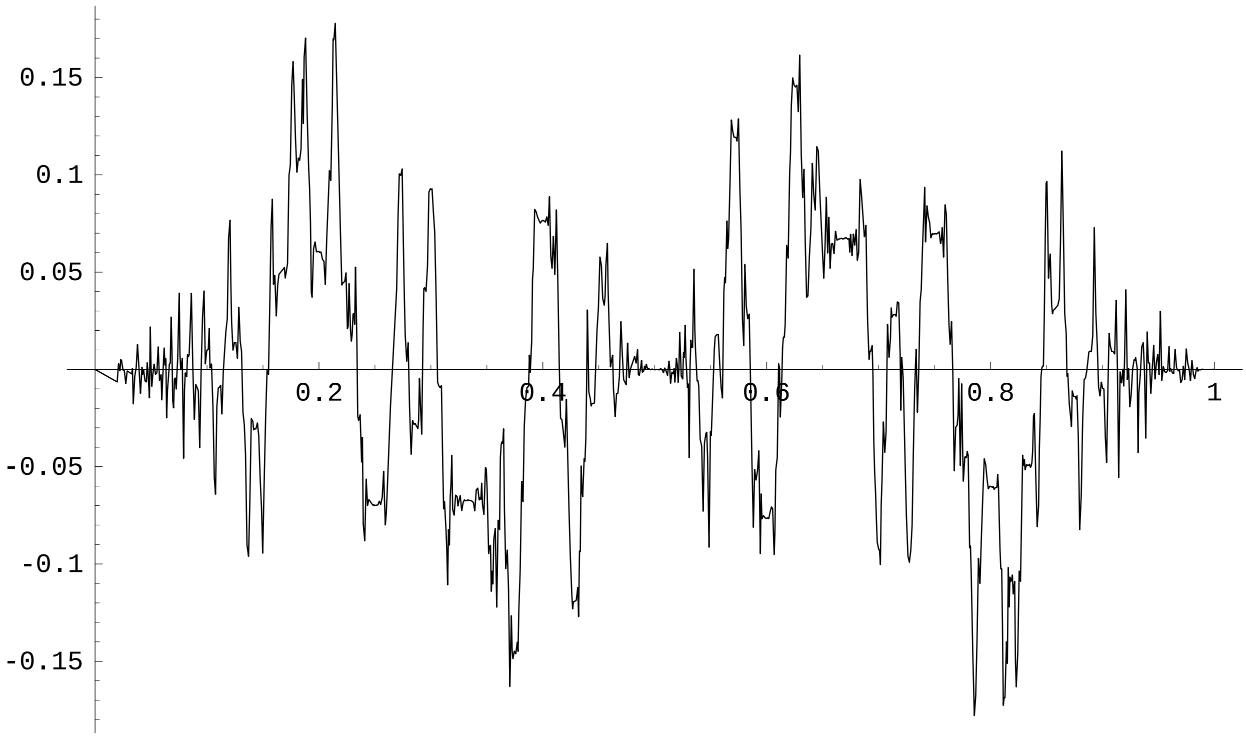

Figure 2: The imaginary part of the antiderivative of the boundary distribution associated to the Maass form for $SL_2(\mathbb{Z})$ with $\lambda \approx 27.56\imath$

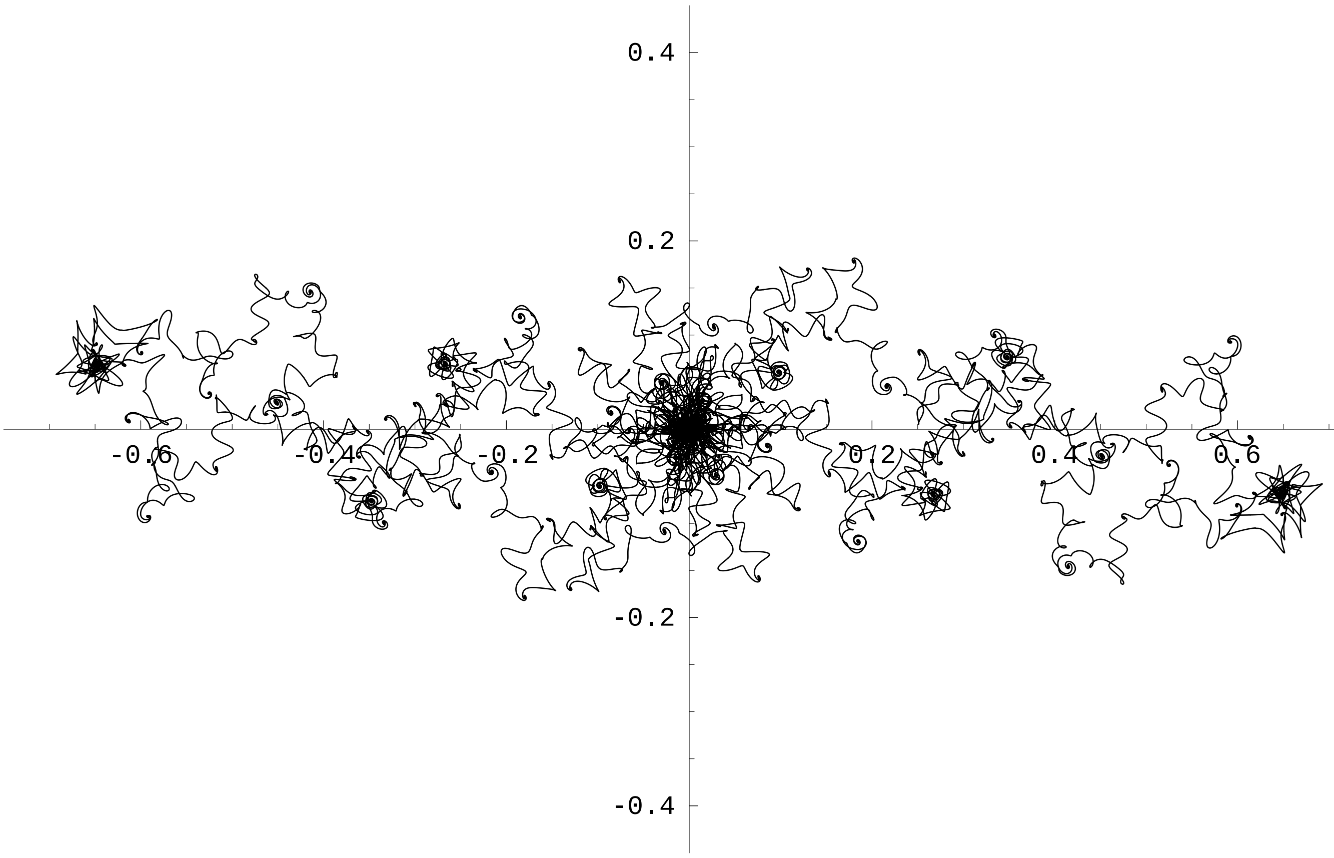

Figure 3: The real part vs. imaginary part of the antiderivative of the boundary distribution associated to the Maass form for $SL_2(\mathbb{Z})$ with $\lambda \approx 27.56\imath$

it does not have a tangent, because even though the derivatives of both coordinates exist and are continuous, they vanish at $t = 0$, so that the curve comes to a corner there.

On the other hand, if we take a curve with tangents and re-parameterize it by a continuous non-differentiable function, it may fail to be differentiable. A simple example can be constructed by taking a continuous non-differentiable function f. Then the curve $(f(t), f(t))$ has no derivative, but it has a tangent line for every t. In fact, there exist Maass forms that behave in precisely this manner.

More interesting, however, are the curves like the one associated to the modular form of weight 1 pictured in figures 5-7. From our result we obtain a large class of curves that, like this one, are continuous but have tangents nowhere. Examples of such curves are not difficult to come by, but they do not often arise from such fundamental objects as modular forms and Maass forms.

We should note here that the graphs of the real vs. imaginary parts of antiderivatives of automorphic distributions that we include in this paper were done in Mathematica with very high resolution, while the remaining graphs, figures 1, 2, 5, 6, 8 and 9, were done at lower resolution, and exhibit minor irregularities, as helpfully pointed out by Stephen Miller.

In this paper, we shall show that tangents to the graphs of curves parameterized by the real and imaginary parts of antiderivatives of automorphic distributions only exist for a small class of these distributions, and even in that case only at certain rational points. Specifically, if ϕ_τ is the antiderivative of an automorphic distribution τ,

1. if ϕ_τ is a constant multiple of a real function, then tangents exist at every point, for trivial reasons;

2. if ϕ_τ is the antiderivative of a non-cuspidal Maass form of eigenvalue less than 1/4, tangents may exist at rational points in the direction of $c_{\gamma\tau,0}$ or $c_{\tau,0}$; here $c_{\gamma\tau,0}$ and $c_{\tau,0}$ are constants that measure the cuspidality of the distribution τ at various cusps;

3. otherwise, tangent lines do not exist.

We shall see that the Weierstrass function behaves somewhat like the Maass forms in the second case, so that tangent lines exist at some rational points, but not at any irrational points.

In section 2, we will give some basic definitions of modular forms, Maass forms and automorphic distributions, and outline the relationships among them. In section 3, we introduce the tools used in our proofs. In section 4, we deal with the existence of tangent lines at rational points, and in section 5 with the more complicated case of irrational points. Our proofs will rely heavily on automorphy, which, as has been shown in [1], [8] and [7], becomes a powerful tool for examining the local properties of automorphic distributions. We conclude with a discussion of how these proofs can be extended to the case of the Weierstrass function and other curves associated to modular forms of weight 1/2. The case of the Weierstrass function will contain some subtleties which do not arise in other cases, and we shall examine it in detail.

2 Modular Forms, Maass Forms and Automorphic Distributions

In this section, we will define modular forms and Maass forms and explain how automorphic distributions arise in connection with them. We will follow [3], [7], [8], [9], [10] in our exposition.

The group $SL_2(\mathbb{R})$ acts on the upper half plane by fractional linear transformations. For $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \to \frac{az+b}{cz+d}.$$

The subgroup $SL_2(\mathbb{Z}) \subset SL_2(\mathbb{R})$ is a discrete subgroup which arises naturally in the study of the discrete modules, i.e. lattices, in the complex plane. A discrete module M (over $\mathbb{Z}$) is a discrete, additive subgroup of the complex numbers.

A discrete module can be either zero alone, the set of integer multiples $n\omega$ of a complex number ω or the set of integer linear combinations of two complex numbers $n_1\omega_1 + n_2\omega_2$ where ω_1, ω_2 are complex numbers such that ω_1/ω_2 is non-real. We shall concern ourselves with the last of these cases–lattices of rank 2.

We wish to identify lattices which differ by a constant, so that $\lambda\omega_1, \lambda\omega_2$ generate a lattice equivalent to the one generated by ω_1, ω_2. Thus a rank two lattice is completely determined by $z = \omega_1/\omega_2$. By interchanging ω_1 and ω_2 if necessary, we can require z to lie in the upper half plane

$$\mathbb{H} = \{z = x + iy : x \in \mathbb{R}, y \in \mathbb{R}^+\}.$$

Additionally, ω_1, ω_2 and ω_1', ω_2' determine the same lattice if

$$\begin{aligned} \omega_1' &= a\omega_1 + b\omega_2 \\ \omega_2' &= c\omega_1 + d\omega_2 \end{aligned}$$

where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. We need that $ad - bc = 1$ because ω_1, ω_2 must also be expressible as an integral linear combination of ω_1, ω_2, so that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ must be invertible in order for ω_1, ω_2 and ω_1', ω_2' to determine the same lattice. Such a matrix with entries a, b, c, d is called a unimodular transformation. Then the action of unimodular transformations on lattices $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ corresponds to the action of the modular group

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

on $\mathbb{H}$ by linear fractional transformations

$$\gamma z = \frac{az+b}{cz+d} \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let us find a fundamental domain of the action of the modular group on the upper half plane. First observe that for any $z \in \mathbb{H}$, there are only finitely many pairs of integers (c, d) such that

$$|cz + d| \leq 1.$$

The reason is that as c, d range over the integers, the numbers $cz + d$ form a discrete lattice, so that only finitely many points lie in any bounded neighborhood of 0. Now, if $z = x + \imath y$, call $y = \text{Im } z$ the "height" of z. For any $\gamma \in SL_2(\mathbb{Z})$, we have that

$$\gamma z = \frac{az+b}{cz+d} = \frac{az+b}{cz+d}\frac{a\bar{z}+b}{c\bar{z}+d} = \frac{\text{Real} + i(ad-bc)\text{Im } z}{|cz+d|^2},$$

so that for all $z \in \mathbb{H}$,

$$\text{Im } \gamma z = \frac{\text{Im } z}{|cz+d|^2}. \tag{1}$$

Then given a complex number z, there are only finitely many (c, d) such that there exists a $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$ with the height of γz greater than the height of z. For a fixed pair (c, d) all such γz differ from one another by a transformation of the form $T^n : z \to z + n$, where $T : z \to z + 1$ corresponds to the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Thus among all the images of z under $SL_2(\mathbb{Z})$, we may choose one of maximum height, i.e., such that $|cz + d| \geq 1$ for all $c, d \in \mathbb{Z}$, and such that $|\text{Re } z| \leq 1/2$. We then claim that

$$D = \{z \in \mathbb{H} | |\text{Re } z| \leq \frac{1}{2}, |z| > 1\}$$

is a fundamental domain (every point z has an image in the closure of D).

