The Stars Above Us:

Regular and Uniform Polytopes up to Four Dimensions

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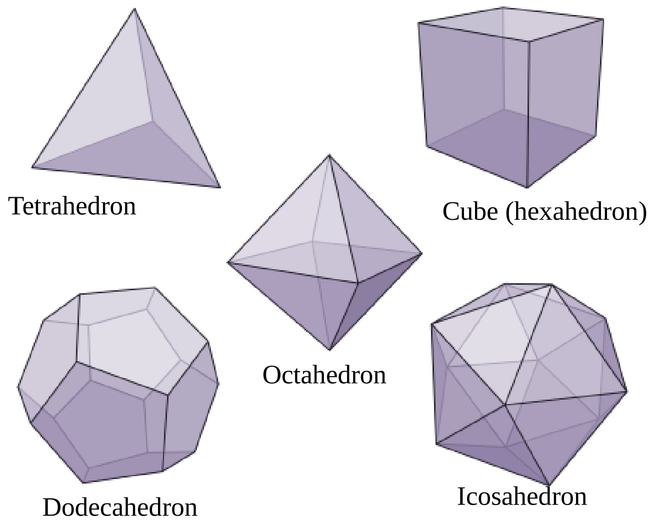
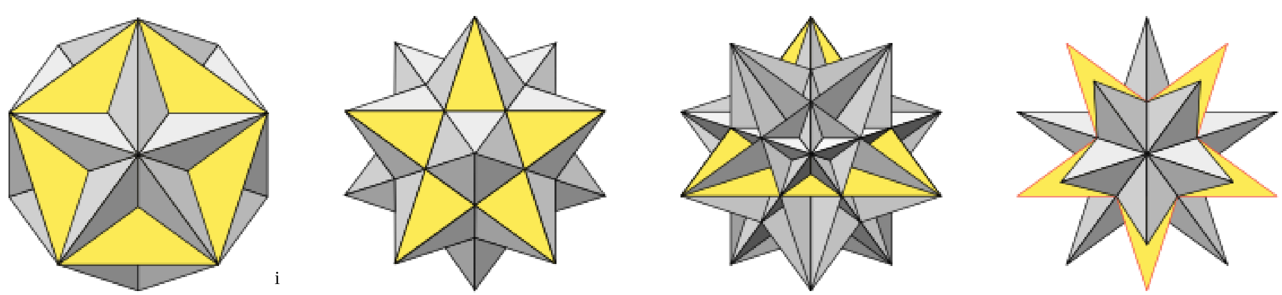
Introduction: Plato's Friends and Relations



It is a remarkable fact, known since the time of ancient Athens, that there are five Platonic solids. That is, there are precisely five polyhedra with identical edges, vertices, and faces, and no self-intersections. While will see a formal proof of this fact in part 1a, it seems strange a priori that the club should be so exclusive. In this paper, we will look at extensions of this family made by relaxing some conditions, as well as the equivalent families in numbers of dimensions other than three. For instance, suppose that we allow the sides of a shape to pass through one another. Then the following figures join the ranks:

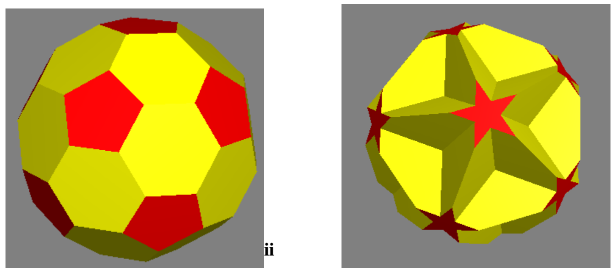


Great Dodecahedron Small Stellated Dodecahedron Great Icosahedron Great Stellated Dodecahedron

Geometer Louis Poinsot in 1809 found these four figures, two of which had been previously described by Johannes Kepler in 1619. These four are the *Regular Star Polyhedra*, and are considered along with the Platonic solids as regular polyhedra by modern geometry[1]. Informally, this means that in three dimensions, these nine figures (subject to some caveats we will see in section four) are the only ones for which each vertex is identical to every other, each edge is identical to every other, and each face is identical to every other. A more complete definition of regular will be given in the next section.

What if we relax some of the conditions? If we only require that the *vertices* are identical to one another, and that the faces remain regular polygons, we include the *Uniform Solids*, which come in both starry and non-starry varieties:

1 "Regular Polyhedron", *Mathworld*

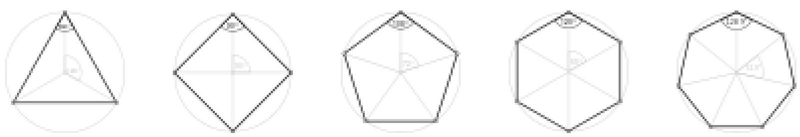


[ii]

Non-star (left) and star (right) Uniform Polyhedra

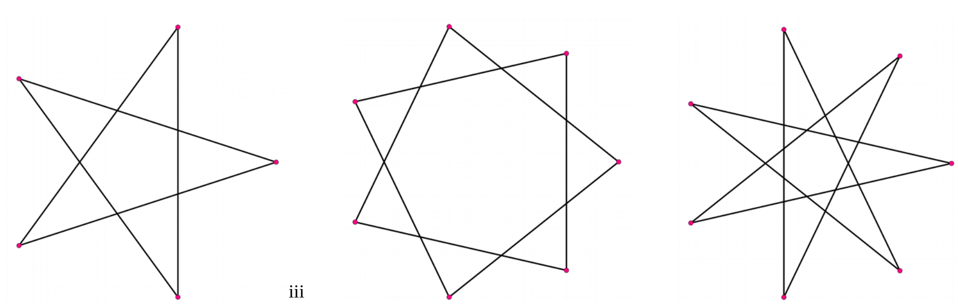
We will examine both of these categories, in sections two and six respectively. This time, in addition to the previously seen regular polyhedra, there are a couple of easily describable infinite families and a small number of exceptional cases, thirteen for the non-stars and 53 for the stars.

What about other numbers of dimensions? The situation in dimensions lower than three has been well-known for millennia, and is simple enough. A zero-dimensional point and a one-dimensional line segment are the only possible shapes in their numbers of dimensions. In two dimensions, there is a single infinite family of regular shapes without self-intersections, differing only in their number of edges:



{3}[2] {4} {5} {6} {7}

If we allow self intersections, we need get the *regular star polygons*[3]. We need just two numbers to describe each of these: one for the number of vertices, and the second for how many vertices away each edge takes us. For the star polygon to be a single closed shape, the second number must be coprime with the first, so we conveniently can make them into fractions:



[iii]

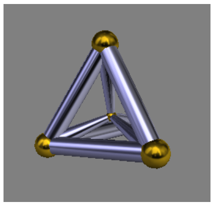
{5/2} {7/2} {7/3}

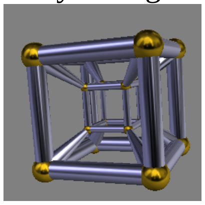
What about in dimensions *higher* than three? Thinking in four dimensions is conceptually somewhat harder since we are not used to working with four dimensional space. Time is a fourth dimension, but it acts differently to the other three since we can only move in one direction along it. It makes more mathematical sense to talk of a hypothetical fourth spatial dimension that behaves like the usual three. We can get a sense of what to expect in a hypothetical 4D world by analogy to two and three dimensions. The analogues of polygons and polyhedra are the *polychora* (singular *polychoron*), which are made of 3D *cells*. In four dimensions, instead of five Platonic solids we have six, with 5, 8,

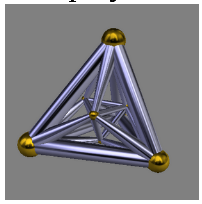
2 These numbers in braces are known as Schläfli symbols. We will only use these symbols occasionally since they apply only to regular shapes and not to their relatives.

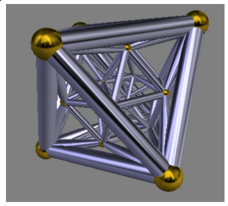
3 "Star Polygon", Mathworld.

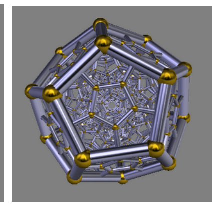
16, 24, 120, and 600 cells each[4]. The 5-cell is roughly an analogue of a tetrahedron, the 8-cell of a cube, the 16-cell of an octahedron, the 120-cell of a dodecahedron, and the 600-cell of a tetrahedron. The 24-cell does not have a regular 3d analogue, but it is similar to the uniform cuboctahedron and a closely related shape, the rhombic dodecahedron. We explore these six polychora in section eight. There are additionally 10 regular star polychora, which we will see in section ten.

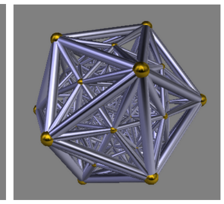

5-cell


8-cell

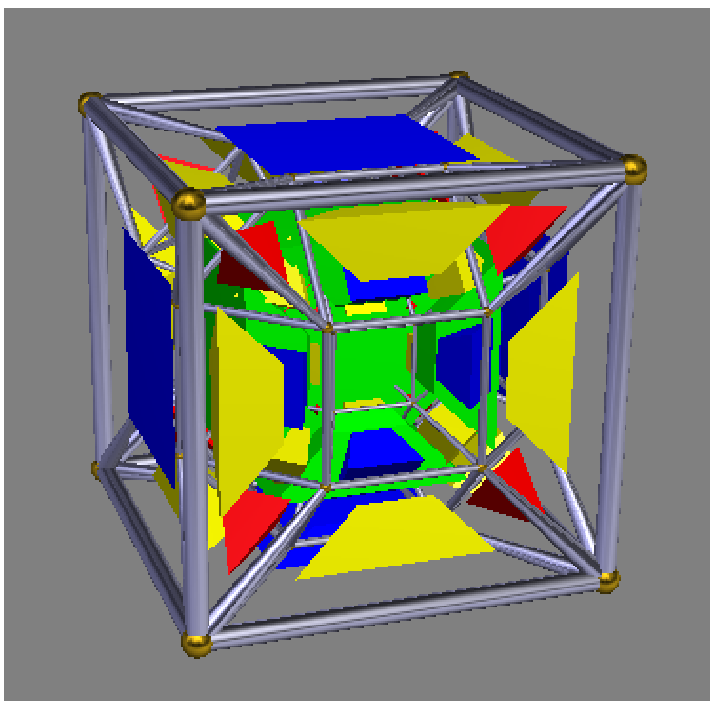

16-cell


24-cell


120-cell


600-cell

The set of 4D uniform shapes is even more expansive: in the non-self intersecting case, excluding infinite families and the six regular shapes there are 58 uniform polychora[5]. If we do allow self-intersection, the number remains unknown. By last count, there are *1849* uniform star polychora, with the most recent four being found in 2006. One of the main motivations of this paper is to understand the methods by which this enumeration might be continued or even completed. We will do this by looking at the simpler cases for which the answer is already known, and in section 11 we will examine how they are (and aren't) applicable to this unsolved case. Hopefully, this will give the reader an enticing glimpse into the world of the next dimension and the strange starry shapes that call it home.