First we show that the set

$$D_1 = \{z \in \mathbb{H} | |\text{Re } z| \leq 1/2, |cz + d| > 1, \forall c, d \in Z, (c, d) \neq (0, 0), (0, 1)\}$$

is the same set as D. Taking $c = 1, d = 0$ in the definition of D_1, we see that $D_1 \subseteq D$. On the other hand, if $z \in D$, then

$$|cz + d|^2 = c^2(x^2 + y^2) + 2cdx + d^2 > c^2 - |cd| + d^2 \geq 1, \tag{2}$$

so that $D \subseteq D_1$. Thus every point has an image in the closure of D by (1) and the preceding paragraph. We have that the points on the boundary are identified by reflection across the line $x = 0$ by the transformations $T : z \to z + 1$ and $S : z \to \frac{-1}{z}$. If two points in the closure of D are equivalent, then we must have that they are related by $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $|cz+d| = 1$. This implies that the inequalities in (2) are in fact equalities, and that either $c = 0, d = \pm 1$ or $c = \pm 1, d = 0$. In the first instance, the two points are related by $\gamma = T$, in the second they are related by $\gamma = S$, as defined above. See figure 4.

From this point forward, let $\Gamma \subseteq SL_2(\mathbb{Z})$ be a subgroup of finite index in the modular group. Then the fundamental domain of Γ is a union of a finite number of copies of D, each one corresponding to a right coset of Γ.

An "unrestricted modular form of weight k" associated to the the group Γ is a meromorphic function f such that for all $\gamma \in \Gamma$,

$$f(\gamma z) = (cz + d)^k f(z). \tag{3}$$

We wish to restrict our attention to holomorphic functions, which are, moreover, "holomorphic at ∞" and holomorphic at all the parabolic points (cusps) of Γ, in a sense which we will soon define. Note that because Γ has finite index in $SL_2(\mathbb{Z})$, there is some smallest $N \in \mathbb{N}$ for which $T^N \in \Gamma$, so that f has period N. When we require f to be "holomorphic at ∞," we mean the following: if we map the strip $0 \leq \text{Re } z \leq N$ of $\mathbb{H}$ to the disc $|z| < 1$ taking $\imath\infty$ to 0 by the map $\zeta = e^{2\pi\imath z/N}$, and let $\hat{f}(\zeta) = f(z)$, then $\hat{f}$ is holomorphic at 0, i.e.

$$\hat{f} = \sum_{m=0}^{\infty} a_m \zeta^m.$$

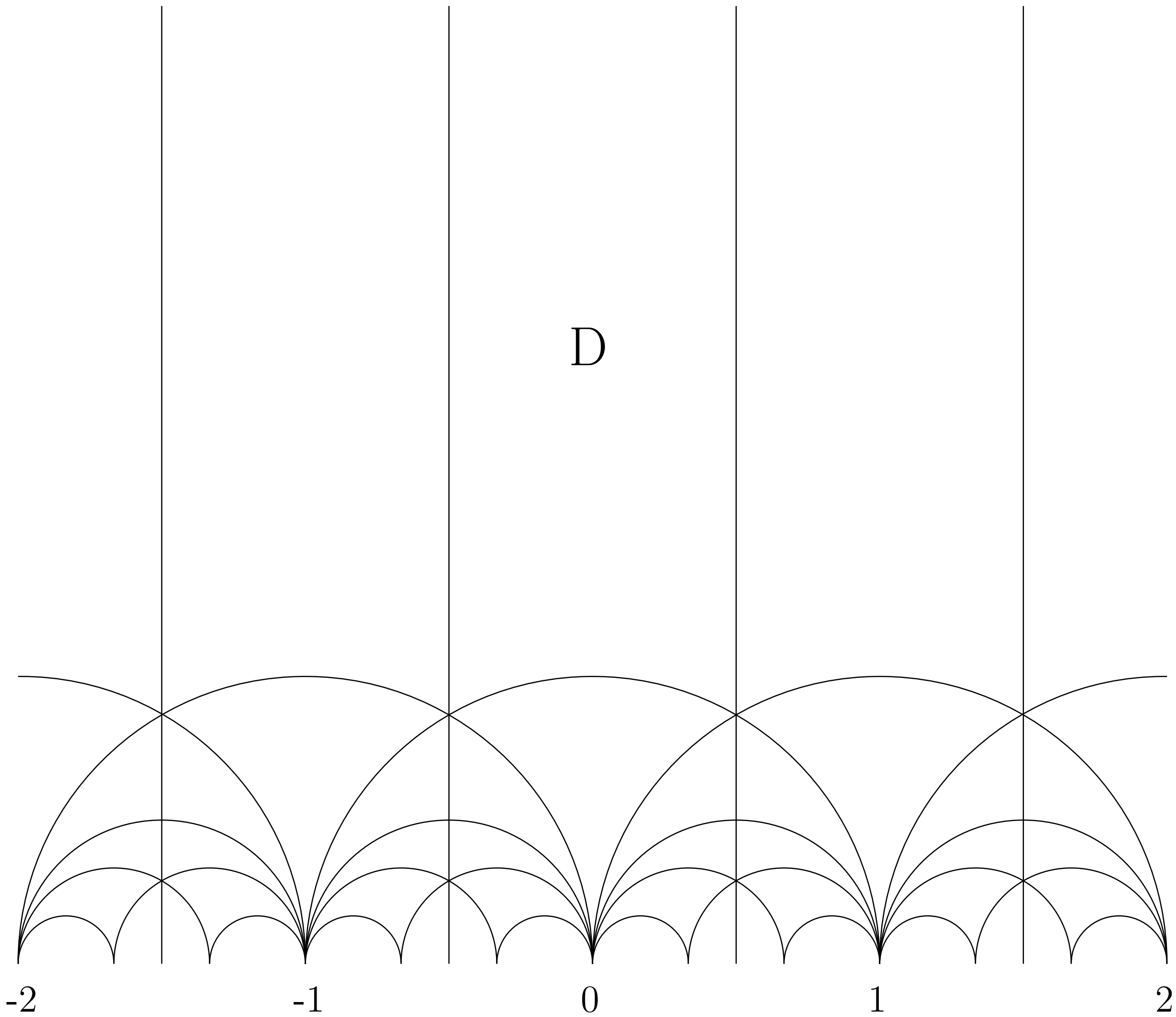

Figure 4: The fundamental region D for the action of $SL_2(\mathbb{Z})$ on the upper half plane H.

This in turn induces a Fourier expansion of f:

$$f = \sum_{m=0}^{\infty} a_m e(mz/N),$$

where $e(mz/N) = e^{2\pi\imath mz/N}$. At other parabolic points a/c, we can choose

$$\gamma = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in SL_2(\mathbb{Z})$$

such that $\gamma(a/c) = \infty$, and require f to be holomorphic at a/c by requiring that the translate of f by γ^{-1} is holomorphic at infinity, i.e. that

$$(cz+d)^k f(\gamma^{-1}z)$$

has a Fourier expansion $f = \sum_{m=0}^{\infty} a_m e(mz/N)$, with $a_m = 0$ for $m < 0$.

Definition 1 *A modular form of weight k for the subgroup Γ is a holomorphic function f satisfying $f(\gamma z) = (cz+d)^k f(z)$ for all $\gamma \in \Gamma$, which is moreover holomorphic at ∞ and at all the parabolic points of Γ.*

We can also give a definition for Maass forms:

Definition 2 *A Maass form F on $\mathbb{H}$ is a Γ-invariant eigenfunction of the hyperbolic Laplace operator $y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ with eigenvalue $\frac{1}{4}(\lambda^2 - 1)$. We require that λ is purely imaginary or that $-1 < \lambda < 1$. Moreover, we require that F has finite L^2 norm:*

$$y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})F(x,y) = \frac{1}{4}(\lambda^2 - 1)F(x,y) \tag{4}$$

$$\int_{\Gamma\backslash\mathbb{H}} |F(x,y)|^2 \frac{dxdy}{y^2} < \infty.$$

We now proceed to define automorphic distributions for $SL_2(\mathbb{R})$.

Let $C^{-\infty}(\mathbb{R})$ denote the space of complex-valued distributions on the real line, or the dual of $C_c^{\infty}(\mathbb{R})$, the space of compactly supported complex-valued C^{∞} functions on $\mathbb{R}$. For $\lambda \in \mathbb{C}$ and $\delta \in \mathbb{Z}/2\mathbb{Z}$, we let $V_{\lambda,\delta}^{-\infty}$ denote vector space of pairs $(\tau, \tilde{\tau}) \in C^{-\infty}(\mathbb{R}) \times C^{-\infty}(\mathbb{R})$ such that

$$\tilde{\tau}(x) = (\text{sgn } x)^{\delta} |x|^{\lambda - 1} \tau\Big(\frac{-1}{x}\Big) \tag{5}$$

for $x \neq 0$. Thus τ and $\tilde{\tau}$ determine each other except at 0. Roughly speaking, $\tilde{\tau}$ is needed so that we can define τ "at ∞." We will want to be able to make sense of $\tilde{\tau}$ even at 0, and we will soon impose a condition on τ so that this will be possible.

For $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, we define an action $\pi_{\lambda,\delta}$ on $V_{\lambda,\delta}^{-\infty}$ as follows:

$$(\pi_{\lambda,\delta}(g)\tau)(x) = (\text{sgn}(cx+d))^{\delta} ||cx+d|^{\lambda-1} \tau(\frac{ax+b}{cx+d}). \tag{6}$$

This identity makes sense for $cx+d \neq 0$. We can make sense of this equation at $cx+d=0$ by using equation (5) to express τ at ∞ in terms of $\tilde{\tau}$ at 0. Note that for $a=d=0, b=-c=-1$, $\pi_{\lambda,\delta}(g^{-1})$ switches τ and $\tilde{\tau}$. From this we can define the corresponding action $\tilde{\pi}_{\lambda,\delta}$ by $\tilde{\pi}_{\lambda,\delta}(g) = \pi_{\lambda,\delta}(SgS^{-1})$, so that

$$(\tilde{\pi}_{\lambda,\delta}(g)\tilde{\tau})(x) = (\text{sgn}(a-bx))^{\delta}||a-bx|^{\lambda-1}\tilde{\tau}(\frac{dx-c}{a-bx}),$$

and the pair $(\pi_{\lambda,\delta}(g)\tau, \tilde{\pi}_{\lambda,\delta}(g)\tilde{\tau})$ is also in $V_{\lambda,\delta}^{-\infty}$. One can check that the action $\pi_{\lambda,\delta}$ gives a representation of $SL_2(\mathbb{R})$ on pairs $(\tau,\tilde{\tau}) \in V_{\lambda,\delta}^{-\infty}$.