The Mysterious Iquipadah: A Uniform Star Polychoron

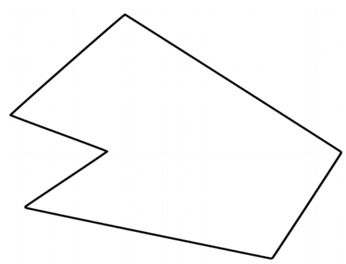
4 Conway, Burgiel, Goodman-Strauss, pp. 387
5 Möller (2004).

DEFINITIONS: What are Regular and Uniform Polytopes?[6]

The only possible shape with no dimensions is a *point*. This is (roughly) a point → .

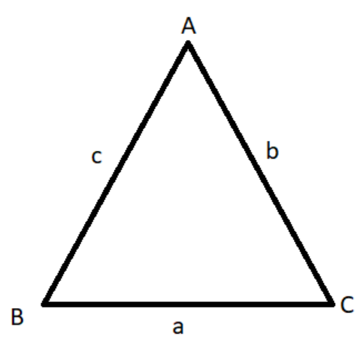
Connecting two points (*vertices*) in one dimension makes a *line segment*. Line segments only differ in their size (length). This is a line segment: ______________

A *polygon* is a shape made of line segments (*edges*) connected at their vertices in a single closed loop, meaning every vertex is on two edges, and any two vertices are connected by a path of edges.

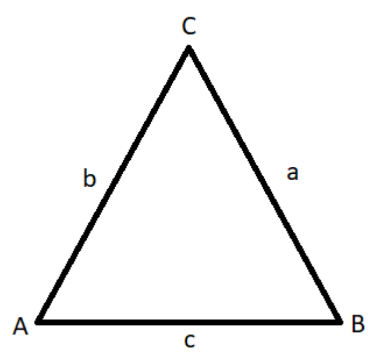


A six-sided polygon

Consider the polygon below, an equilateral triangle with vertices ABC and edges abc.



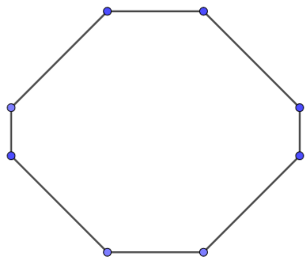
A triangle



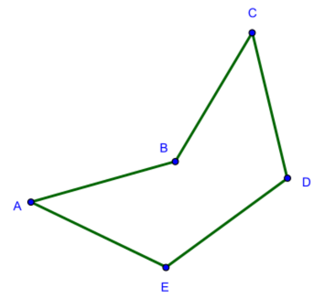
A rotated triangle

Now we rotate this triangle a third turn counterclockwise. Notice that, other than the labels, the triangle looks identical. Since A lies where B was, and C lies where A was, we can say that A, B and C are indistinguishable vertices. Since these are all of the vertices this means that our shape is *vertex-transitive*: any vertex can be moved onto any other vertex while keeping the overall triangle covering the same part of the plane. Note that this is a stricter condition than just having equal angles: the octagon below has equal angles at each vertex, but some of the vertices lie on a long and short side while some lie on a long and medium side, and these two types cannot be moved onto each other without changing how the overall shape lies. Another way to think about this is that we can move any vertex of the triangle onto any other vertex by spinning the triangle or flipping it around its lines of symmetry (these are called *rigid transformations* because we can do them to a rigid shape).

6 Versions of these definitions can be found many places in the bibliography, e.g. Olshevesky (2008).



iv

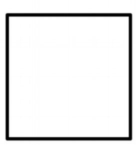


v

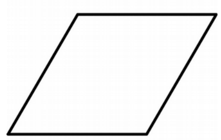
An equal-angled but not vertex-transitive octagon (left)
and an equal-sided but not edge transitive pentagon (right)

Going back to the equilateral triangle, edges a, b, and c can be moved onto one another while preserving the overall shape (excluding the labels). Thus that polygon is *edge-transitive*. If we allow rigid transformations on the triangle, any edge is indistinguishable from any other edge. Again, this is a stricter condition than having the edges be equal length: the above pentagon has equal length edges but for instance the ones that are connected to the concave angle cannot be moved onto the others without changing how the shape lies in the plane:

A polygon that is both vertex-transitive and edge-transitive is a *regular polygon*. Look at the quadrilaterals below: which are vertex-transitive? Edge-transitive? Regular?



Square



Rhombus

Dart

Rectangle

The dart and rhombus have differing angles, and are therefore not vertex-transitive. The square clearly is. The rectangle is also vertex transitive, but you may have to use reflections rather than just rotations to move one corner onto another.

Similarly, the dart and rectangle have different edge lengths, and are therefore not edge-transitive, while the square is. The rhombus is as well, but again you may have to use reflections. Since the square is vertex and edge transitive, it is regular.

A shape is *convex* if there is no line segment that starts and ends in the shape and goes outside of it. This is meant to be a more formal definition of the intuitive “doesn’t have any in-ish parts”. Of the above quadrilaterals, only the dart is not convex – it is therefore *concave*.[7]

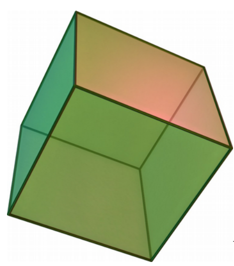
7 Because self-intersecting forms are not convex, and all non-self intersecting uniform polytopes are convex, this paper will often contrast convex uniform or regular polytopes with star polytopes, even though there are some shapes (like the dart) that are neither convex nor self-intersecting.

Moving to 3D, a *polyhedron* is a shape made of polygons (faces) connected at their edges in a single closed surface, meaning every edge is on two faces and any two vertices are connected by a path of edges.

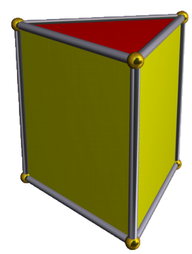

[vi]

A five-sided polyhedron

We say a polyhedron is *regular* if all of its 0d vertices are identical to each other (it is vertex-transitive), along with all of its 1ds edges (edge-transitive) and 2d faces (*face-transitive*). A cube is regular. A triangular prism is not, since some of its faces are squares while the others are equilateral triangles. A triangular bipyramid is also not regular, since some of its vertices are on three faces while some are on four faces. There are five regular polyhedra.


[vii]

Cube[8]


[viii]

Triangular prism


[ix]

Triangular bipyramid

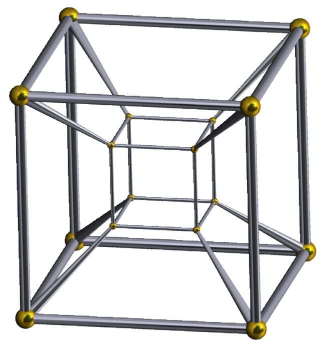
We say a polyhedron is *uniform* if all of its vertices are identical (it is vertex-transitive), whether or not its edges and faces are, and its faces are regular polygons[9]. A triangular prism is uniform: we can move any vertex to any other vertex by spinning the prism and/or flipping it over. A triangular bipyramid is still not uniform, since we can distinguish the vertices on three faces from those on four. It is, however, face-transitive: each equilateral triangle face is on one corner with three faces and two corners with four faces. All regular polyhedra are uniform. In addition to the five regular polyhedra, there are two infinite families of uniform polyhedra (the prisms and the antiprisms), and 13 others.

—

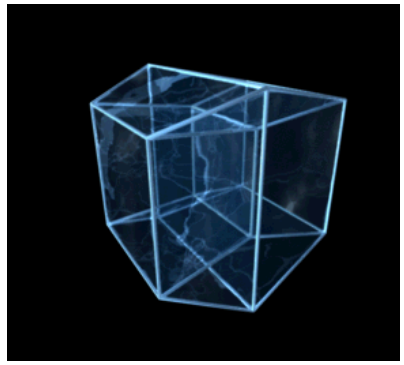
Now we move to 4D. In 4D, a *polychoron* is a shape made up of polyhedra (cells) connected along their faces in a single closed shape, meaning every face is on two cells, and any two vertices are connected by a path of edges. The tesseract below is an example of a polychoron.

8 This is, of course, not a cube, but a drawing of a cube. A cube lives in 3d and cannot be truly represented on a 2d page. We can make 2d drawing, nevertheless, to get a sense of the cube's overall shape.

9 If the faces are not regular polygons, the polyhedron is only *semi-uniform*.

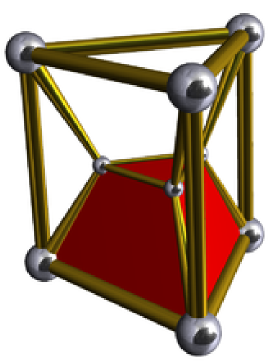


[x]

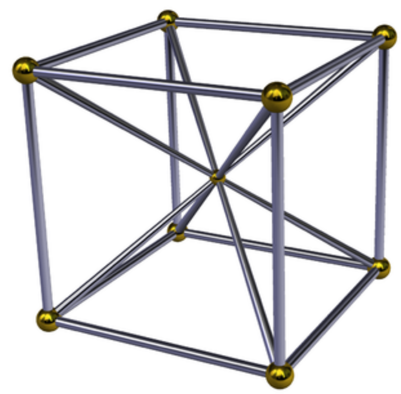


A tesseract[10], a polychoron — A tesseract mid-rotation

We say a polychoron is *regular* if it is vertex- edge- and face-transitive, and all of its 3d cells are identical to each other in the same way (*cell-transitive)*. This means the 3d cells of a regular polychoron are regular polyhedra. Tesseracts are regular: if we toss around and flip a tesseract (or any regular polychoron), there is no way to distinguish any pair of cells, faces, edges, or vertices (notice for instance that four cubic cells meet at each vertex). A 3,3 Duoprism is not regular, since some of its faces are triangular and some are square (we discuss duoprisms in section nine, but the threes in its name roughly refer to its triangular faces). A cubic pyramid is not regular, since one of its cells is a cube while the others are square pyramids. There are six regular polychora.



A 3,3 Duoprism: not regular, but uniform



[xi]

A Cubic pyramid: not regular nor uniform

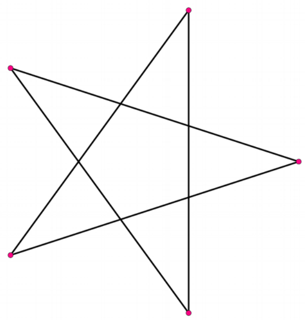
We say a polychoron is *uniform* if all of its vertices are identical (it is vertex-transitive), whether or not its cells, faces, or edges are, and its cells are uniform polyhedra. A 3,3 duoprism is uniform: notice e.g. that each of its vertices is at the intersection of four of its triangular prism cells. A cubic pyramid is not uniform: the vertex at the tip of the square pyramid faces can be distinguished from the vertices at the bases. In addition to the six regular polychora, there are two infinite families of uniform polychora (antiprism prisms and duoprisms), and 58 others. The final one (the Grand Antiprism) was discovered by John Conway and Michael Guy in 1965, and the list was proven complete by Marco Möller in his 2004 dissertation.

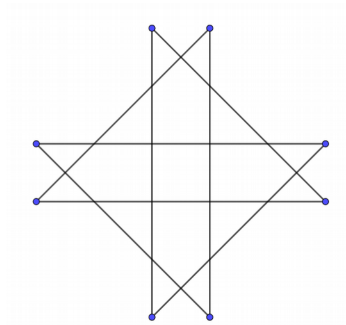
In general, shapes like line segments, polygons, polyhedra and polychora are known as *polytopes.* An *n*-dimensional polytope is made out of lower dimensional *facets,* the general term for things like

10 This is, of course, not a tesseract. A tesseract lives in 4d and cannot be truly represented in 2 or 3d. We can make a 3d drawing, nevertheless, to get a sense of the tesseract's overall shape. This is a 2d drawing of one such 3d drawing.

vertices, edges, faces, and cells. Regular and uniform polytopes in any number of dimensions are defined similarly to the definitions for two, three, and four dimensions.

The five-edged shape below is a polygon that intersects with itself; we call it a *star* polygon. Only the endpoints of a line segment are vertices of the segment: thus if two edges intersect in the middle we can reasonably claim this doesn't make any new vertices. Looking at the five sided self-intersecting polygon below, we can see then that all of the "true" vertices are still indistinguishable from each other, as are the five edges. We thus say that this shape, a pentagram, is also technically regular, a *regular star polygon*. There are infinitely many of these.



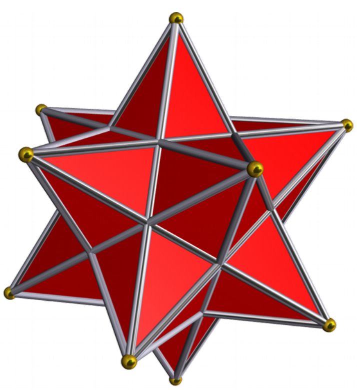


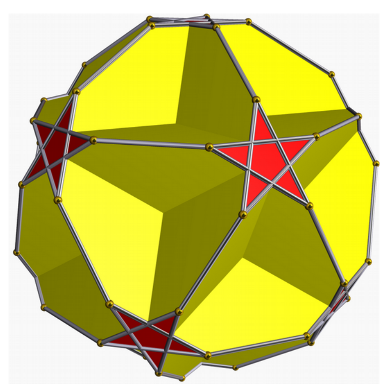
[xii]

A pentagram: a regular star polygon   A vertex-transitive (but not regular) octogram

This eight sided star polygon (octogram) is not regular, since its "outer" edges are shorter than its "inner" ones. Its vertices, however, each connect one outer and one inner edge, and are identical to one another if we can rotate and reflect the octogram. This is thus a *vertex-transitive star polygon*. Again, there are infinitely many of these.

In 3d, there are *star polyhedra*. This small stellated dodecahedron (sissid for short[11]) is a *regular star polyhedron* made up of 12 regular pentagrams. There are four regular star polyhedra.



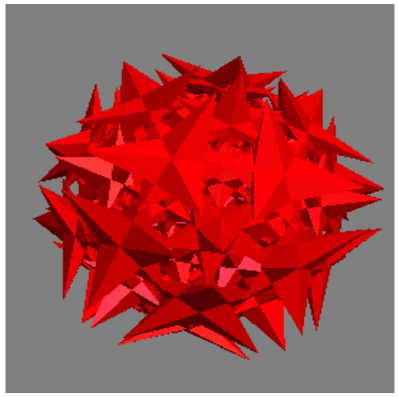


The sissid: a *regular star polyhedron*   The tigid: a uniform star polyhedron

There are also *uniform star polyhedra*, which must be vertex transitive and must have regular (or regular star) polygons as faces. Besides the four regular ones, there are two infinite families (again prisms and antiprisms), and 53 others, including the truncated great dodecahedron (tigid for short).

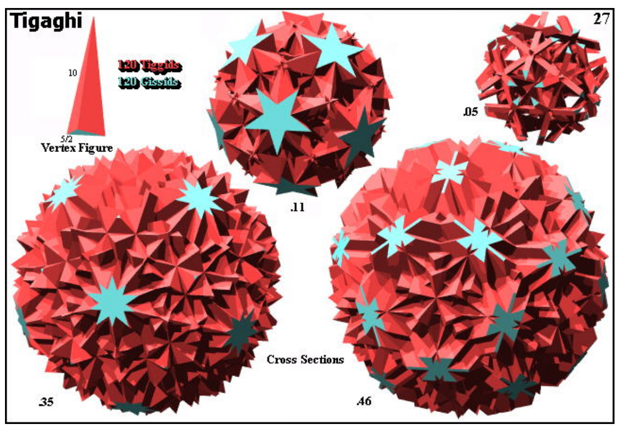
11 Short form names and tigaghi image credited to Jonathan Bowers (polytope.net). Other images original or Wikimedia Commons.

These concepts of course generalize to 4d as well. Below is one of the 10 *regular star polychora*, the Small stellated 120-cell (Sishi for short).



The sishi: a regular star polychoron

Finally, we reach the broadest of the classifications that we will examine in this paper, the *uniform star polychora:* polychora that are vertex-transitive and whose faces are regular polygons, with self-intersection allowed. Even their shortened names sound increasingly Klingon: this one is known as the Tigaghi:



The tigaghi, a uniform star polychoron

How many of these exist? Again, all ten of the regular star polychora are uniform, and there are infinite families based on the prisms and antiprisms. Excluding the infinite families but including all other stars and non-stars, there are at least 1849 uniform polychora, with the most recently known four (named Ondip, Gondip, Sidtindip, and Gidtindip) being discovered in 2006[12]. The set of uniform star polychora is in some sense the simplest sort of polytope for which it is unknown whether or not we have found them all.

In this thesis, we will explore the methods used to *enumerate* (find a complete list of) these categories of polytopes, primarily in the three and four dimensional cases. This will help the reader understand the difficulties in solving the open question of enumerating the uniform star polychora.

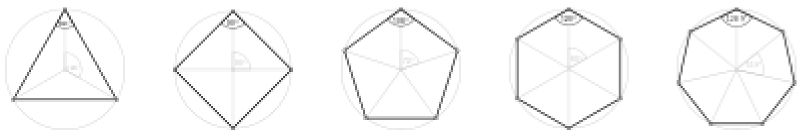
One more note on terminology: because all non-self intersecting uniform polytopes are convex, we will often contrast "convex" and "star" polytopes, even though there are some (non-uniform) polytopes that are neither convex nor self-intersecting.

12 Bowers (http://www.polytope.net/hedrondude/polychora.htm)

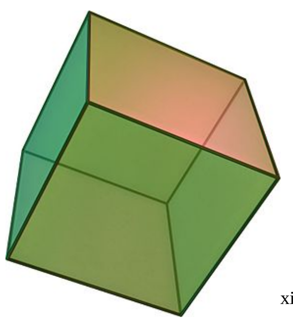
1. 2d and 3d regular shapes

Let us first examine what it means for a shape to be *regular*. In our introduction, we claimed a 2d polygon is regular if all of its 1d vertices and 2d edges are identical. What does this mean more formally?

In 2D, a regular shape is one that is both vertex- and edge-transitive. There is an infinite family of regular convex polygons: one for each number of sides[13]. The first few are below, hopefully it is clear that this family is infinite.



How can we extend this definition to three (or even more) dimensions? One approach would be to directly copy the definition from two dimensions: that a shape be vertex-transitive and edge-transitive. Certainly this includes something like the cube:



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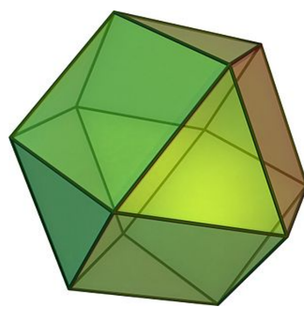

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A cube A cuboctahedron

The cube is vertex-transitive because we can rotate and reflect any of its eight vertices onto any other while keeping the cube lying within the same portion of 3-d space. You can also do the same with any pair of its twelve edges, so it is edge-transitive.

By contrast, consider the cuboctahedron. It has twelve vertices and 24 edges. Each edge is between one triangle face and one square face. Each vertex is between two opposite triangular faces and two opposite square faces. It turns out that this shape is also both vertex-transitive and edge-transitive. It is missing something compared to the cube though: it is not *face-transitive*. Similar to edge or vertex-transitive, a shape is face-transitive when any face can be moved onto any other face while keeping the overall shape lying in the same part of space. You can think of it like rolling dice: other than the labels, each face of the die is indistinguishable from all the others, which makes it a fair die. Geometers call a 3d shape *regular* if it is vertex, edge, and face-transitive, and *quasi-regular*[14] if it is vertex and edge transitive but not face-transitive. A shape that is vertex and face-transitive is *noble*, while one that is edge and face transitive is a *quasi-regular dual*[15]

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13 This is counting rotations, reflections, translations and scalings of a shape as the same shape. A square but a bit to the left, tilted, and smaller is still a square.

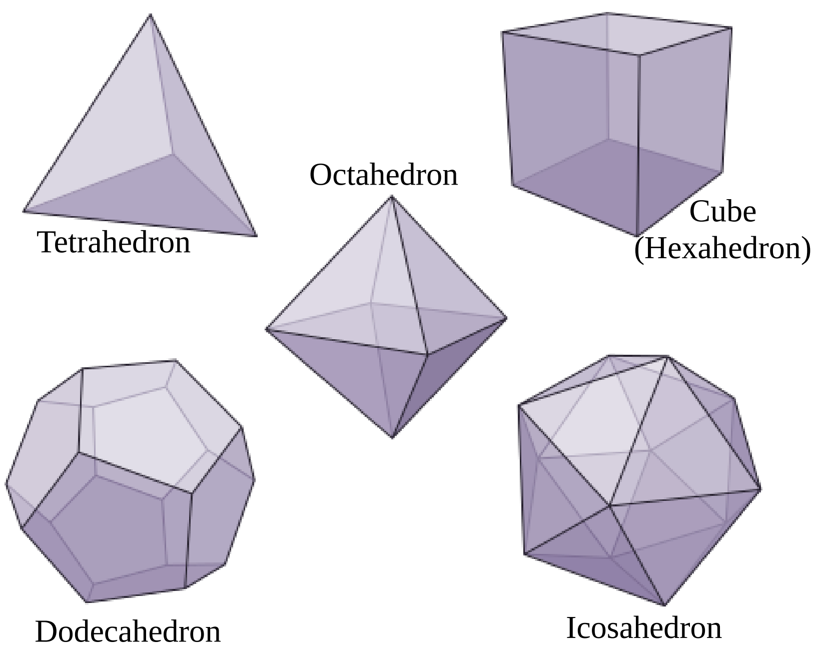
14 Coxeter (1973) gives a different but equivalent definition.

15 We shall expound upon dual shapes in section three

[xv]

Polyhedral dice: all these shapes are face-transitive. One is not regular.

The dice in the above photo provide an excellent example of face-transitivity. The topmost (red) die is not vertex-transitive: two of its corners are on the narrow angles of its kite-shaped faces, while the other corners are on the broad angles. All the other dice in the photograph are vertex-transitive and edge-transitive as well: this makes them regular. Let us examine these five shapes:

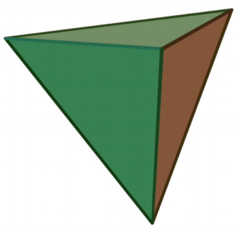
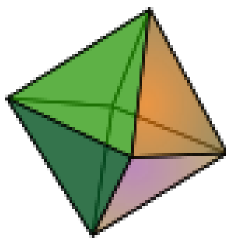
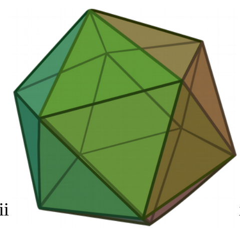
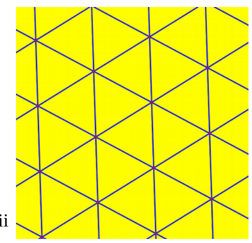


We have the *Tetrahedron*, with four triangular faces. Next is the *Cube* or *Hexahedron* with six square faces. Then the *Octahedron*, with eight triangular faces. We have the *Dodecahedron* with twelve pentagon faces, and finally the *Icosahedron* with twenty triangle faces. These are the five *Platonic Solids*, named for the Greek philosopher Plato, and they are the only five convex regular polyhedra. We shall next explore why this is the case.