We let $(V_{\lambda,\delta}^{-\infty})^{\Gamma}$ be the subspace of Γ-invariant vectors. Since Γ has finite index in $SL_2(\mathbb{Z})$, there exists a smallest positive integer N such that $\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma$. Then because of Γ-invariance, we have that $\tau(x)$ is periodic with period N. Thus we have a Fourier expansion for $\tau(x)$:

$$\tau(x) = c_0 + \sum_{n\neq 0} c_n e^{2\pi \imath n x/N}. \tag{7}$$

In order for τ to determine $\tilde{\tau}$, we need to be able to extend the distribution

$$c_0(\text{sgn } x)^{\delta}|x|^{\lambda-1} + (\text{sgn } x)^{\delta}|x|^{\lambda-1}\sum_{n\neq 0} c_n e^{2\pi \imath n/Nx}$$

across $x=0$. The first summand can be extended across zero by analytic continuation in the variable λ unless $\lambda \in (2\mathbb{Z}+\delta)\cap\mathbb{Z}_{\leq 0}$, while the second summand can be extended across 0 by integration by parts. In order to ensure that the first summand can be extended, we require that

$$c_0 = 0 \text{ unless } \text{Re } \lambda > 0. \tag{8}$$

When the two summands have been extended in this manner, we obtain a natural extension of $\tilde{\tau}$ across $x=0$, or a natural extension of τ across ∞, and we say that "τ agrees with its natural extension across ∞." This condition guarantees that τ is completely determined by its Fourier expansion (7) both as a distribution and as an element of $(V_{\lambda,\delta}^{-\infty})^{\Gamma}$. Because we will be working with all the $SL_2(\mathbb{Z})$-translates of τ, we shall suppose that (8) holds for the $SL_2(\mathbb{Z})$-translates of τ, and that all of the $SL_2(\mathbb{Z})$-translates of τ agree with their natural extension across ∞. Under these conditions, τ and $\tilde{\tau}$ determine each other completely, so we now identify the distributions $(\tau,\tilde{\tau})$ by the first member of the pair. Finally, we shall exclude the space of constants from the space $(V_{1,0}^{-\infty})^{\Gamma}$ to avoid having to treat this simple case separately. We shall take these conditions to be part of the definition of an automorphic distribution for our purposes:

Definition 3 *An automorphic distribution $\tau \in (V_{\lambda,\delta}^{-\infty})^{\Gamma}$ associated to the subgroup Γ is a distribution with all its $SL_2(\mathbb{Z})$-translates satisfying (2) and agreeing with their natural extensions across ∞, which additionally satisfies*

$$\tau(x) = (\pi_{\lambda,\delta}(g)\tau)(x) = (sgn(cx+d))^{\delta}|cx+d|^{\lambda-1}\tau(\frac{ax+b}{cx+d})$$

for all $g\in\Gamma$. We shall exclude the space of constants from $(V_{1,0}^{-\infty})^{\Gamma}$.

We give one more definition:

Definition 4 *We say that an automorphic distribution* $\tau \in (V_{\lambda,\delta}^{-\infty})^{\Gamma}$ *is* cuspidal at ∞ *if* $c_0 = 0$. *We shall say that* τ *is* cuspidal *if all its* $SL_2(\mathbb{Z})$*-translates are cuspidal at* ∞.

We can now connect automorphic distributions to modular forms and Maass forms.

Theorem 5 *([7], [8]) The space* $(V_{\lambda,\delta}^{-\infty})^{\Gamma}$ *corresponds bijectively to the space of*

1. *cuspidal Maass forms for* Γ *with eigenvalue* $\frac{1}{4}(1-\lambda^2) \geq \frac{1}{4}$, *when* $\lambda \in \imath\mathbb{R}$ *and* $\delta = 0$;
2. *cuspidal Maass forms for* Γ *with eigenvalue* $\frac{1}{4}(1-\lambda^2) < \frac{1}{4}$, *when* $-1 < \lambda < 0$ *and* $\delta = 0$;
3. *square-integrable (not necessarily cuspidal) Maass forms for* Γ *with eigenvalue* $\frac{1}{4}(1-\lambda^2) < \frac{1}{4}$, *when* $0 < \lambda < 1$ *and* $\delta = 0$;
4. *cuspidal odd-weight Maass forms for* Γ *with eigenvalue* $\frac{1}{4}(1-\lambda^2) > \frac{1}{4}$, *when* $\lambda \in \imath(\mathbb{R} - \{0\})$ *and* $\delta = 1$;
5. *cuspidal holomorphic modular forms of weight* $k \geq 1$, *when* $\lambda = 1 - k$ *and* $\delta \equiv k$ *(mod 2).*

In order to see more explicitly this connection between automorphic distributions and modular forms and Maass forms, we must give another description of these distributions. Automorphic distributions associated to a discrete subgroup $\Gamma \subset SL_2(\mathbb{R})$ arise from representations of $SL_2(\mathbb{R})$. The irreducible representations of $SL_2(\mathbb{R})$ come in five types [6], [9]:

1. Finite dimensional representations σ_n, $n \in \mathbb{Z}_{\geq 0}$, with weights $-n, -n+2, \ldots, n-2, n$
2. Infinite dimensional representations π_n^+, $n \in \mathbb{Z}_{>0}$, with weights $n, n+2, n+4, \ldots$
3. Infinite dimensional representations π_n^-, $n \in \mathbb{Z}_{>0}$, with weights $-n, -n-2, -n-4, \ldots$
4. Even weight infinite dimensional representations $\pi_{s,\text{even}}$, $s \neq 1(2)$, $s \in \mathbb{C}$
5. Odd weight infinite dimensional representations $\pi_{s,\text{odd}}$, $s \neq 1(2)$, $s \in \mathbb{C}$

Note that $\pi_{s,\text{even}} \simeq \pi_{-s,\text{even}}$ and $\pi_{s,\text{odd}} \simeq \pi_{s,\text{odd}}$. Also, $\pi_{2k+1,\text{even}}$ contains π_{2k+2}^+ and π_{2k+2}^- as subrepresentations and σ_{2k} as a quotient, while $\pi_{-2k-1,\text{even}}$ contains σ_{2k} as a subrepresentation and π_{2k+2}^+ and π_{2k+2}^- as quotients. Similarly, $\pi_{2k,\text{odd}}$ contains π_{2k+1}^+ and π_{2k+1}^- as subrepresentations and σ_{2k-1} while $\pi_{-2k,\text{odd}}$ contains σ_{2k-1} as a subrepresentation and π_{2k+1}^+ and π_{2k+1}^- as quotients.

We are interested only in irreducible unitary representations. Given this restriction, our list becomes

1. The trivial representation
2. Infinite dimensional representations π_n^+ and π_n^-
3. Even weight representations $\pi_{s,\text{even}}$, $s \neq 1(2)$, $s \in \mathbb{C}$
4. Odd weight representations $\pi_{s,\text{odd}}$, $s \in \imath\mathbb{R}\backslash 0$
5. Even weight representations $\pi_{s,\text{even}}$, $-1 < s < 0$ or $0 < s < 1$

The representations in 2. are known as discrete series representations; those in in 3. and 4. are known as principal series representations; those in 5. are known as complementary series representations [6], [9].

Given an irreducible unitary representation (π, V) of $SL_2(\mathbb{R})$, we define $V^\infty \subset V$ as the set of vectors such that $g \to \pi(g)v$ is a C^∞ function from $SL_2(\mathbb{R})$ to the Hilbert space V. Call this map f. We refer to V^∞ as the space of C^∞ vectors. V^∞ is dense inside V, and via the identification

$$V^\infty \simeq \{f \in C^\infty(SL_2(\mathbb{R}), V) | f(g) = \pi(g)f(e) \forall g \in SL_2(\mathbb{R})\}$$

where $v \leftrightarrow f(e)$, V^∞ carries a natural Fréchet topology. There is an irreducible unitary representation (π', V') dual to (π, V). We can similarly define V'^∞, the space of C^∞ vectors in V'. The space of continuous linear functionals on V'^∞ is the space $V^{-\infty}$ of distribution vectors. We have $V^\infty \subset V \subset V^{-\infty}$ and, dually, $V'^\infty \subset V' \subset V'^{-\infty}$. This is consistent with the convention that distributions on a manifold are dual to compactly supported smooth measures. Thus distributions include all continuous and all L^2 functions.

Now let $\Gamma \subseteq SL_2(\mathbb{Z})$ as before, and $K \simeq SO_2(\mathbb{R}) \subset SL_2(\mathbb{R})$ be a maximal torus. Then $SL_2(\mathbb{R})$ acts on the right via the right regular representation on the space of Γ-invariant functions on $SL_2(\mathbb{R})$, denoted $L^2(\Gamma \backslash SL_2(\mathbb{R}))$. Then because this representation is unitary and because $\Gamma \backslash SL_2(\mathbb{R})/K$ is compact, we can decompose it as a direct sum of unitary representations. Suppose that (π, V) occurs as a direct summand in $L^2(\Gamma \backslash SL_2(\mathbb{R}))$. Then because $L^2(\Gamma \backslash SL_2(\mathbb{R}))$ is self-dual, (π', V') also occurs as a direct summand. Then given any inclusion

$$i : V' \hookrightarrow L^2(\Gamma \backslash SL_2(\mathbb{R})),$$

we have that that the inclusion sends C^∞ vectors to C^∞ functions, so that we get a $SL_2(\mathbb{R})$-invariant map

$$i : V'^\infty \hookrightarrow C^\infty(\Gamma \backslash SL_2(\mathbb{R})).$$

Define

$$\langle \tau, v' \rangle = i(v')(e),$$

for all $v' \in V'^\infty$ so that τ is a linear functional on V'^∞, i.e., $\tau \in V^{-\infty}$. Because $v \in V'^\infty$ is Γ-invariant, composition with i and evaluation at the identity determines a Γ-invariant distribution vector, so that in fact

$$\tau \in (V^{-\infty})^\Gamma.$$

Note that we use the convention that the action $\pi'(g)$ of $SL_2(\mathbb{R})$ on V' is compatible with right translation $r(g)$ of functions on $L^2(\Gamma \backslash SL_2(\mathbb{R}))$, while the action $\pi(g^{-1})$ of $SL_2(\mathbb{R})$ on V is compatible with left translation $l(g)$ of functions on $L^2(\Gamma \backslash SL_2(\mathbb{R}))$. Also τ completely determines the embedding i: for $v' \in V'^\infty$ and $g \in SL_2(\mathbb{R})$, $i(v')(g) = r(g)i(v')(e) = i(\pi(g)v')(e) = \langle \tau, \pi(g)v' \rangle$. Thus τ determines the embedding of V'^∞ in $L^2(\Gamma \backslash SL_2(\mathbb{R}))$, and because V'^∞ is dense V', τ completely determines the embedding. The space $(V^{-\infty})^\Gamma$ corresponds bijectively to the space of $SL_2(\mathbb{R})$-invariant homomorphisms of $(V')^\infty$ into $C^\infty(\Gamma \backslash SL_2(\mathbb{R}))$ [8].

We are now ready to describe the connection between automorphic distributions and modular forms and Maass forms. Given an embedding of a discrete series representation

$$i : V_{-k} \hookrightarrow L^2(\Gamma \backslash SL_2(\mathbb{R}))$$

take the highest SO_2-weight vector v_0.

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \in SO_2$$

acts on v_0 by $e^{-ik\theta}$. Then $i(v_0)(gk) = e^{-ik\theta} i(v_0)(g)$ for any $k \in SO_2$, and $i(v_0)(\gamma g) = i(v_0)(g)$ for any $\gamma \in \Gamma$. If we let $F(g) = i(v_0)(g)(g'(\imath))^{-k/2}$, where $g'(\imath)$ is the derivative of the transformation g at $\imath$, we find that

$$F(gk) = i(v_0)(gk)(((gk)')(\imath))^{-k/2} = e^{-ik\theta} i(v_0)(g)(g'(k\imath))^{-k/2} k'(\imath)^{-k/2} =$$
$$e^{-ik\theta} i(v_0)(g)(g'(\imath))^{-k/2} e^{ik\theta} = F(g),$$

and for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$F(\gamma g) = i(v_0)(\gamma g)((\gamma g')(\imath))^{-k/2} = i(v_0)(g)(g'(\imath))^{-k/2} \gamma'(g\imath) = F(g)(cg\imath + d)^k.$$

Now, the stabilizer of $\imath$ under the action of the group $SL_2(\mathbb{R})$ on the upper half plane $\mathbb{H}$ is $SO_2(\mathbb{R})$, so that $SL_2(\mathbb{R})/SO_2(\mathbb{R}) \simeq \mathbb{H}$. Thus F can be viewed as a function of $g\imath$, so that $F \in L^2(\backslash\mathbb{H})$. By (9), $F(z)$ satisfies the transformation rule (3) for elements of Γ, and $F(z)$ is a cuspidal modular form of weight k on $\mathbb{H}$. There is a natural bijection between non-zero cuspidal modular forms of weight k and embeddings of V_k in $L^2(\Gamma\backslash SL_2(\mathbb{R}))$, both spaces isomorphic to the vector space $(V'^{-\infty}_{-k})^\Gamma \simeq (V_k^{-\infty})^\Gamma$, because $V'_{-k} = V_k$ [8].

If we allow

$$\tau(x) = \lim_{y \to 0^+} F(x + \imath y),$$

then the distribution τ is a concrete realization of the distribution vector corresponding to the embedding

$$V_k \hookrightarrow L^2(\Gamma\backslash SL_2(\mathbb{R})).$$

The distribution $\tau(x)$ completely determines $F(z)$ and vice-versa. τ inherits this automorphy from F, so that

$$\tau(x) = \frac{1}{(cx+d)^k} \tau(\gamma x). \tag{9}$$

Thus τ is invariant under the action of Γ described in equation (6). The space $(V_k^{-\infty})^\Gamma$ is precisely the space $(V_{k,\delta}^{-\infty})^\Gamma$, $\delta \equiv k \bmod 2$ defined previously, covering case 5 of Theorem 5.