1a. Why are there only five Platonic solids?

The Platonic Solids, the convex regular polyhedra, are a very exclusive club compared to the infinite families of regular polygons and uniform polyhedra. There are the triangle-faced tetrahedron, octahedron, and icosahedron (from Greek meaning 4-base, 8-base, and 20-base), the square-faced cube (or 6-sided hexahedron), and the pentagon-faced dodecahedron (12-base). The proof that these represent all the possible shapes is fairly understandable, and likely goes back at least to Theaetetus in 4th-century BCE Athens[16]. The proof is as follows:

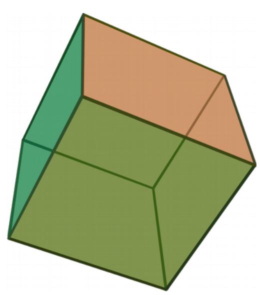
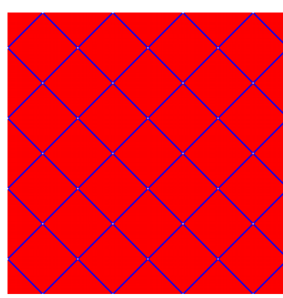
Each vertex of a regular polyhedron must have at least three faces: if there were two or fewer, the 'polyhedron' would fit in the plane. We can build up a polyhedron given what faces and edges meet at each vertex (its *vertex figure*) by taking one copy, and simply copying that figure on each new vertex until we get a closed shape, if there is one for that vertex figure. Consider the regular polygon with the fewest sides: the equilateral triangle. Exactly three at a point form a tetrahedron, Four form an octahedron, and five form an icosahedron. Six lie flat in the plane:


[xvi]

[xvii]

[xviii]


Tetrahedron Octahedron Icosahedron Flat triangle tiling

We can see that if we add any more triangles beyond six they will push up into a saddle shape rather than folding in on themselves, and will no longer be convex shapes. Thus there are only three convex regular polyhedra with triangular faces.

We repeat this process for squares. Three squares at each corner form a cube, while four tile a plane, meaning any more than four will not be convex:



[xix]

Cube Flat square tiling

Thus, the only Platonic solid with square faces is the cube.

We do this again with pentagons, finding the dodecahedron with three pentagons at each corner, and no more beyond three. Three hexagons already lie flat in the plane, so there are no Platonic solids with faces of six edges or more.

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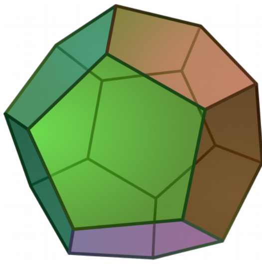
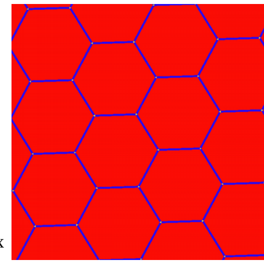
16 Allman, G.J (1889) pp. 206 cites the *Suda* (a 10th century Byzantine encyclopedia), which credits Theaetetus

 [xx]  [xxi]

Dodecahedron   Flat hexagon tiling

Thus we have found all five Platonic solids, as well as all three regular tilings of the plane.

The more formal version of the same argument relies on the idea of a *vertex configuration*. This is simply a list of the faces meeting at the vertex of a polyhedron. For instance, the octahedron has four triangles meeting at each vertex so its vertex configuration is (3.3.3.3)[17].

Theorem: There are five Platonic Solids (regular convex polyhedra.)

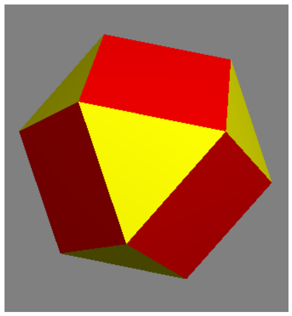
Proof: The vertex configuration of a regular polyhedron must consist of identical polygons. That of a convex polyhedron must have at least three polygons and must not have the sum of their angles be equal to or greater than 360 degrees. Since six triangles make 360 degrees, the triangle-faced vertex configurations are (3.3.3), (3.3.3.3), and (3.3.3.3.3). Since four squares make 360 degrees, the only square-faced vertex configuration is (4.4.4). Since four pentagons make more than 360 degrees, the only pentagon-faced vertex configuration is (5.5.5). Since three hexagons make 360 degrees, there is no vertex configuration of 6 or greater sided faces that makes a regular convex polyhedron. Thus there are five possible vertex configurations of regular convex polyhedra. As each of these is observed to produce a polyhedron, there are precisely five regular convex polyhedra.

This process of looking at the vertex configurations extends relatively easily to more complex cases, where it is very useful. We shall see this next with a slightly more general class of shapes: the Uniform polytopes.

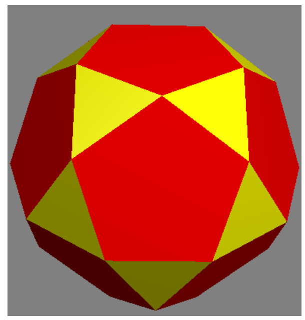
17 Since this is three occurring four times, it could be abbreviated in the Schläfli symbol {3,4}

2. Uniform Polyhedra

Regular polyhedra must be face-transitive, edge-transitive, and vertex-transitive. There are only a few such polyhedra. If we relax some of these conditions, we can get broader categories. If we require only edge and vertex transitivity, we get the *quasiregular solids*[18]. Note that these automatically have regular polygons as faces, since their faces must be edge and vertex transitive as well. However, they need not have *only one type* of regular polygon as a face. The cuboctahedron that we saw earlier fits in this category, as well as the similar icosidodecahedron.

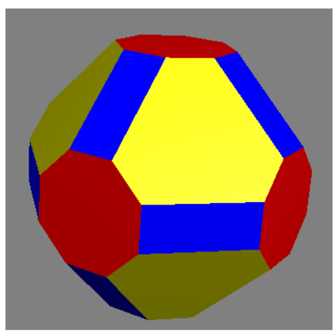


Cuboctahedron



Icosidodecahedron

The two above are the only two convex quasiregular solids that are not also regular[19]. If we want more solids, we can look at those that are vertex-transitive but not necessarily edge or face-transitive. If we don't specify that we require regular polygons as faces, that could include shapes like this:

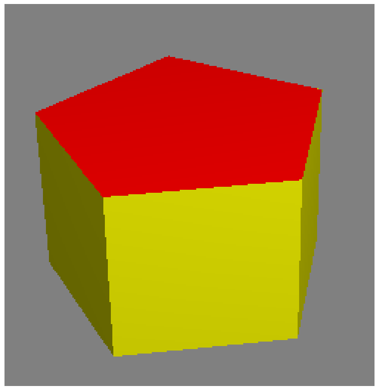


A wonky vertex-transitive shape

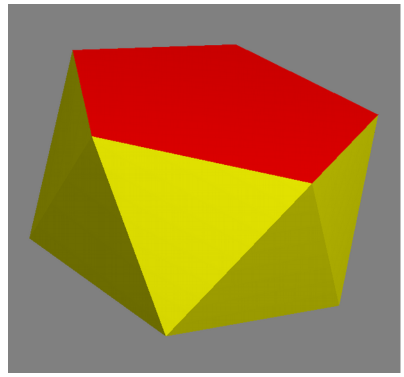
If we *do* require regular faces along with vertex-transitivity, there is a more limited set of shapes to pick from. These are the *uniform polytopes*. Of the convex ones, we have two infinite families. The first are the *prisms*, which are formed from two copies of a regular polygon connected by a row of squares. The second are the *antiprisms*, where the squares are replaced with alternating triangles, and the top and bottom faces are angled relative to one another

18 Coxeter (1973), pp. 18

19 Amusingly, both of these shapes feature in *Star Trek*, the Cuboctahedron as a form of suspended animation in the Original Series episode "By Any Other Name", and the Icosidodecahedron as the shape of the Vulcan game of Kal-toh in the Voyager episode "Alter Ego"

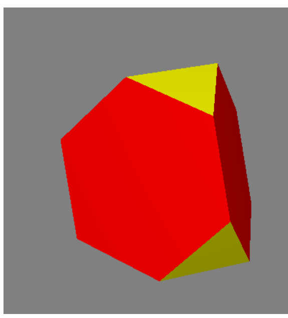


A Pentagonal prism

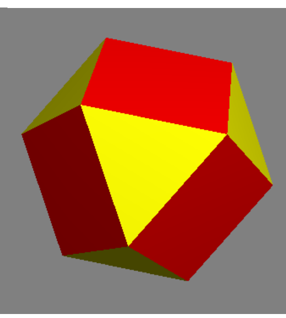


A pentagonal Antiprism

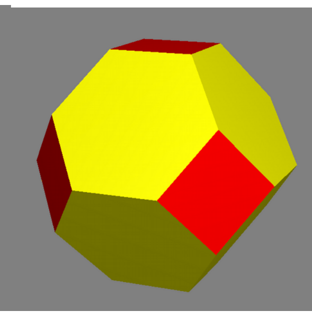
These shapes generally only have one axis of rotational symmetry and one plane of reflectional or rotoreflectional symmetry. This makes prisms and antiprisms much less symmetric than the *Archimedean Solids*[20], which are the 13 other non-regular uniform convex polyhedra[21]:



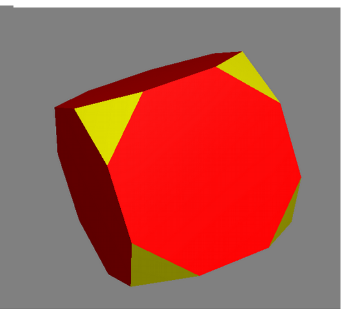
Truncated Tetrahedron



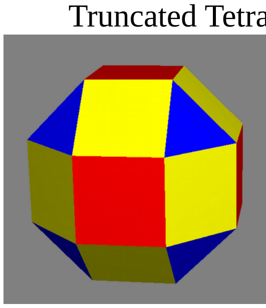
Cuboctahedron



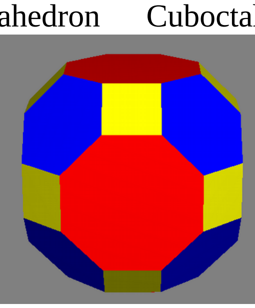
Truncated Octahedron



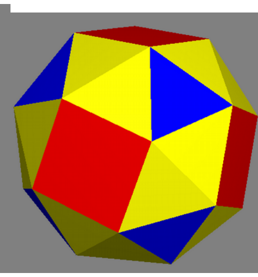
Truncated Cube



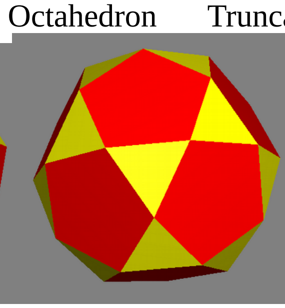
Rhombicuboctahedron



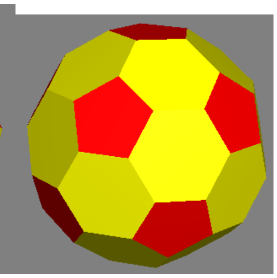
Great Rhombicuboctahedron



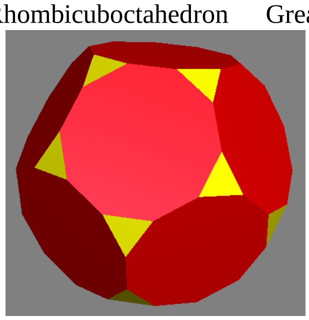
Snub Cube



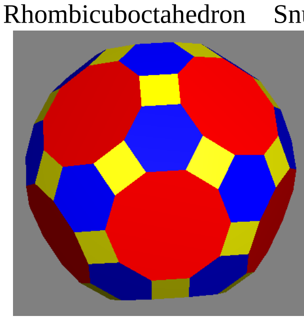
Icosidodecahedron



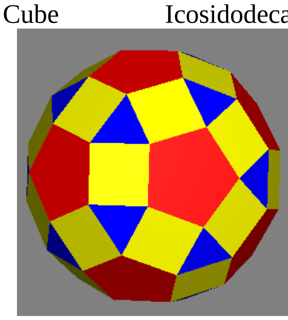
Truncated Icosahedron



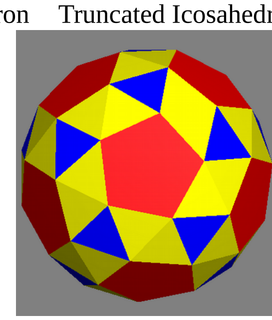
Truncated Dodechedron



Great Rhombicosidodecahedron



Rhombicosidodecahedron



Snub Dodechedron

20 Conway, Burgiel, Goodman-Strauss pp. 251

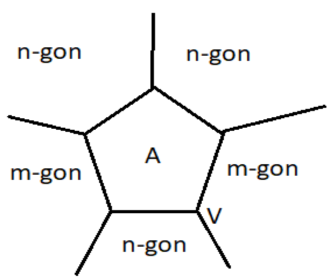
21. Note that the snub cube and snub dodecahedron come in both left and right handed versions, which if counted separately would make 15 Archimedean solids.

These thirteen solids were those enumerated by Archimedes in a now-lost work. Some of these are likely familiar to you: the Truncated Icosahedron for instance is a very common shape for soccer balls. Others of them you may be seeing for the first time. The one thing connecting all of them is that these shapes, along with the Platonic solids, prisms, and antiprisms, are the only convex polyhedra that are vertex transitive and have regular faces. The proof that these are all of them is similar to the one that there are only five Platonic Solids: look at which combinations of faces can make up a vertex, and then try to build up polyhedra using that vertex configuration.

Theorem: There are thirteen Archimedean solids (Uniform convex polyhedra, excluding prisms and antiprisms).

Lemma: If a uniform polyhedron has an odd-sided face, that face must be surrounded at each vertex by two faces that are identical to each other.

Proof of Lemma: Consider polygons meeting at a vertex *V* of a uniform shape, with one an odd-sided polygon *A*. AWLOG that one of the polygon neighboring A has n sides. *A* neighbors the *n*-gon at two vertices, *V* and another *V'*. Since the polyhedron is uniform, the face other than the *n*-gon which shares an edge with *A* at *V* must be identical to the corresponding shape at *V'*. Let them be *m*-gons. Again, because the polyhedron is uniform all of its vertices are identical, so the shapes neighboring *A* sharing a vertex with both these *m*-gons must be *n*-gons. Continue in this manner until we reach the vertex of *A* opposite the first *n*-gon. The two polygons neighboring *A* at this vertex must be identical, either both *n*-gons or both *m*-gons. Since the polyhedron is uniform, all the vertices then must be of this sort. This proves the lemma.



Proof of theorem: Consider the arrangement of faces meeting at any vertex. For a convex polyhedron, we need at least three faces per vertex, and we need the angles at each vertex to add up to less than 360 degrees. For this polyhedron to be uniform, we need every vertex to be identical up to rotation and reflection. By our lemma, we also must surround any odd-sided polygon face with identical faces.

Then all that is left is to enumerate the possible vertex configurations and see which ones make polyhedra. If (k.l.m.n) means a vertex with a k-gon, l-gon, m-gon, then n-gon, the possible vertices (and their corresponding shapes) are:[22]

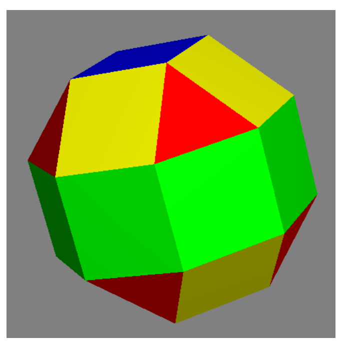
22 Enumeration and conditions from Wolfram Mathworld. Third condition proof from ywhmaths.webs.com

(3.3.3)	Tetrahedron	(5.6.6)	Truncated Icosahedron
(3.4.4)	Triangular prism	(3.3.3.*n*)	*n*-gonal antiprism
(3.6.6)	Truncated tetrahedron	(3.3.3.3)	Octahedron
(3.8.8)	Truncated cube	(3.4.3.4)	Cuboctahedron
(3.10.10)	Truncated dodecahedron	(3.5.3.5)	Icosidodecahedron
(4.4.*n*)	*n*-gonal prism	(3.4.4.4)	Rhombicuboctahedron
(4.4.4)	Cube	(3.4.5.4)	Rhombicosidodecahedron
(4.6.6)	Truncated octahedron	(3.3.3.3.3)	Icosahedron
(4.6.8)	Great Rhombicuboctahedron	(3.3.3.3.4)	Snub Cube
(4.6.10)	Great Rhombicosidodecahedron	(3.3.3.3.5)	Snub dodecahedron
(5.5.5)	Dodecahedron		

We find that there are the two infinite families corresponding to the prisms and antiprisms, then several others, thirteen of which make single solids that are not the Platonic solids.

2a. Solid 14 and Vertex Transitivity

Before we move on, we need to address a point about the meaning of *vertex-transitive*. The formal definition we are using involves reflecting and rotating a shape to match corners with each other, but an intuitive meaning is that each corner “looks the same” as every other. One check that you might do is to see if each vertex has the same *vertex figure*, that is the combination and order of faces around each vertex. This is a necessary condition for vertex transitivity, but is not quite sufficient. A counterexample is following solid (the elongated square gyrobicupola or pseudorhombicuboctahedron)[23]:

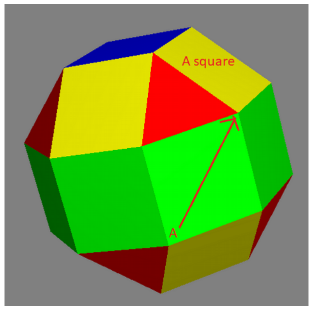


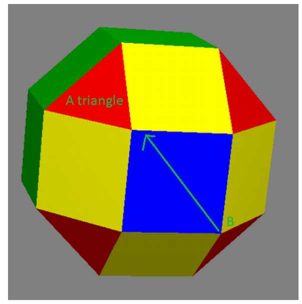
Elongated Square Gyrobicupola – Solid 14

We will call this solid 14 since elongated square gyrobicupola or pseudorhombicuboctahedron is rather long. The vertex figure of solid 14 is the same at each vertex, and is the same as the rhombicuboctahedron: three squares and one equilateral triangle. Even though the vertex configuration at each vertex is the same, some pairs of vertices are still distinguishable.

Theorem: Not every shape that has the same configuration at each vertex is vertex-transitive.
Proof: A counterexample is Solid 14. Every vertex has the configuration (3.4.4.4). Consider vertex A in the figure below: if we look across the square opposite the triangle, we find another square. If we do the same with vertex B, we instead find a triangle:

23 Grünbaum, B. (2008).



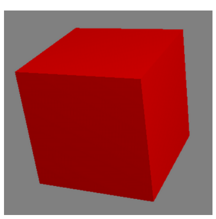


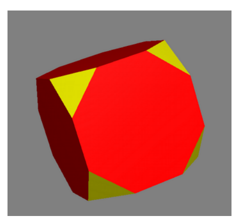
We can thus see that no reflection or rotation can bring vertex A onto vertex B without changing how Solid 14 lies in space. Solid 14 is therefore not truly vertex-transitive and thus not truly a uniform polyhedron. It should not be counted as an Archimedean solid, although it is still a very interesting shape.[24]

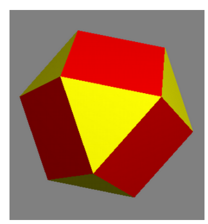
2b. Polyhedron Transformations[25]

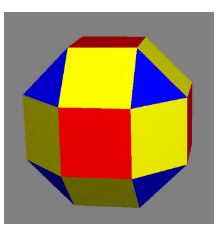
You may have noticed that several of the Archimedean solids bear the names of Platonic Solids, with modifiers like “Truncated” “Rhombi-” or “snub”. These polyhedra have the same set of underlying symmetries as the Platonic Solids they are named after, but have been modified in some way. For instance, in each of the pictures below, the red faces are derived from the six faces of a cube.

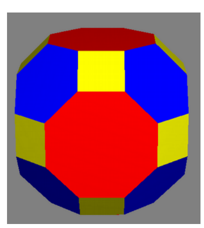
Truncation is the simplest to understand, and consists of cutting the corners off of a shape until the new faces are regular polygons: you can see this for instance in the truncated cube. If we cut off all the way to the midpoints of the edges, we call it *rectification*. The Cuboctahedron is a rectified cube, and also a rectified octahedron. This is because of duality, which we explore in the next section.

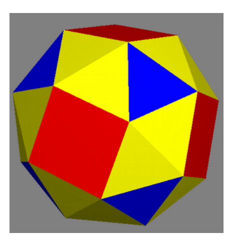

Cube


Truncated Cube


Rectified Cube


Cantellated Cube


Truncated Rectified Cube


Snub cube

A slightly more complex modification is cantellation, when both the vertices and edges are cut. The Rhombicuboctahedron for instance is the cantellation of either a cube or an octahedron. We can

24 Grünbaum argues that this solid should be called “Archimedean” but not “Uniform”. As Archimedes did not include this shape in his 13 solids, that seems like a foolish classification.
25 Definitions can be found e.g. Olshevesky (2008).

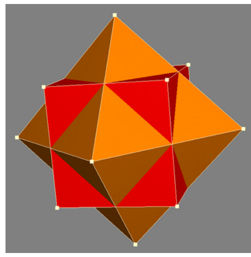
stack these modifications together: the Great Rhombicuboctahedron is a (slightly distorted) truncation of a cuboctahedron, which itself is the rectified cube.

Finally, we have snubbing, which involves taking the faces of a regular solid, moving them apart, rotating them, and connecting the gaps with equilateral triangles. This can result in a non mirror-symmetric shape, so there are left and right handed versions of snub cubes and snub dodecahedra.

Beyond just the naming scheme, understanding these transformations can help get a sense of how these shapes are related to one another and why the set of Archimedean solids contains only these particular members.