Now suppose that we have an embedding of an even principal series or complementary series representation

$$i : V_{-\lambda,+1} \hookrightarrow L^2(\Gamma\backslash SL_2(\mathbb{R})),$$

where the $+1$ refers to the even parity condition. Then there is a unique (up to scalars) SO_2-invariant vector $v_0 \in V_{\lambda,+1}$ (v_0 has weight 0). Then let $F(g) = i(v_0)$. Because v_0 is invariant under the action of SO_2, F is invariant under the action of SO_2 on the right. Thus

$$F(g) \in L^2(\Gamma\backslash SL_2(\mathbb{R})/SO_2(\mathbb{R})).$$

We find that $F \in L^2(\Gamma\backslash\mathbb{H})$, so that $F(x,y)$ is an Γ-invariant function on the upper-half plane. In fact, F is a Maass form with eigenvalue λ, and satisfies equation (14), so that the space of square-integrable Maass forms corresponds bijectively to embeddings $V_{-\lambda,+1} \hookrightarrow L^2(\Gamma\backslash SL_2(\mathbb{R}))$, both spaces isomorphic to $(V_{\lambda,+1}^{-\infty})^\Gamma$ [8].

The Maass form F has an asymptotic expansion near the real axis:

$$F(x,y) \sim y^{\frac{1-\lambda}{2}} \sum_{n \geq 0} \tilde{\tau}_{\lambda,n}(x) y^n + y^{\frac{1+\lambda}{2}} \sum_{n \geq 0} \tilde{\tau}_{-\lambda,n}(x) y^n.$$

The distribution $\tilde{\tau}_{\lambda,0}$ completely determines the Maass form $F(x,y)$ and vice-versa, so the distribution $\tilde{\tau}_{\lambda,0}$ is a concrete realization of the space $(V_{\lambda,+1}^{-\infty})^{\Gamma}$ of Γ-invariant distributions on $SL_2(\mathbb{R})$ arising from embeddings of $V_{-\lambda,+1}$ in ${}^2(\Gamma\backslash SL_2(\mathbb{R}))$. This concrete realization of the space $(V_{\lambda,+1}^{-\infty})^{\Gamma}$ corresponds precisely to the space of distributions on the real line which we also called $(V_{\lambda,+1}^{-\infty})^{\Gamma}$. Note that $y^{\frac{1-\lambda}{2}}\tilde{\tau}_{\lambda,0}(x)$ is Γ invariant, so that $\tau(x)=\tilde{\tau}_{\lambda,0}(x)$ carries the automorphy property

$$\tau(x) = |cx+d|^{\lambda-1}\tau(\frac{ax+b}{cx+d}). \tag{10}$$

This covers cases 1, 2 and 3 of Theorem 5. Odd-weight Maass forms arise in a similar, but slightly more complicated, way from odd-weight representations [8].

Because the discrete series representation V_{-k} occurs as part of the principal series representation $V_{\lambda,\delta}$ for $\lambda = k-1$ and δ of the appropriate parity, $1-k$ and λ play analogous roles in the definition of automorphic distributions corresponding to modular forms and Maass forms, as we can see from the equations of automorphy in (9) and (10).

3 Preliminaries

We now recall some results from [8] and [7]. A function $f \in C(\mathbb{R})$ is Hölder continuous of index α, $0<\alpha\leq 1$ if

$$|f(x)-f(y)| < C|x-y|^{\alpha}$$

for all $x,y\in\mathbb{R}$, where $C>0$ can be chosen locally uniformly in x,y. We define $C^{\alpha}(\mathbb{R})\subset C^{\infty}(\mathbb{R})$ for $0<\alpha<1$ to be the set of Hölder continuous functions of index α, and $C^{0}(\mathbb{R})$ to be the space of continuous functions. We extend this definition to all real α and to distributions as well as functions by the equation

$$C^{\alpha}(\mathbb{R}) = \frac{d}{dx}C^{\alpha+1}(\mathbb{R}).$$

Then $C^{k}(\mathbb{R})$ for $k\in\mathbb{N}$ coincides with its usual definition, and $C^{-k}(\mathbb{R})$ is the space of distributions expressible as the k-th derivative of a continuous function. We shall also define

$$C^{<\alpha}(\mathbb{R}) = \bigcap_{\beta<\alpha} C^{\beta}(\mathbb{R}),$$

$$C^{>\alpha}(\mathbb{R}) = \bigcup_{\beta>\alpha} C^{\beta}(\mathbb{R}),$$

so that for $\alpha<\beta<\gamma$,

$$C^{\gamma}(\mathbb{R})\subset C^{>\beta}(\mathbb{R})\subset C^{\beta}(\mathbb{R})\subset C^{<\beta}(\mathbb{R})\subset C^{\alpha}(\mathbb{R}).$$

Theorem 6 *([8]) For an automorphic distribution $\tau\in(V_{\lambda,\delta}^{-\infty})^{\Gamma}$,*

1. *$\tau\in C^{\lambda-1}(\mathbb{R})$ if τ is noncuspidal;*
2. *$\tau\in C^{\frac{Re\ \lambda-1}{2}}(\mathbb{R})$ if τ is cuspidal, λ not an odd integer;*
3. *$\tau\in C^{<\frac{\lambda-1}{2}}(\mathbb{R})$ if τ is cuspidal, λ an odd integer.*

From this we conclude that only automorphic distributions corresponding to modular forms of weight 1 or Maass forms have continuous antiderivatives. From now on we only consider such distributions, so that $\lambda \in \imath\mathbb{R}$ or $-1 < \lambda < 1$. For an automorphic distribution τ which can be expressed as the antiderivative of a continuous function, we can choose a unique antiderivative ϕ_τ such that

$$\phi_\tau \in C^0(\mathbb{R}/N\mathbb{Z}), \int_0^N \phi_\tau(x)dx = 0. \tag{11}$$

In other words, we choose ϕ_τ so that its Fourier series has no constant term.

Then
$$\tau(x) = c_{\tau,0} + \phi'_\tau(x),$$
and for any $\gamma \in SL(2,\mathbb{Z})$, we can analogously define $\phi_{\gamma\tau}(x)$ by

$$\pi_{\gamma,\delta}(\gamma)\tau(x) = c_{\gamma\tau,0} + \phi'_{\gamma\tau}(x),$$

where $\phi_{\gamma\tau}(x)$ is chosen as in (11). For $k \geq 1$, we denote the k-th antiderivative of $\phi_{\gamma\tau}$ by $\phi_{\gamma\tau}^{(-k)} \in C^k(\mathbb{R}/N\mathbb{R})$, again with the antiderivatives chosen so that they have no constant term.

Fix a rational number p/q, with p and q relatively prime. We can choose $r, s \in \mathbb{Z}$ so that $pr - qs = 1$. Then we let
$$\gamma = \begin{pmatrix} r & -s \\ -q & p \end{pmatrix} \in SL(2,\mathbb{Z}).$$

Note that γ maps the point p/q to ∞. Then for such a γ, we have the following equation:

$$\begin{gathered}\phi_\tau(x) - \phi_\tau(p/q) = \frac{c_{\tau,0}}{q}(p - qx) - \frac{c_{\gamma\tau,0}}{\lambda q}(\operatorname{sgn}(p-qx))^{\delta+1}|p-qx|^\lambda + \\ + \sum_{k=0}^n q^k (\operatorname{sgn}(p-qx))^{\delta+k}\Big(\prod_{1\leq j\leq k}(\lambda+j)\Big)\phi_{\gamma\tau}^{-k}(\gamma x)|p-qx|^{\lambda+k+1} - \\ -q^{n+1}(\operatorname{sgn}(p-qx))^{\delta+n}\Big(\prod_{1\leq j\leq n}(\lambda+j+1)\Big)\times \\ \times \int_{\operatorname{sgn}(p-qx)\gamma x}^{+\infty} (qt + r\operatorname{sgn}(p-qx))^{-\lambda-n-2}\phi_{\gamma\tau}^{(-n)}(\operatorname{sgn}(p-qx)t)dt\end{gathered} \tag{12}$$

Note that

$$\gamma x = -\frac{r}{q} + \frac{1}{q(p-qx)}, \tag{13}$$

so that as t ranges from $\operatorname{sgn}(p-qx)\gamma x$ to ∞, $qt + r\operatorname{sgn}(p-qx)$ ranges from $\frac{1}{|p-qx|}$ to ∞, and the integral in the final term converges.

The above equation is really just an integrated version of the equation

$$\tau(x) = (\operatorname{sgn}(p-qx))^\delta |p-qx|^{\lambda-1}(\pi_{\lambda,\delta}(\gamma))\tau(\gamma x).$$

Thus to see that our equation holds, we must check that both sides agree at $x = p/q$, which is clear, and then take derivatives. The derivative of the left hand side is $\phi'_\tau(x)$. The derivative of the right-hand side is

$$-c_{\tau,0} + c_{\gamma\tau,0}(\mathrm{sgn}(p-qx))^\delta |p-qx|^{\lambda-1} +$$

$$+\sum_{k=0}^{n} q^{k+1}(\mathrm{sgn}(p-qx))^{\delta+k+1}(\prod_{1\le j\le k+1}(\lambda+j))\phi_{\gamma\tau}^{-k}(\gamma x)|p-qx|^{\lambda+k} +$$

$$+\sum_{k=0}^{n} q^{k}(\mathrm{sgn}(p-qx))^{\delta+k}(\prod_{1\le j\le k}(\lambda+j))\phi_{\gamma\tau}^{-k+1}(\gamma x)|p-qx|^{\lambda+k-1} -$$

$$-q^{n+1}\mathrm{sgn}(p-qx)^{\delta+n}(\prod_{1\le j\le n}(\lambda+j+1))\times$$

$$\times|p-qx|^{-2}|p-qx|^{\lambda+n+2}\phi_{\gamma\tau}^{(-n)}(\gamma x),$$

which simplifies to

$$-c_{\tau,0} + c_{\gamma\tau,0}(\mathrm{sgn}(p-qx))^\delta |p-qx|^{\lambda-1} + (\mathrm{sgn}(p-qx))^\delta |p-qx|^{\lambda-1}\phi'_{\gamma\tau}(\gamma x),$$

so that taking derivatives of both sides leaves

$$\tau(x) = (\mathrm{sgn}(p-qx))^\delta |p-qx|^{\lambda-1}(\pi_{\lambda,\delta}(\gamma))\tau(\gamma x),$$

which is an identity, as we can see by using equation (6).

Because $\phi_{\gamma\tau}$ can be written as a Fourier series without a constant term, we find that the integral in (12) is $O(|p-qx|^{\mathrm{Re}\ \lambda+n+2})$. More precisely, we have that by integration by parts

$$\int_{\mathrm{sgn}(p-qx)\gamma x}^{+\infty}(qt + r\mathrm{sgn}(p-qx))^{-\lambda-n-2}e(nx/n)dt =$$

$$-\frac{N}{2\pi\imath n}e(n\mathrm{sgn}(p-qx)\gamma x/N)(qt+r\mathrm{sgn}(p-qx))^{-\lambda-n-2} +$$

$$+q(\lambda+n+2)\frac{N}{2\pi\imath n}\int_{\mathrm{sgn}(p-qx)\gamma x}^{+\infty}(qt+r\mathrm{sgn}(p-qx))^{-\lambda-n-3}e(nx/N)dt.$$

The second term is bounded by

$$\left|q(\lambda+n+2)\frac{N}{2\pi n}\int_{\mathrm{sgn}(p-qx)\gamma x}^{+\infty}(qt+r\mathrm{sgn}(p-qx))^{-\lambda-n-3}\right| \le (\lambda+n+2)\frac{N}{2\pi n}|p-qx|^{\mathrm{Re}\ \lambda+n+2}.$$

Thus, we have

$$|\phi_\tau(x) - \phi_\tau(p/q) - \frac{c_{\tau,0}}{q}(p-qx) + \frac{c_{\gamma\tau,0}}{\lambda q}(\mathrm{sgn}(p-qx))^{\delta+1}|p-qx|^\lambda \tag{14}$$

$$-\sum_{k=0}^{n} q^k(\prod_{1\le j\le k}(\lambda+j))\phi_{\gamma\tau}^{-k}(\gamma x)|p-qx|^{\lambda+k+1}| \le Cq^{n+1}|p-qx|^{\mathrm{Re}\ \lambda+n+2},$$

where C depends on N, n, λ and the maximum absolute value of the $\phi_{\gamma\tau}^{(-n)}$ as γ ranges over $SL(2,\mathbb{Z})$, but not otherwise on τ, Γ, p or q. This equation is expresses an *asympotic expansion* for $\phi_\tau(x)$ as $x \to p/q$. For $n = 0$, we have

$$\left|\phi_\tau(x) - \phi_\tau(p/q) - \frac{c_{\tau,0}}{q}(p-qx) + \frac{c_{\gamma\tau,0}}{\lambda q}(\operatorname{sgn}(p-qx))^{\delta+1}|p-qx|^\lambda - |p-qx|^{\lambda+1}\phi_{\gamma\tau}(\gamma x)\right|$$

$$\leq Cq|p-qx|^{\mathrm{Re}\ \lambda+2}. \tag{15}$$

We shall use this formula to determine the points where the graph of $\phi_\tau(x)$ has tangent lines.