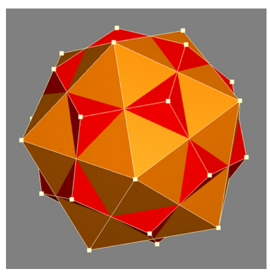
3. Dice Duals

An important idea in the classification of polytopes is the *dual*. Replacing a polyhedron's faces with vertices and vertices with faces gives you the dual of that polyhedron[26]. An edge between two faces in a polyhedron becomes an edge between two vertices in its dual. As you might expect, the dual of a dual is the original shape.


[xxii]

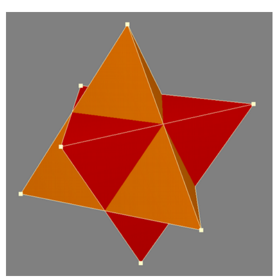
A cube superimposed with its dual, an octahedron

Though most easily seen with polyhedra, the concept of duals also applies to polygons (where a dual replaces vertices with edges and vice versa) and higher dimensional polytopes (an n-dimensional dual replaces n-1 dimensional facets with vertices and vice versa). Superimposing a shape and its dual makes it easy to see that they share the same axes and planes of symmetry. The cube and octahedron are duals, as seen above. So are the dodecahedron and icosahedron:



A dodecahedron superimposed with its dual, an icosahedron

The dual of a tetrahedron is another tetrahedron. This makes it *self-dual*. All regular polygons are also self-dual, which is why they make bad examples.

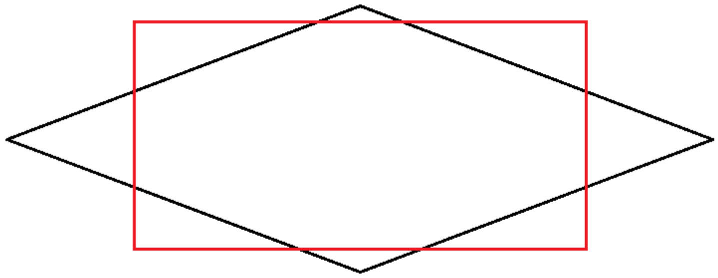


Two tetrahedra, a self-dual shape

The dual of a vertex-transitive polygon will be edge-transitive and vice versa. For instance, the dual of a rectangle is a rhombus:

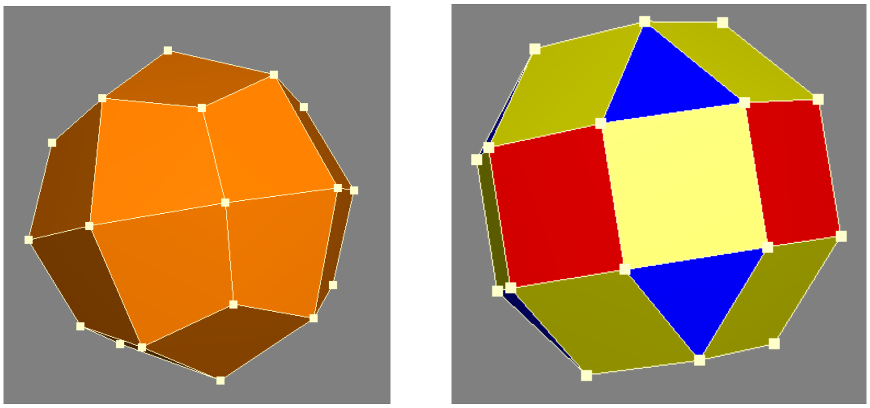
---

26 "Dual Polyhedron", *Wolfram Mathworld*



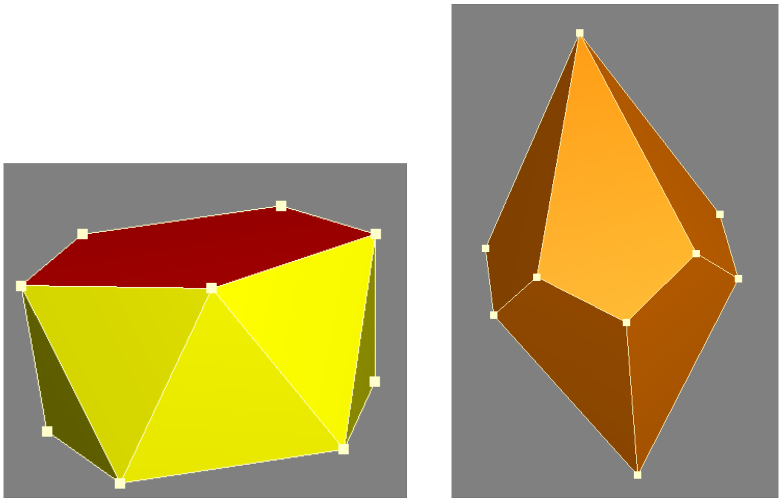
A rhombus and a rectangle, dual shapes.

Similarly, the dual of a vertex-transitive polyhedron will be face-transitive. This means that the duals of the Archimedean solids are all fair dice:



The Strombic icosatetrahedron (left) is the dual of the rhombicuboctahedron (right)

For instance, the rhombicuboctahedron has 24 indistinguishable vertices, so its dual has 24 indistinguishable faces. If you have ever played *Dungeons and Dragons*, you will recognize its ten-sided die as the dual of a pentagonal antiprism:



The pentagonal antiprism and its dual, the d10.

The pentagonal antiprism has ten vertices: five on the top, and five on the bottom, angled halfway between those on the top. Its dual has ten faces arranged in the same pattern: five on top, and five on the bottom angled halfway between those on top. It is easy to see why a face-transitive shape cut from a solid piece of material will be a fair dice: the odds of the shape landing on any one side must be the same as the odds of landing on any other side, otherwise there would be two distinguishable sides.

As we explore further, the idea of duals will help us categorize regular and near-regular shapes. Firstly, it shows that there must be one face-transitive polyhedron to match each convex vertex-transitive one, so there are two infinite families, the five Platonic solids, and thirteen others. Since the dual of a convex shape is convex, these are all the convex face-transitive polyhedra: there are others corresponding to non-convex shapes. This principle will apply to four dimensions as well once we examine shapes there: the duals of vertex-transitive 4-polytopes will be cell-transitive.

We can also make some uniform shapes by combining a regular shape and its dual in different ways, for instance the cuboctahedron by combining the faces of a cube and octahedron, and similarly the icosidodecahedron. This is equivalent to rectifying one of the regular shapes. Combining the faces of a tetrahedron with itself in this manner will result in an octahedron.

4. Filthy degenerates: Beach Balls, Sandwiches, and Lines

In the introduction, we used a simplistic definition of a polygon:

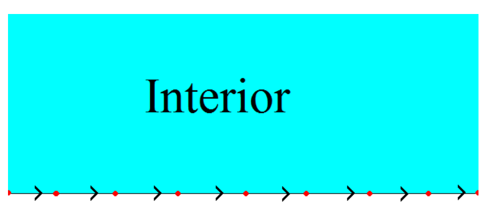
> A *polygon* is a shape made of line segments (edges) connected at their vertices in a single closed loop, meaning every vertex is on two edges, and any two vertices are connected by a path of edges.

This worked for our purposes at that point in the thesis, and fits neatly with the intuitive definition of a polygon, but it has some weak points. Consider the following definition of a polygon from Wikipedia:

> *A* ***polygon*** *(/ˈpɒlɪɡɒn/) is a plane figure that is described by a finite number of straight line segments connected to form a closed polygonal chain or polygonal circuit. The solid plane region, the bounding circuit, or the two together, may be called a* ***polygon***[27].

The Wikipedia definition has a convenient pronunciation guide, and helpfully clarifies that either the interior, the bounding circuit of edges, or both together may be called a polygon. Additionally, notice that a pair of conditions have been added. It must have a finite number of edges, and it must fit in a plane. What purpose does each of these qualifications serve?

Take a line, and divide it into an infinite number of segments of equal length, strung end to end. These divide the plane into two parts. Declare one part to be the interior:



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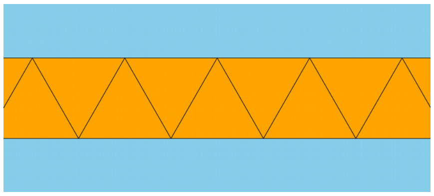
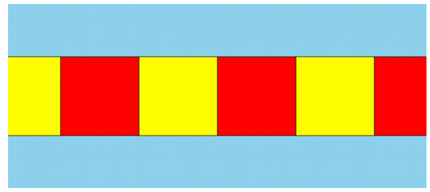
Does this meet our simpler definition of polygon? Every vertex is on exactly two edges, and you can get from any vertex to any other vertex through a path of edges. This strange construction would seem to be a polygon. We name it an *apeirogon*, from the Greek "apeiros" meaning "boundless"[28]. By translating this particular apeirogon some length left or right we can match any vertex or edge onto any other vertex or edge, so this is vertex and edge-transitive, and thus is a regular apeirogon.

The apeirogon doesn't quite act like all the usual polygons: for one, it has infinite area. If we have two apeirogons, one for the top half-plane and one for the bottom half-plane, we can tessellate the plane. If we define a prism as two of the same polygon connected by a row of squares, an apeirogonal prism fits into two dimensions, as does the similarly defined antiprism:

27 Wikipedia Contributors (2020)
28 "Apeirogon", *Wolfram Mathworld*



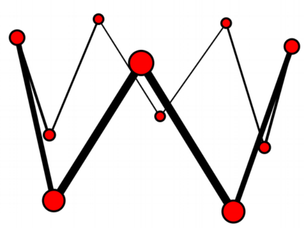
Apeirogonal antiprism



Apeirogonal prism

In general, when talking about polygons, we want to exclude apeirogons, so we say that polygons need to have a finite number of edges.

Next Consider this crown-like figure:

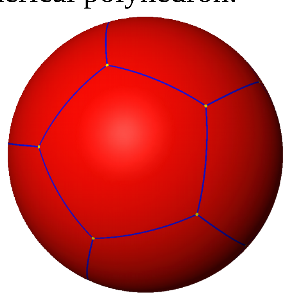


xxiv

It has ten edges, meeting at ten vertices, in a closed loop. The angle at each vertex is identical. This also meets our simplistic definition of a polygon, specifically a regular decagon. The only trouble is that it does not lie in the plane, and thus suffers from the same drawing problems as polyhedra. This sort of polygon is known as a *skew* polygon[29], so the above is a regular skew decagon. In general, when talking about polygons we want to exclude skew polygons, so we say that polygons must lie in a flat plane.

Polyhedra also have some special cases that we may wish to exclude. The regular tilings of the plane and anything with an apeirogon face should be excluded by requiring a finite number of edges. Although it is a bit harder to visualize, if we allow polyhedra to bend into the fourth dimension we can also get skew polyhedra, so these must be excluded as well by requiring our polyhedra to lie perfectly in three dimensions. There are another class that should be dealt with: *improper polyhedra*.