4 Rational points

We now state our main result:

Theorem 7 *In the notation of (3), a tangent line exists at the rational point p/q only in the following cases:*

1. $c_{\gamma\tau,0} \neq 0$
2. $c_{\gamma\tau,0} = 0$, *but* $c_{\tau,0} \neq 0$
3. ϕ_τ *is a constant multiple of a real function*

In the first case, the tangent will be in the direction of the complex number $c_{\gamma\tau,0}$; in the second in the direction of $c_{\tau,0}$. When Re $\lambda \leq 0$, we are in the cuspidal case, so that $c_{\gamma\tau,0} = c_{\tau,0} = 0$, and tangent lines do not exist. There are no tangent lines at irrational points.

We will first consider the question of whether tangents exist at rational points and separate our analysis into the case when Re $\lambda \leq 0$ and two cases when $\lambda > 0$.
Case 1: Re $\lambda \leq 0$

In this case, τ is cuspidal, so we have $c_{\tau,0} = 0$ and $c_{\gamma\tau,0} = 0$. Thus our asymptotic equation becomes

$$|\phi_\tau(x) - \phi_\tau(p/q) - |p-qx|^{\lambda+1}\phi_{\gamma\tau}(\gamma x)| \leq Cq|p-qx|^{\mathrm{Re}\ \lambda+2}.$$

We wish to determine whether the limit

$$\lim_{x\to p/q} \frac{\mathrm{Im}\ \phi_\tau(x) - \phi_\tau(p/q)}{\mathrm{Re}\ \phi_\tau(x) - \phi_\tau(p/q)}$$

exists and the value of the limit when it does.

It is clear that as $x \to p/q$, the error term $Cq|p-qx|^{-\lambda+2}$ becomes negligible. Thus, the limit we seek is

$$\lim_{x\to p/q} \frac{\mathrm{Im}\ |p-qx|^{\lambda+1}\phi_{\gamma\tau}(\gamma x)}{\mathrm{Re}\ |p-qx|^{\lambda+1}\phi_{\gamma\tau}(\gamma x)}.$$

When $\lambda \in \mathbb{R}$, this simply becomes

$$\lim_{x\to p/q} \frac{\mathrm{Im}\ \phi_{\gamma\tau}(\gamma x)}{\mathrm{Re}\ \phi_{\gamma\tau}(\gamma x)}.$$

In this case, note that because $\phi_{\gamma\tau}(x)$ is periodic with period N and $\gamma x = -\frac{r}{q} + \frac{1}{q(p-qx)}$ (from (13)), so as $x \to p/q$, $\phi_{\gamma\tau}(\gamma x)$ assumes all values of $\phi_{\gamma\tau}(x)$ infinitely often. Thus, unless $\phi_{\gamma\tau}(x)$ moves along a line, i.e., unless it is a constant multiple of a real function, a tangent does not exist to the graph of $\phi_\tau(x)$ at the point $\phi_\tau(p/q)$. Note that $\phi_{\gamma\tau}(x)$ is a constant multiple of a real function if and only if the same is true of $\phi_\tau(x)$. For $\lambda = 0$ and $\delta = 1$–the case of the boundary distribution of a modular form of weight 1–there are no τ with $\phi_\tau(x)$ a constant multiple of a real function, so that there is no tangent line at any rational point.

When λ is imaginary, we have that $\phi_{\gamma\tau}(\gamma x)$ assumes all values of $\phi_{\gamma\tau}$ with approximate spacing $N(p-qx)^2$, while $|p-qx|^{\lambda+1}$ goes through an entire phase over intervals of length on the order of $2\pi|x-p/q|(\text{Im }\lambda)^{-1}$. Thus as $x \to p/q$,

$$\frac{\text{Im }|p-qx|^{\lambda+1}\phi_{\gamma\tau}(\gamma x)}{\text{Re }|p-qx|^{\lambda+1}\phi_{\gamma\tau}(\gamma x)}$$

takes on all possible values, definitely ruling out the possibility of a tangent line at p/q.

Case 2: $\lambda \geq 0$

Here we have that

$$\lim_{x\to p/q} \frac{\text{Im }\phi_\tau(x) - \phi_\tau(p/q)}{\text{Re }\phi_\tau(x) - \phi_\tau(p/q)} =$$

$$= \lim_{x\to p/q} \frac{\text{Re }\frac{c_{\tau,0}}{q}(p-qx) + \frac{c_{\gamma\tau,0}}{\lambda q}(\text{sgn}(p-qx))^{\delta+1}|p-qx|^\lambda - |p-qx|^{\lambda+1}\phi_{\gamma\tau}(\gamma x)}{\text{Im }\frac{c_{\tau,0}}{q}(p-qx) + \frac{c_{\gamma\tau,0}}{\lambda q}(\text{sgn}(p-qx))^{\delta+1}|p-qx|^\lambda - |p-qx|^{\lambda+1}\phi_{\gamma\tau}(\gamma x)}.$$

If $c_{\gamma\tau,0} \neq 0$, then the $|p-qx|^\lambda$ term dominates as $x \to p/q$, so that the limit quotient becomes

$$\lim_{x\to p/q} \frac{\text{Im }\frac{c_{\gamma\tau,0}}{\lambda q}|p-qx|^\lambda}{\text{Re }\frac{c_{\gamma\tau,0}}{\lambda q}|p-qx|^\lambda}.$$

λ is real, so this limit exists and is just $\frac{\text{Im }c_{\gamma\tau,0}}{\text{Re }c_{\gamma\tau,0}}$.

If $c_{\gamma\tau,0} = 0$, but $c_{\tau,0} \neq 0$, then the quotient becomes $\lim_{x\to p/q} \frac{\text{Im }\frac{c_{\tau,0}}{q}(p-qx)}{\text{Re }\frac{c_{\tau,0}}{q}(p-qx)}$, because the $(p-qx)$ term dominates. Thus in this case, too, the limit exists, and is equal to $\frac{\text{Im }c_{\gamma\tau,0}}{\text{Re }c_{\gamma\tau,0}}$. Finally, if $c_{\gamma\tau,0} = c_{\tau,0} = 0$, then we are in a situation analogous to $\lambda \leq 0$: there exists a tangent if and only if ϕ_τ equals some constant times a real function.

To summarize, we have that a tangent line exists at the point p/q only in the following conditions:

1. $c_{\gamma\tau,0} \neq 0$
2. $c_{\gamma\tau,0} = 0$, but $c_{\tau,0} \neq 0$
3. ϕ_τ is a constant multiple of a real function

The first two conditions occur only when $\lambda \geq 0$. The final one cannot occur for boundary distributions of modular forms of weight 1. Thus we see that tangent lines do not exist for curves which graph the antiderivatives of modular forms of weight 1. Figures 5, 6, and 7, show graphs

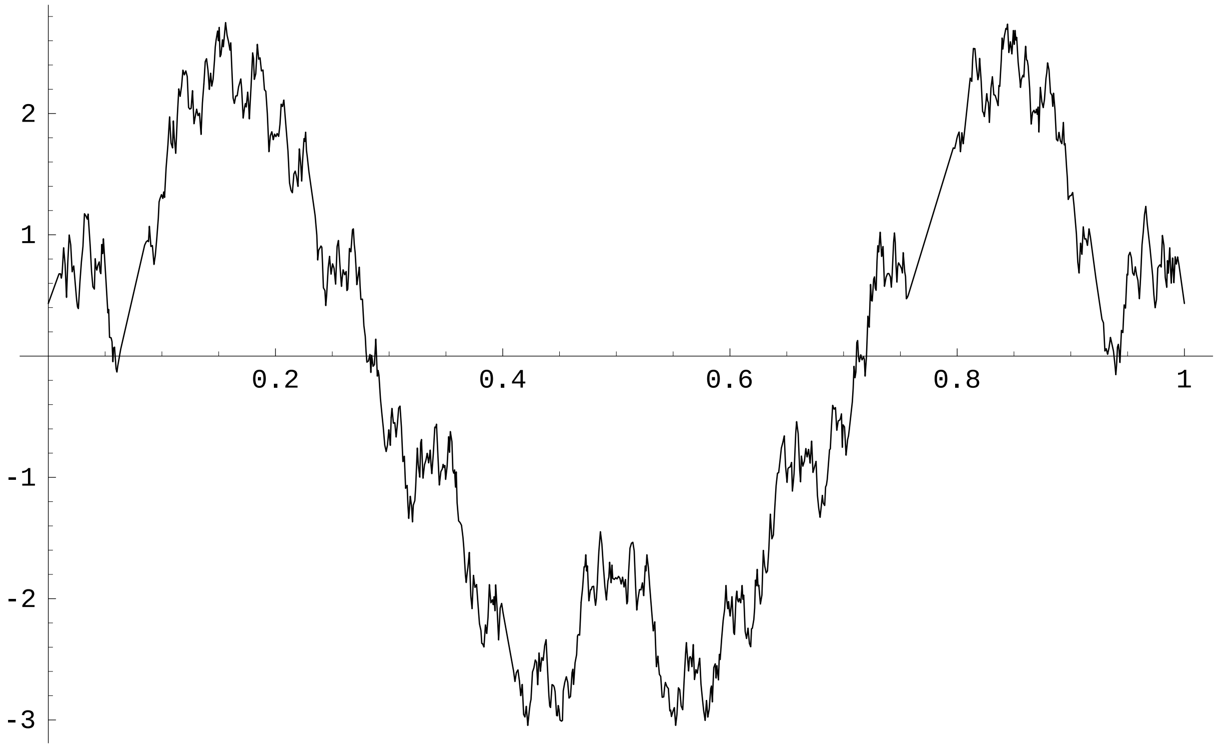

Figure 5: The real part of the antiderivative of the boundary distribution of a modular form of weight 1

of the real part, imaginary part and real part vs. imaginary part for ϕ_τ arising from the modular form

$$F(z) = \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2} \Big(e((m^2 + mn + 6n^2)z) - e((2m^2 + mn + 3n^2)z) \Big).$$

This is a cuspidal modular form of weight 1 associated to the group usually denoted $\Gamma_0(23)$.

5 Irrational points

We shall see that at irrational points tangents to the graph of ϕ_τ do not exist unless ϕ_τ is a constant multiple of a real function. This does not happen unless λ is real. Even then, if $\lambda = 0$ and $\delta = 1$, then we are in the case of boundary distributions of modular forms of weight 1, whose antiderivatives are never a constant multiple of a real function.

Given an irrational x_1, we shall show that unless ϕ_τ is real,

$$\lim_{x\to x_1} \frac{\text{Im } \phi_\tau(x) - \phi_\tau(x_1)}{\text{Re } \phi_\tau(x) - \phi_\tau(x_1)} \neq 0.$$

This is sufficient to show that no tangent exists to ϕ_τ at x_1, because if one does exist, we may multiply ϕ_τ by a constant to make the tangent horizontal.

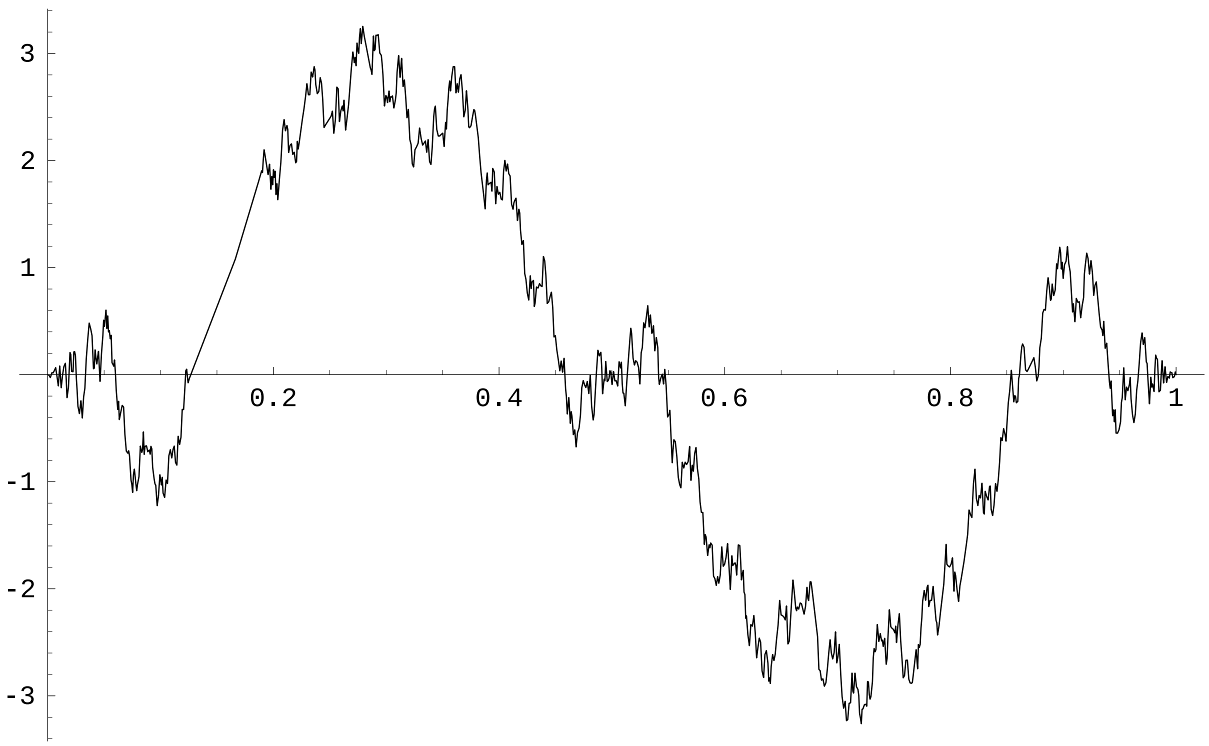

Figure 6: The imaginary part of the antiderivative of the boundary distribution of a modular form of weight 1

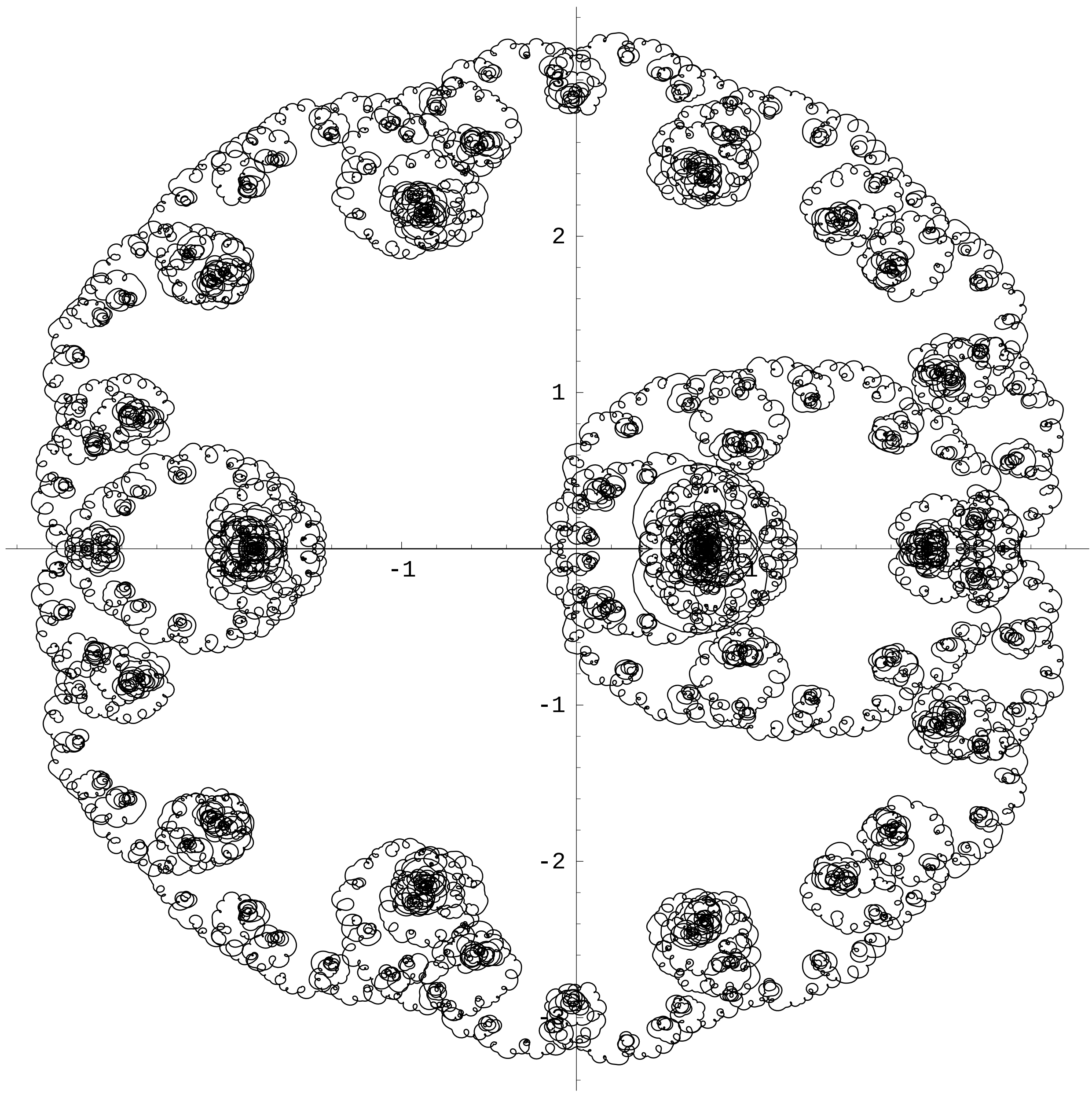

Figure 7: The real part vs. imaginary part of the antiderivative of the boundary distribution of a modular form of weight 1

Using the continued fraction expansion of x, we can find a sequence of rational numbers p/q such that

$$|p - qx_1| < q^{-1}.$$

We take such an approximating sequence. Because the rational numbers in this sequence are alternately greater than and less than x, we may take a sequence such that

$$p/q > x_1$$

for all rational numbers p/q in our approximating sequence. The argument is identical for $p/q < x_1$, so that our argument shows that

$$\lim_{x \to x_1} \frac{\text{Im } \phi_\tau(x) - \phi_\tau(x_1)}{\text{Re } \phi_\tau(x) - \phi_\tau(x_1)}$$

does not have a right-sided or left-sided limit unless ϕ_τ is a constant multiple of a real function.

Our approach will be to show that there exists some constant M, such that for rational numbers p/q close enough to x in an approximating sequence, there is an x, $x_1 < x \leq p/q$, with

$$\left| \frac{\text{Im } \phi_\tau(x) - \phi_\tau(x_1)}{\text{Re } \phi_\tau(x) - \phi_\tau(x_1)} \right| > M,$$

unless ϕ_τ is real. Usually, we will choose x "close" (in a sense which will become clear later on) to p/q, although in the last case we consider, we will need to choose x close to x_1.

Case 1: Re $\lambda \leq 0$.

Taking $p/q > x_1$, we define x_η, $0 \leq \eta \leq 1$ by

$$p - qx_\eta = \eta(p - qx).$$

Then for any x_η, $x_1 \leq x_\eta \leq p/q$, by (15),

$$\phi_\tau(x_\eta) - \phi_\tau(p/q) = \phi_{\gamma\tau}(\gamma x_\eta)|p - qx_1|^{\lambda+1}\eta^{\lambda+1} + O(q|p - qx_1|^{\text{Re } \lambda+2})\eta^{\text{Re } \lambda+2} \tag{16}$$

where the implied constant in the last term is fixed as p/q ranges through the approximating sequence. Now for all x_η, $0 \leq \eta \leq 1$ we have that

$$\text{Re } \phi_\tau(x_\eta) - \phi_\tau(x_1) < B|p - qx|^{\lambda+1}, \tag{17}$$

for some B not depending on p, q. However, we may choose a range for η, say $a < \eta < b$ in which

$$\phi_{\gamma\tau}(\gamma x_\eta)|p - qx_1|^{\lambda+1}\eta^{\lambda+1} \gg q|p - qx_1|^{\text{Re } \lambda+2}\eta^{\text{Re } \lambda+2}. \tag{18}$$

Assume for now that $\phi_{\gamma\tau}$ is not a multiple of a real function, otherwise we would be in a trivial case. Then for any θ, $0 \leq \theta \leq 1$, we have that $e(\theta)\phi_{\gamma\tau}(x)$ has some point x where its imaginary part achieves some maximal positive value $A_\theta > 0$ (we can always make the imaginary part positive because the integral of $\phi_{\gamma\tau}(x)$ over one cycle is 0, and $\phi_{\gamma\tau}(x)$ is not a multiple of a real function by assumption). Let $A' = \min_\theta A_\theta > 0$, which is positive by compactness. Thus for any θ, there exists some x_θ such that Im $e(\theta)\phi_{\gamma\tau}(x_\theta) > A'$. Moreover, because of the boundedness of the curve given by $\phi_{\gamma\tau}(x)$, there is some ϵ such that if $|\theta' - \theta| < \epsilon$, then Im $e(\theta')\phi_{\gamma\tau}(x_\theta) > A$ for a possibly smaller, but still positive, A. Because there are finitely many translates $\phi_{\gamma\tau}(x)$, we may assume that this A suffices for all the $\phi_{\gamma\tau}(x)$ as γ ranges over $SL_2(\mathbb{Z})$.

Then if Im $\lambda \neq 0$, we may choose a, b, so that a/b is small and $(b-a)/ab$ is large. This will require that $a, b \ll 1$. We have that

$$\gamma x_a - \gamma x_b = \frac{b-a}{ab(p-qx)q} \geq \frac{b-a}{ab}.$$

Thus, by making $(b-a)/ab$ large enough, we can force $\phi_{\gamma\tau}(\gamma x_\eta)$ to undergo several cycles, while the argument of $|p-qx_1|^{\lambda+1}\eta^{\lambda+1}$ varies little (say, less than ϵ), if we keep

$$\frac{\lambda}{2\pi} \log \frac{a}{b} < \epsilon.$$

Then in the range $a \leq \eta \leq b$, we have some η for which

$$\text{Im } \phi_{\gamma\tau}(\gamma x_\eta)|p-qx_1|^{\lambda+1}\eta^{\lambda+1} \geq A|p-qx_1|^{\text{Re } \lambda+1}a^{\text{Re } \lambda+1}.$$

Adjusting for the $O(q|p-qx_1|^{\text{Re } \lambda+2})$ term using (18), we get

$$|\text{Im } \phi_\tau(x_\eta) - \phi_\tau(p/q)| > \frac{1}{2}A|p-qx_1|^{\text{Re } \lambda+1}a^{\text{Re } \lambda+1}.$$

But

$$|\text{Re } \phi_\tau(x_\eta) - \phi_\tau(x_1)| < B|p-qx_1|^{\text{Re } \lambda+1},$$
$$|\text{Re } \phi_\tau(p/q) - \phi_\tau(x_1)| < B|p-qx_1|^{\text{Re } \lambda+1}.$$

Thus for either $x = p/q$ or $x = x_\eta$, we have that

$$\left|\frac{\text{Im } \phi_\tau(x) - \phi_\tau(x_1)}{\text{Re } \phi_\tau(x) - \phi_\tau(x_1)}\right| > \frac{Aa^{\text{Re } \lambda+1}}{4B}. \tag{19}$$

This is true for all p/q in our approximating sequence, so there are infinitely many values of x approaching x_1 that prevent ϕ_τ from having a horizontal tangent at $\phi_\tau(x_1)$. Because $a, b \ll 1$, x_η is "close" to p/q. The argument in the case where Im $\lambda = 0$ is even simpler: we merely take $A = \min_\gamma \max_x \text{Im } \phi_{\gamma\tau}(x)$, and choose a, b so that between x_η passes through at least one cycle of $\phi_{\gamma\tau}(x)$.

Now we turn to the case $\lambda > 0$. Here we have that $c_{\tau,0}, c_{\gamma\tau,0}$ may not be zero. By going to a subsequence of rationals p/q we can arrange that either

1. for all the $c_{\gamma\tau,0}$, we have Im $c_{\gamma\tau,0} > I$ or

2. all the $c_{\gamma\tau,0}$ are real (where we allow $c_{\gamma\tau,0} = 0$ as well).

This is possible because there are only finitely many values for the $c_{\gamma\tau,0}$. We shall now examine these two cases in turn.

Case 2: $\lambda > 0$ and in the approximating sequence for x_1, we have that for all the $c_{\gamma\tau,0}$, Im $c_{\gamma\tau,0} > I$.