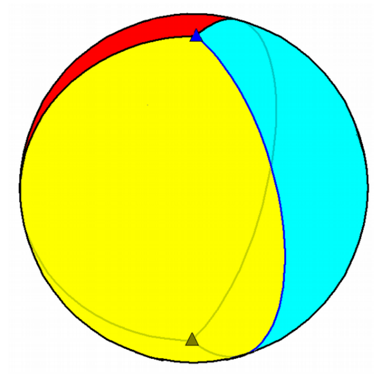
One convenient way to think about polyhedra is as tilings of the surface of the sphere. For instance, here is a dodecahedron as a spherical polyhedron:



A spherical dodecahedron

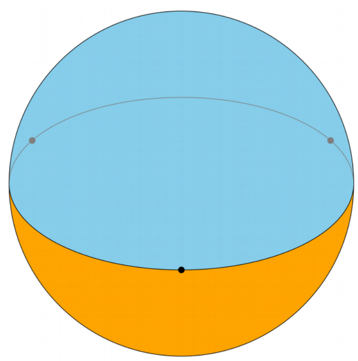
29 "Skew Polygon", *Wolfram Mathworld*

Most of these spherical polyhedra can be made as normal polyhedra as well, but there are some that cannot. For instance, see this beach-ball like device:



A three-sided hosohedron

This is known as a *hosohedron*[30]. Its faces are *lunes*, which have two spherical arcs as edges. In spherical geometry, this is a perfectly valid tiling of a sphere, and it is in fact vertex-, edge- and face-transitive. If we tried to correspond it to a normal polyhedron we would just end up with a bunch of line segments on top of each other, enclosing no volume. We can similarly look at its dual shape, the *dihedron*[31]:



A three-cornered dihedron

This shape is also a perfectly valid tiling of a sphere, and is also vertex- edge- and face-transitive. If we try to correspond it to a polyhedron, we end up with two triangles stuck back to back, again enclosing no volume in a sort of polygon sandwich. We can exclude the infinite families of many-sided hosohedron beach balls and many-cornered dihedron sandwiches by saying that a *proper* polyhedron must enclose non-zero volume. Hosohedra and dihedra are then *improper polyhedra*. In general, when we refer to polyhedra we will mean only proper polyhedra.

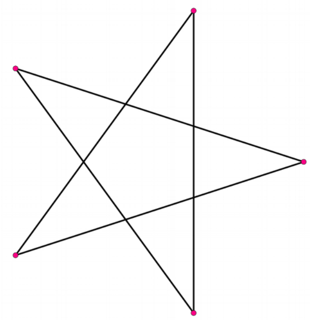
30 Coxeter (1973), pp. 12

31 Ibid

THE STARS: Self-intersecting Polytopes

5. Regular stars

You may recall the following shape from the introduction:



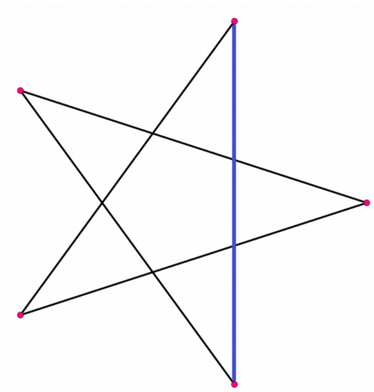
The Pentagram

Is this a polygon? Let us look once more at our stronger definition of a polygon:

> *A **polygon** (/ˈpɒlɪɡɒn/) is a plane figure that is described by a finite number of straight line segments connected to form a closed polygonal chain or polygonal circuit. The solid plane region, the bounding circuit, or the two together, may be called a **polygon**.*

Certainly, the pentagram lies in the plane. We can describe it with five line segments (hence the name pentagram) that form a closed circuit. The line segments do intersect with each other at some points, but nothing explicitly excludes this. This suggests that it is indeed a polygon.

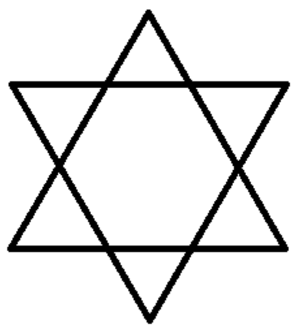
Let us consider the pentagram's properties. Firstly how many vertices does it have? Remember that an edge is a line segment, meaning it is formed by connecting two points, which are vertices. The blue edge below connects two of the five red vertices, as does each of the other four edges:



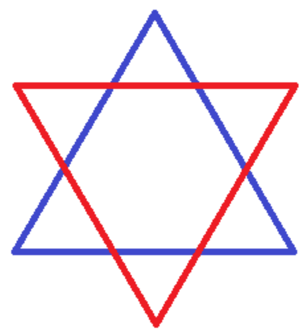
Notice that to describe this pentagram, we do not need more than these five vertices. Specifically, there are no edges defined by the interior intersections. Since we do not need them to describe the shape, these other intersection points should not be considered vertices.

The pentagram thus has five edges and five vertices. Since some of its edges intersect at places other than their endpoints, we say it is a *star polygon*. If we rotate this pentagram by fifths of a full turn, we can map any vertex onto any other vertex or any edge onto any other edge, and the overall shape lies the same in the plane. Thus the pentagram is a *regular star polygon*.

Now let's examine this next shape: a hexagram



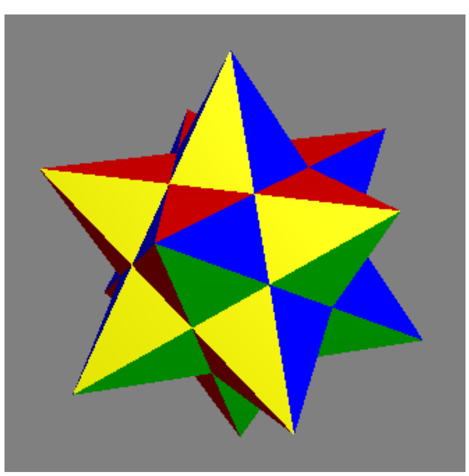
Hexagram



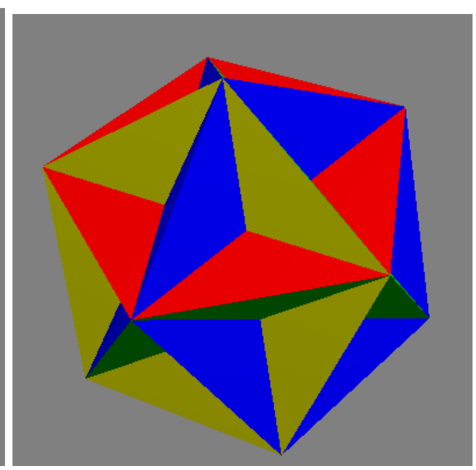
Two triangles

This shape has six edges and six vertices (again, we don't need the interior intersections to describe the shape, so they shouldn't be counted as vertices). The edges do not, however, form a single closed loop. Instead, they form two such loops. We call such a figure a *compound*, in this case a compound of two triangles. Because we can map any vertex of this compound onto any other and any edge onto any other by a series of sixth turns, leaving the shape lying the same in the plane, we call this a *regular compound*. While compounds can be interesting shapes in their own right, we do not count them among the star polytopes.

There are three dimensional star polytopes and compounds as well[32]. This includes those whose faces are star polygons, and those whose faces intersect at points other than their edges. The Small stellated dodecahedron (sissid) is an example of the former, with twelve pentagrams as faces, and its dual, the great dodecahedron (gad), is an example of the latter, with twelve intersecting pentagons as faces:



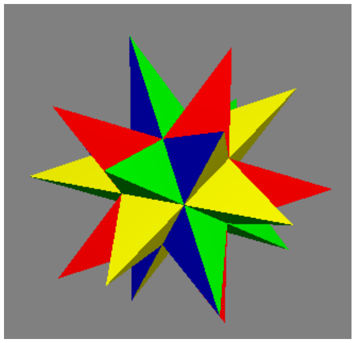
Small Stellated Dodecahedron (sissid)

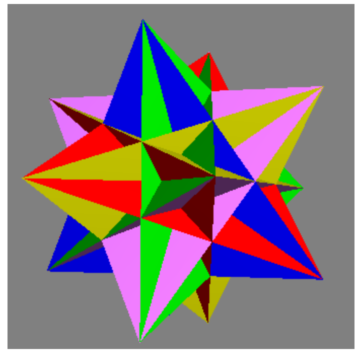


Great Dodecahedron (gad)

32 Conway, Burgiel, Goodman-Strauss (2008), pp. 404

Both of these two shapes fit our criterion for being regular polyhedra, and are two of the four *regular star polyhedra*, sometimes known as the *Kepler-Poinsot polyhedra*, after Johannes Kepler and Louis Poinsot, who played important roles in describing their properties. The other two are the great stellated dodecahedron (gissid), which also has twelve pentagram faces, and its dual the great icosahedron (gike), with twenty triangular faces:





Great Stellated Dodecahedron   Great Icosahedron

There are also some regular polyhedral compounds, such as this compound of two tetrahedra, known as the stella octangula:

Stella Octangula: a compound of two tetrahedra

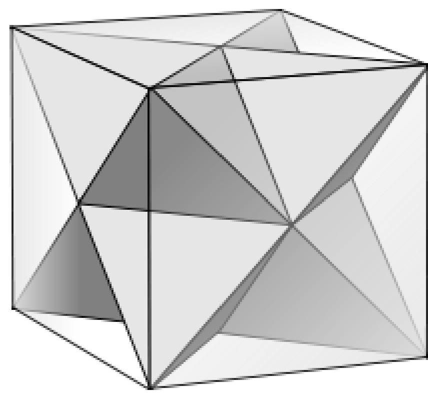
If you look closely, you may notice that the Sissid, Gad, and Gike all share the same vertex arrangement as the icosahedron, while the Gissid shares its with the dodecahedron. The stella octangula shares its vertex arrangement with the cube. It turns out that every regular star polyhedron or compound has the same vertex arrangement as one of the Platonic solids. This fact will allow us to enumerate all the regular star polyhedra and compounds, which we shall do next.

5a. Why are there only four Kepler-Poinsot Polyhedra?

When Poinsot identified the great icosahedron and great dodecahedron, he was not certain whether these two, along with Kepler's small and great stellated dodecahedra, were all the possible regular star polyhedra. It turns out this is true, that is to say the five Platonic solids and the four regular star solids are the only nine regular polyhedra. There are also four regular polyhedral compounds. We could try to prove this using the same method as the Platonic solids, that of examining the possible vertex configurations, however since the star polyhedra are not convex there is no limit to the sum of the face angles meeting at any given vertex. Instead, we will enumerate the regular star polyhedra (and compounds, by a different technique, that of *faceting*. One such enumeration[33] goes as follows:

Theorem: There are four regular star polyhedra (and three compounds)

> *Lemma:* Any regular star polyhedron (or compound) shares its vertices with one of the Platonic solids.
>
> *Proof of Lemma:* Coxeter proves this by noting that the polyhedral rotation groups are the only ones with multiple axes of more than 2-fold rotational symmetry. Joseph Bertrand, who originated this proof, argues similarly without explicitly using groups:
>
> Given any finite set of points in space there is a (non-strictly[34]) convex polyhedron that has its vertices at some of the points and which encloses the remaining points in its interior. This is its "convex hull". We can place a regular star polyhedron inside its convex hull, with the vertices matching up one to one.



xxv

The regular compound Stella Octangula inside its convex hull, a cube

> We can treat the whole polyhedron-hull system as one shape *P* and we can make a copy of it *Q*. Since the star polyhedron in *P* is regular, we can rotate or reflect *P* so that any vertex *v* of *P* goes onto any vertex *v'* of *Q*, and each of the other vertices of *P* will match up with a vertex of *Q*. Thus *P* must be vertex transitive and the convex hull a uniform polyhedron.   We can even match this alignment of *v* to *v'* in at least three different ways, by matching up each of the at least three identical faces that meet at *v*. Thus the convex hull in question must be a convex uniform polyhedron with at least threefold symmetry around every vertex, which excludes all but the Platonic solids.