This case can be dealt with in a manner similar to the first case. Define x_η as before. Using

$$\phi_\tau(x_\eta) - \phi_\tau(p/q) = \frac{c_{\tau,0}}{q}(p-qx_1)\eta - \frac{c_{\gamma\tau,0}}{\lambda q}|p-qx_1|^\lambda\eta^\lambda + \tag{20}$$

$$+\phi_{\gamma\tau}(\gamma x)|p-qx_1|^{\lambda+1}\eta^{\lambda+1} + O(q|p-qx_1|^{\lambda+2})\eta^\lambda$$

we see that for all $0 \leq \eta \leq 1$, we have that

$$|\text{Re } \phi_\tau(x_\eta) - \phi_\tau(x_1)| \leq Bq^{-1}|p - qx|^\lambda.$$

However, for η small enough, we can force

$$\text{Im } \frac{c_{\gamma\tau,0}}{\lambda q}|p - qx_1|^\lambda \eta^\lambda > 2(|\frac{c_{\tau,0}}{q}(p - qx_1)\eta| + |\phi_{\gamma\tau}(\gamma x)|p - qx_1|^{\lambda+1}\eta^{\lambda+1}| + Cq|p - qx_1|^{\lambda+2})\eta^\lambda), \quad (21)$$

because Im $c_{\gamma\tau,0} > I$ for all γ as $p/q \to x$ in our sequence. Combining (21) and (22), the result is that

$$\text{Im } \phi_\tau(x_\eta) - \phi_\tau(p/q) > \frac{I}{2\lambda q}|p - qx_1|^\lambda \eta^\lambda.$$

Thus for either $x = p/q$ or $x = x_\eta$, we have in analogy to (19) that

$$\left| \frac{\text{Im } \phi_\tau(x) - \phi_\tau(x_1)}{\text{Re } \phi_\tau(x) - \phi_\tau(x_1)} \right| > \frac{I\eta^{\text{Re } \lambda}}{4B\lambda}. \quad (22)$$

Case 3: $\lambda > 0$ and all the $c_{\gamma\tau,0}$ are real.

Again passing to a subsequence, we can arrange that either

1. $p(q - px_1) > D$ for some fixed constant D in the approximating sequence or
2. $p(q - px_1)$ approaches zero as p/q approaches x_1.

The first case is simple to deal with. In this case, all the terms in (20) are bounded by some multiple of $|p - qx_1|^{\lambda+1}$. By choosing η as before, we may arrange that the $\phi_{\gamma\tau}(\gamma x)|p - qx_1|^{\lambda+1}\eta^{\lambda+1}$ term dominates the $O(q|p - qx_1|^{\lambda+2})\eta^\lambda$ term. This, in addition to $c_{\gamma\tau,0}$ all being real, will ensure that $|\text{Im } \phi_\tau(x_\eta) - \phi_\tau(p/q)|$ is larger than some multiple of $|p - qx_1|^{\lambda+1}$, while the real part is bounded by some multiple of $|p - qx_1|^{\lambda+1}$. Then we can again find a sequence values for x approaching x_1 with $\frac{\text{Im } \phi_\tau(x) - \phi_\tau(x_1)}{\text{Re } \phi_\tau(x) - \phi_\tau(x_1)}$ bounded away from 0 as in case 1.

In the later case, we take the equation

$$\phi_\tau(x) - \phi_\tau(p/q) = \frac{c_{\tau,0}}{q}(p - qx) - \frac{c_{\gamma\tau,0}}{\lambda q}|p - qx|^\lambda +$$

$$+\phi_{\gamma\tau}(\gamma x)|p - qx|^{\lambda+1} + O(q|p - qx_1|^{\lambda+2})\eta^\lambda$$

and take the difference for the two values x_1 and x. This yields

$$\phi_\tau(x) - \phi_\tau(x_1) = c_{\tau,0}(x_1 - x) - \frac{c_{\gamma\tau,0}}{\lambda q}(|p - qx|^\lambda - |p - qx_1|^\lambda) + \quad (23)$$

$$+\phi_{\gamma\tau}(\gamma x)|p - qx|^{\lambda+1} - \phi_{\gamma\tau}(\gamma x_1)|p - qx_1|^{\lambda+1} + O(q|p - qx_1|^{\lambda+2}).$$

Next, we will restrict x to a range over which $\phi_\tau(\gamma x)$ undergoes a complete cycle. If ϕ_τ has period N, we take x' such that

$$\gamma x' - \gamma x_1 = \frac{1}{q(p - qx')} - \frac{1}{p - qx_1} = \frac{x' - x_1}{(p - qx')(p - qx_1)} = N. \quad (24)$$

From the last inequality, we see that for $x_1 < x < x'$, $x - x_1 < N(p-qx)^2$. Thus we can bound the first term of (23) $c_{\tau,0}(x_1 - x)$ by $A(p-qx)^2$ for some constant A. The second term can be rewritten as

$$\frac{c_{\gamma\tau,0}}{\lambda q}(|p-qx|^\lambda - |p-qx_1|^\lambda) = c_{\gamma\tau,0}\int_{x_1}^{x}(p-qt)^{\lambda-1}dt \le C(p-qx_1)^{\lambda+1},$$

for some C, because $x - x_1 < N(p-qx)^2$ and $p - qt < p - qx_1$ on the integral of integration. The final term $O(q|p-qx_1|^{\lambda+2})$ of (23) will be much smaller than $(p-qx_1)^{\lambda+1}$ for p/q close to x, because, by assumption, $q(p-qx_1) \to 0$. Thus we bound the real part by a multiple of $|p-qx_1|^{\lambda+1}$.

Then we are left with the middle terms

$$\phi_{\gamma\tau}(\gamma x)|p-qx|^{\lambda+1} - \phi_{\gamma\tau}(\gamma x_1)|p-qx_1|^{\lambda+1}.$$

Note that $\frac{x-x_1}{p/q-x} < Nq(p-qx)$ by the last equality in (24). Thus for p/q close to x, $\frac{|p-qx|^{\lambda+1}}{|p-qx_1|^{\lambda+1}}$ can be made arbitrarily close to 1. If ϕ_τ is not a multiple of a real function, by choosing x appropriately in the interval $[x_1, x']$, we ensure that the imaginary part of $\phi_{\gamma\tau}(\gamma x)|p-qx|^{\lambda+1} - \phi_{\gamma\tau}(\gamma x_1)|p-qx_1|^{\lambda+1}$ is at least some multiple of $(p-qx_1)^{\lambda+1}$. Then we may proceed with the argument as in the other cases. Note that this is the one case where we choose x "close" to x_1.

6 Modular Forms of Weight $\frac{1}{2}$ and the Weierstrass Function

Modular forms of weight $\frac{1}{2}$ arise not from representations of the group $G = SL_2(\mathbb{R})$, but from representations of its twofold covering group $\tilde{G} \to G$, also known as the *metaplectic cover.* The principle series representations of $\tilde{G}$ are parameterized by pairs (λ, δ), where $\lambda \in \mathbb{C}$, $\delta \in \mathbb{Z}/4\mathbb{Z}$. When $\delta = \pm 1$, we get "genuine" representations of $\tilde{G}$, whereas for $\delta = 0, 2$, we merely get liftings of representations $(\lambda, \delta/2)$ of G. We can then consider the space of $\tilde{\Gamma}$-automorphic distribution vectors $(V_{\lambda,\delta}^{-\infty})^{\tilde{\Gamma}}$ for subgroups $\tilde{\Gamma} \in \tilde{G}$ which project down to subgroups Γ of finite index in $SL_2(\mathbb{Z})$. This space corresponds bijectively to the space of (not necessarily cuspidal) modular forms of weight $\frac{1}{2}$ when $(\lambda, \delta) = (\frac{1}{2}, \pm 1)$. The boundary value on the real line of these modular forms is a realization of the automorphic distribution $\tau \in (V_{\frac{1}{2},\pm 1}^{-\infty})^{\tilde{\Gamma}}$ as a distribution on the real line. τ, realized as a distribution on the real line, inherits automorphy from its invariance as a vector in $V_{\lambda,\delta}^{-\infty}$ under $\tilde{\Gamma}$:

$$\tau(x) = \chi|cx+d|^{\lambda-1}\tau(\frac{ax+b}{cx+d}),$$

$\chi \in \{\pm 1, \pm\imath\}$ depending on λ, δ and $\mathrm{sgn}(cx+d)$. We again require that the $SL_2(\mathbb{R})$ translates of τ agree with their natural extensions across ∞. The arguments of the previous two sections can be adapted with little change to determine the points where tangents to the antiderivatives of these distributions exist.

The eta function is constructed from the discriminant function,

$$\Delta(z) = g_2^3 - 27g_3^2,$$

where

$$g_2 = 60G_4 = 60\sum_{(m,n)\neq(0,0)}(mz+n)^{-4},$$

$$g_3 = 140G_6 = 140\sum_{(m,n)\neq(0,0)}(mz+n)^{-6}.$$

$g_2(z)$ and $g_3(z)$ are the coefficients of the differential equation for the function $\wp$ associated to the lattice generated by z and 1:

$$\wp'(u)^2 = 4\wp(u)^3 - g_2\wp(u) - g_3,$$

$$\wp(u) = u^{-2} + \sum_{(m,n)\neq(0,0)} (u - (mz+n))^{-2} - (mz+n)^{-2}.$$

Then

$$\Delta = (2\pi)^{12} e(z) \prod_1^\infty (1 - e(nz))^{24}$$

is a cuspidal modular form of weight 12 for the modular group. Since Δ does not vanish on $\mathbb{H}$, we can define a holomorphic function

$$\eta(z) = (2\pi)^{-\frac{1}{2}} \Delta(z)^{\frac{1}{24}} = e(z/24) \prod_1^\infty (1 - e(nz)).$$

Then for $\gamma \in SL_2(\mathbb{Z})$, we have that

$$\eta(\gamma z) = \theta(\gamma)(cz+d)^{1/2}\eta(z)$$

for some multiplier system θ for $SL_2(\mathbb{Z})$. The function η was used by Hardy and Ramanujan to establish the asymptotic formula

$$p(n) \sim (4\sqrt{3}n)^{-1} \exp \pi\sqrt{2n/3}$$

for the partition function $p(n)$. Because η is cuspidal and not a multiple of a real function (it is holomorphic), the graph of the curve described by the antiderivative of the boundary distribution of η has tangents nowhere, because η is not a multiple of a real function.

The theta-function

$$\theta(z) = \sum_{n\in\mathbb{Z}} e^{2\pi\imath n^2 z}$$

is a modular form of weight $\frac{1}{2}$ for the group $\tilde{\Gamma}_0(4)$. It is invariant under the group $\tilde{\Gamma}_1(4)$ and transforms under $\tilde{\Gamma}_0(4)$ according to the equation

$$\theta(\gamma z) = \bar{\epsilon}_d \left(\frac{c}{d}\right) \theta(z),\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4).$$

$\epsilon_d = 1$ is $d = 1 \bmod 4$ and $\epsilon_d = \imath$ is $d = -1 \bmod 4$. $\left(\frac{c}{d}\right)$ is the Jacobi symbol for positive, odd d. For negative d, we define

$$\left(\frac{c}{d}\right) = \frac{c}{|c|}\left(\frac{c}{-d}\right)$$

and for $c = 0$, we define the Jacobi symbol to be 1 is $d = \pm 1$ and 0 otherwise. From these equations, we can derive an asymptotic expansion for the antiderivative

$$\phi_\theta = \frac{1}{2\pi\imath} \sum_{n\neq 0} n^{-2} e(n^2 x)$$

of $\theta - 1$.

Near points p/q where q is odd or $4|q$, we have ([1])

$$\phi_\theta(x) - \phi_\theta(p/q) = -(x - p/q) + \frac{2e^{\frac{\pi\imath}{4}m}}{q}(qx-p)^{1/2} + e^{\frac{\pi\imath}{4}m}(qx-p)^{3/2}\phi_\theta(\gamma x) + O(q|p - qx|^{5/2}).$$

Here m is an integer that depends on the rational number p/q, and, as before, γ sends p/q to ∞. θ is not cuspidal, so we would expect θ to have tangents. However, because of the half-integer exponents in the asymptotic expansion, the left and right limiting secants approach perpendicular lines at p/q:

$$(x - r)^{1/2} = \imath|x - r|^{1/2} \text{ if } x < r.$$

In other words, near rational points p/q where q is odd or $4|q$, the limits

$$\lim_{x \to p/q^+} \frac{\text{Im } \phi_\tau(x) - \phi_\tau(p/q)}{\text{Re } \phi_\tau(x) - \phi_\tau(p/q)},$$

$$\lim_{x \to p/q^-} \frac{\text{Im } \phi_\tau(x) - \phi_\tau(p/q)}{\text{Re } \phi_\tau(x) - \phi_\tau(p/q)}$$

both exist, but give slopes of tangent lines that are perpendicular to each other. This phenomenon does not arise in any of the antiderivatives of distributions arising from $SL_2(\mathbb{R})$; in those cases, if the either of the above left or right limits exist, they both exist and are equal. Duistermaat [1] gives the following list of values for m at rational points p/q, q odd or divisible by 4. Thus for $m = 0$, as $x \to p/q^-$, $\phi_\theta(x)$ approaches $\phi_\theta(p/q)$ from above, while for $x \to p/q^+$, $\phi_\theta(x)$ approaches $\phi_\theta(p/q)$ from the right. For other values of m, the same local behavior is replicated, only rotated by $e^{\frac{\pi\imath}{4}m}$. By convention, we take $q > 0$:

- $m = 0 \bmod 8$ if $q \in 4\mathbb{Z}$, $p \in 4\mathbb{Z} + 3$, $\left(\frac{q}{|p|}\right) = 1$
- $m = 1 \bmod 8$ if $p \in \mathbb{Z}$, $q \in 4\mathbb{Z} + 1$, $\left(\frac{p}{q}\right) = 1$
- $m = 2 \bmod 8$ if $q \in 4\mathbb{Z}$, $p \in 4\mathbb{Z} + 1$, $\left(\frac{q}{|p|}\right) = 1$
- $m = 3 \bmod 8$ if $p \in \mathbb{Z}$, $q \in 4\mathbb{Z} + 3$, $\left(\frac{p}{q}\right) = 1$
- $m = 4 \bmod 8$ if $q \in 4\mathbb{Z}$, $p \in 4\mathbb{Z} + 3$, $\left(\frac{q}{|p|}\right) = -1$
- $m = 5 \bmod 8$ if $p \in \mathbb{Z}$, $q \in 4\mathbb{Z} + 1$, $\left(\frac{p}{q}\right) = -1$
- $m = 6 \bmod 8$ if $q \in 4\mathbb{Z}$, $p \in 4\mathbb{Z} + 1$, $\left(\frac{q}{|p|}\right) = -1$
- $m = 7 \bmod 8$ if $p \in \mathbb{Z}$, $q \in 4\mathbb{Z} + 3$, $\left(\frac{p}{q}\right) = -1$

For derivations of the asymptotic expansions around rational points p/q and the calculation of the different values for m, see [1].

Near points $p/2q$ where both p and q are odd, ϕ_τ has asymptotic expansion

$$\phi_\theta(x) - \phi_\theta(p/2q) = -(x - p/2q) + O(|p - 2qx|^{3/2})$$

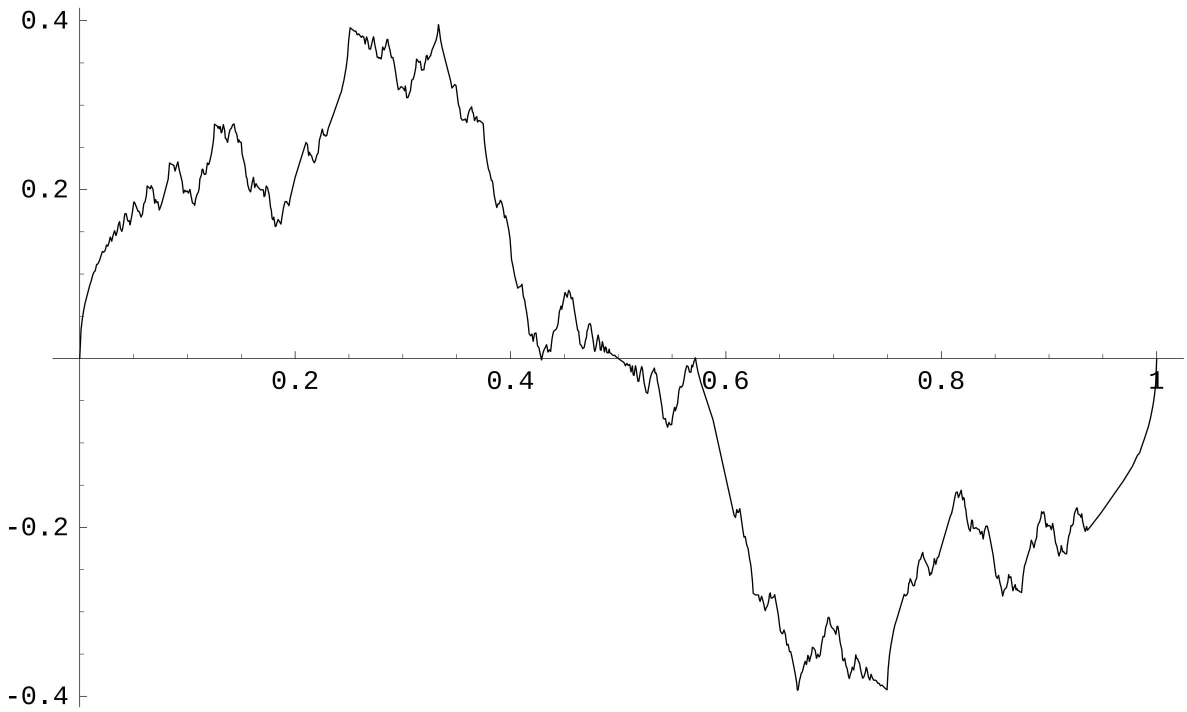

Figure 8: The real part of the antiderivative of the boundary distribution of the θ-function, also known as the Weierstrass function

and derivative equal to -1, so that ϕ_τ has a horizontal tangent at such points. Thus we find that the curve related to the Weierstrass function has tangents or one-sided tangents at rational points and nowhere else. We close with diagrams of the real, imaginary, and real vs. imaginary parts of the antiderivative of the boundary distribution of the θ-function. In Figure 10 the perpendicular tangents that exist at some rational points come across vividly.

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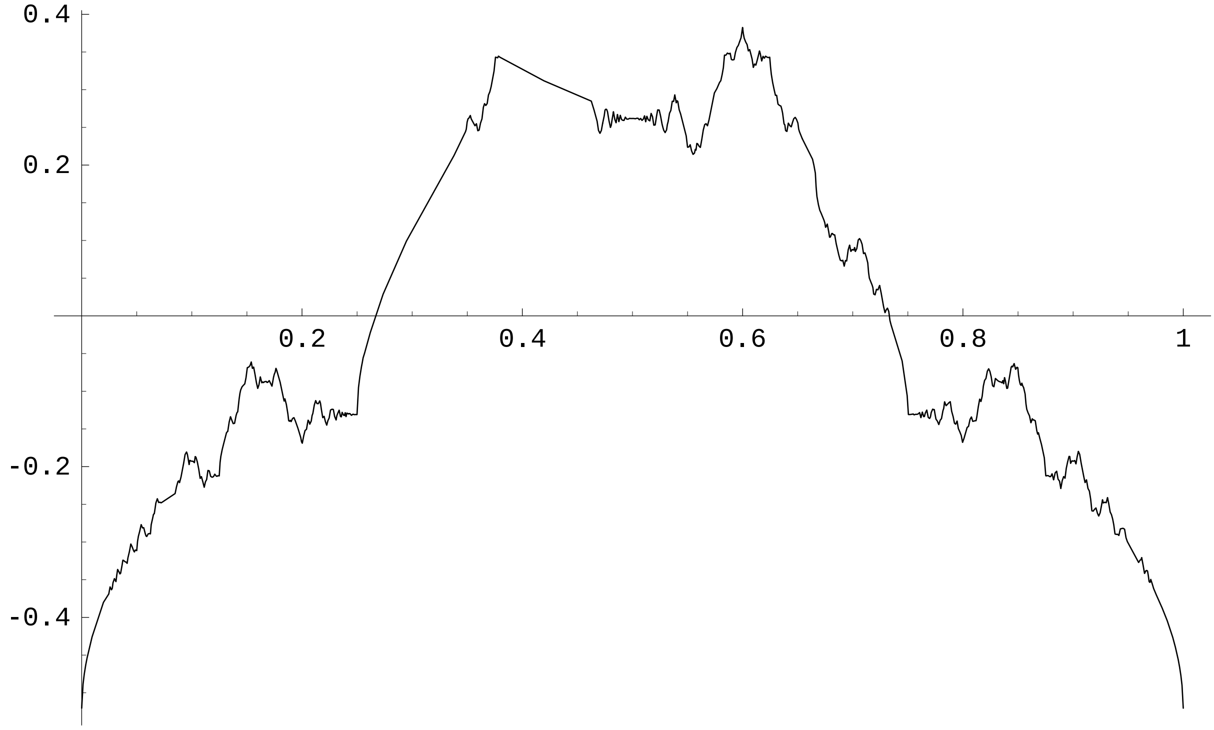

Figure 9: The imaginary part of the antiderivative of the boundary distribution of the θ-function

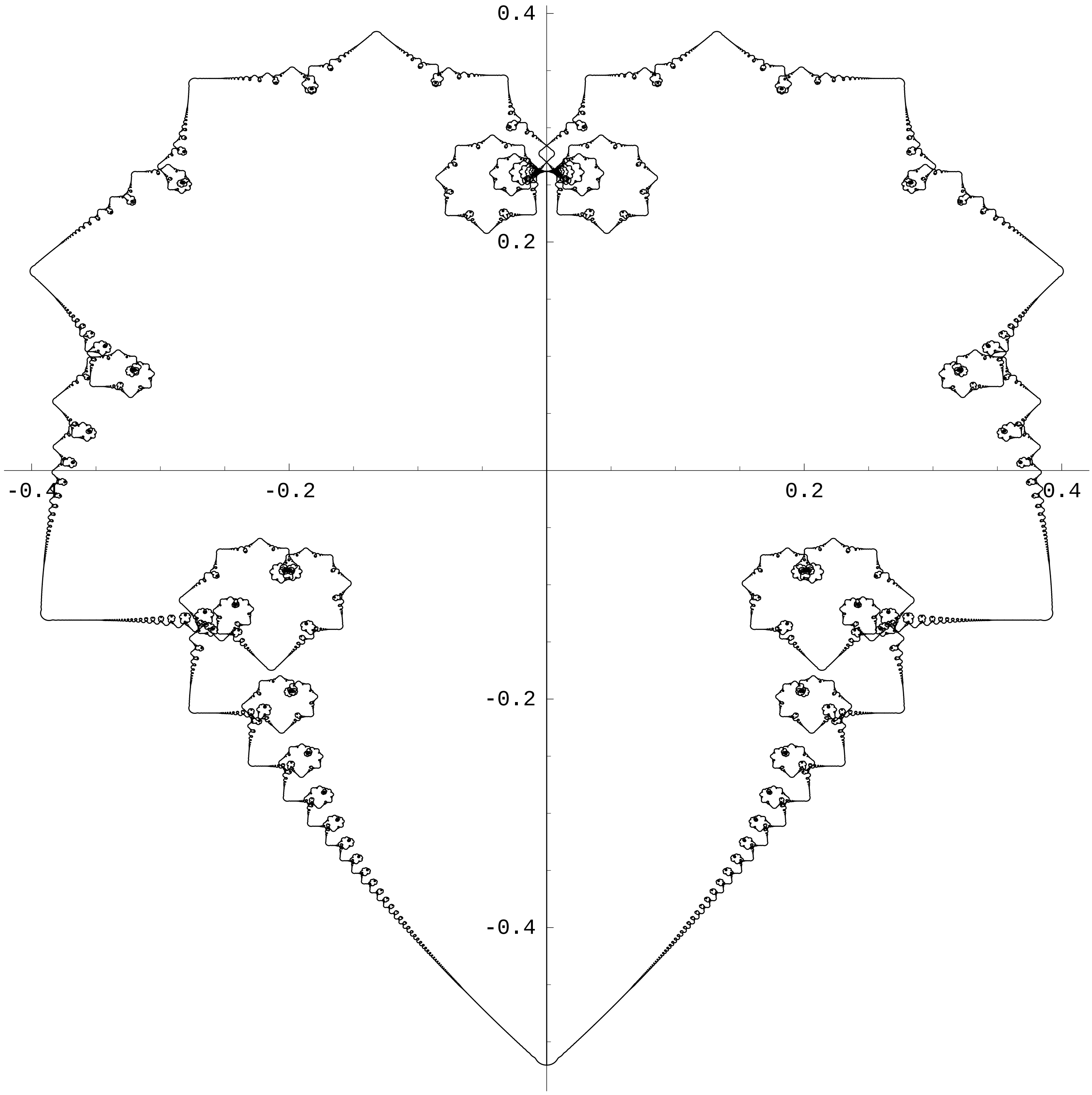

Figure 10: The real part vs. imaginary part of the antiderivative of the boundary distribution of the θ-function

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