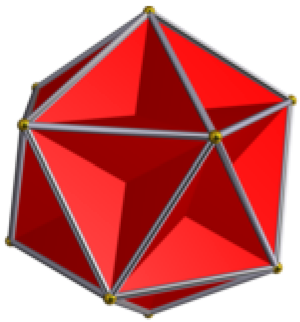
Proof of theorem: Given the Lemma, we simply look at all the regular *facetings* of the Platonic solids: the regular shapes that can be drawn interior to them using their vertices. For each of the Platonic solids, there are only finitely many vertices, and there are only finitely many ways to connect those vertices, so there are only a finite number of cases that we have to look through. If we limit ourselves to only shapes whose faces are regular polygons or regular star polygons, there are even fewer.

33 Modified from Bertrand (1858).

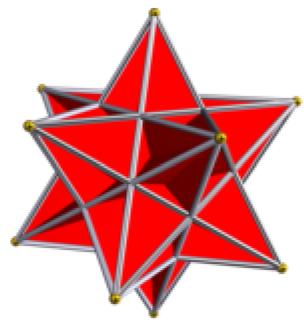
34 This means that we allow neighboring faces to share a plane, so for instance a scattering of points in the same plane can be 'enclosed' by a dihedron (polygon sandwich).

The tetrahedron has no diagonals, so it is useless for faceting. The interior shapes of the octahedron are three perpendicular squares, which do not form any regular star polyhedra or compounds. For a cube, there aren't enough space diagonals to be helpful, while the face diagonals form the stella octangula, which is a compound of two tetrahedra.

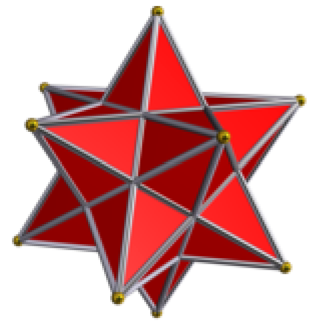
This leaves the icosahedron and dodecahedron. The icosahedron can be faceted into 12 pentagons (Great dodecahedron), 12 pentagrams (Small stellated dodecahedron) or 20 triangles (Great icosahedron), making three of the four Kepler-Poinsot polyhedra. The dodecahedron can be faceted into compounds of tetrahedra or cubes, as well as the Great stellated dodecahedron, the final of the four Kepler-Poinsot polyhedra. Since these are all the possible regular facetings of Platonic solids, these are all the regular star polyhedra and polyhedral compounds.


[xxvi]

Great Dodecahedron

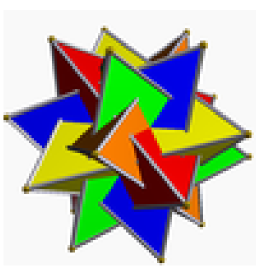


Great Icosahedron

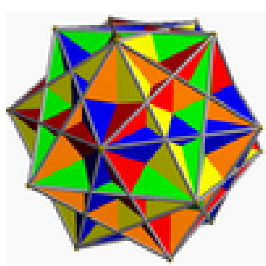


Small Stellated Dodecahedron

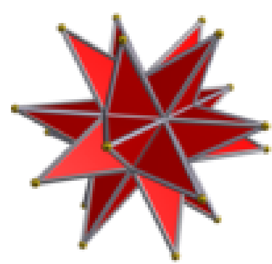
The facetings of the icosahedron



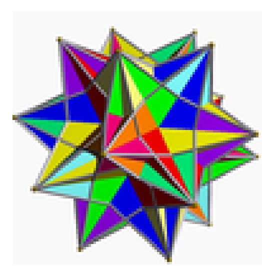
5 tetrahedra



5 cubes



Great Stellated Dodecahedron



10 tetrahedra

The facetings of the dodecahedron

5b. Mirror-regular compounds

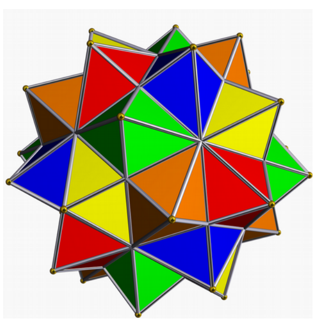
There is one additional regular polyhedral compound which bends the rules a little: the compound of five octahedra[35]. Its convex hull is not regular, but the quasiregular icosidodecahedron. It is not initially clear why the proof of the lemma above should not apply to this compound, but a possible explanation arises from examining the precise symmetries of its faces. In the lemma, we assumed that we could always freely match any face of our shape to any other face in any orientation. In particular, this assumes that the shape nor its facets have any sort of chirality or handedness. This applies to the regular star polyhedra, but not to all the compounds. The compound of five tetrahedra is chiral but it

35 Coxeter (1973) pp. 49

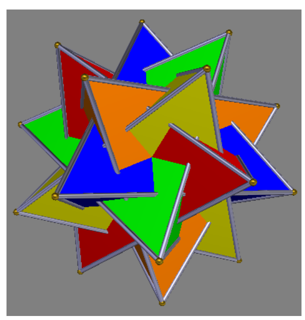
shares the same chirality in all places, so it is a faceting of a regular shape (the dodecahedron) and comes in separate left and right handed versions.

In the compound of five octahedra and that of ten tetrahedra, some pairs of faces can only be moved onto mirrored versions of each other, meaning that they cannot be identified in the unmirrored shape. Thus while it is still vertex-, edge-, and face-transitive, it is somehow *less* transitive than a compound without this mirroring issue. The same problem occurs with the compound of two triangles below, which resembles the pairs of coplanar faces on the compounds in question: we can see for instance that an edge on one triangle and an edge on the other can only be identified using a mirror symmetry, and also that an edge cannot be sent onto itself reflected. No name for this type of compound was available in the sources of this paper, but a good name for these compounds would be "mirror-regular", since their distinguishing property is that in order to fulfill the vertex, edge, and face transitivity needed to be regular you must use mirror symmetries.

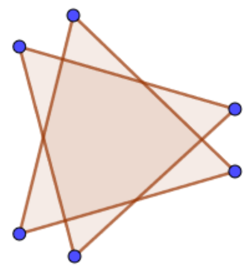
Mirror-regular compounds essentially have half as many usable faces at each vertex. The compound of ten tetrahedra has six faces meeting at each vertex, so that still leaves three and the lemma holds. The compound of five octahedra has only four faces at each vertex, so that leaves only *two* usable faces, and the lemma no longer holds. The compound of five octahedra thus is not a faceting of a regular solid but a quasiregular one. A quick check of the convex quasiregular shapes does not reveal any more unlisted regular (not even mirror-regular) star polyhedra or compounds, so we know that our expanded list is complete.



5 octahedra



5 tetrahedra mirrored



[xxvii]

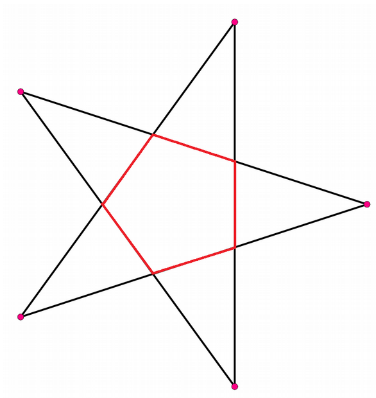
2 triangles

5c. Stellation and Greatening

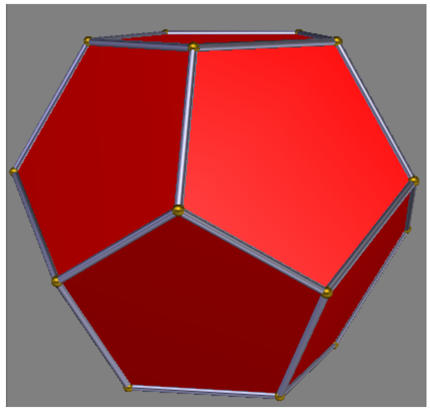
The names of the Kepler-Poinsot polyhedra betray another method of generating them: stellation and greatening[36]. *Stellation* is the process of lengthening the *edges* of a polytope until those edges meet in a new closed shape. The regular pentagram is a stellation of a regular pentagon. *Greatening* is the three-dimensional analogue of the two-dimensional stellation, in which one grows the *faces* of a shape to create a new closed shape. The small stellated dodecahedron is the stellation of the dodecahedron, while the great dodecahedron and great icosahedron are the greatening of the dodecahedron and icosahedron respectively. The great stellated dodecahedron is the stellation of the great dodecahedron, and the greatening of the stellated dodecahedron.

36 Olshevsky (2008)

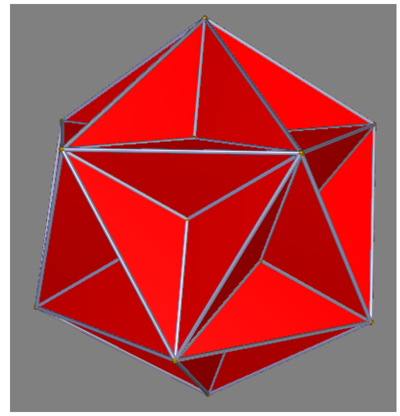
In general, stellating and greatening, when they result in valid polytopes, are useful techniques for identifying unknown polytopes, although it is more difficult than faceting to prove if this will result in all the shapes in a given class.



A red pentagon, stellated



A dodecahedron

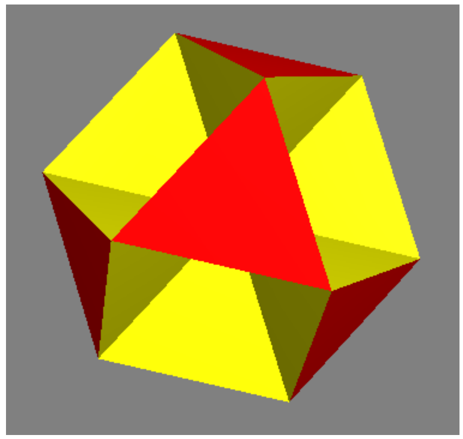


A greatened dodecahedron

6. 57 Varieties: The Uniform Star Polyhedra

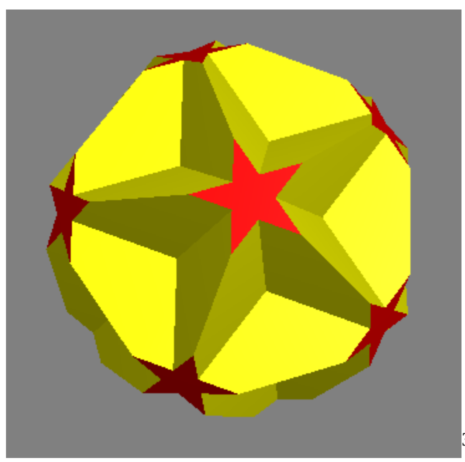
The next class of polytopes we will examine are the most expansive of 3d shapes we will study. These are the *Uniform Star Polyhedra*. They are *polyhedra* because they are three-dimensional shapes composed of faces, they are *star* because they are allowed to self-intersect, and they are *uniform* because they are vertex-transitive and each of their faces is a regular polygon, star or not.

Some of these polyhedra have convex faces that intersect, for instance the Octahemioctahedron, which has triangular and hexagonal faces:



Octahemioctahedron

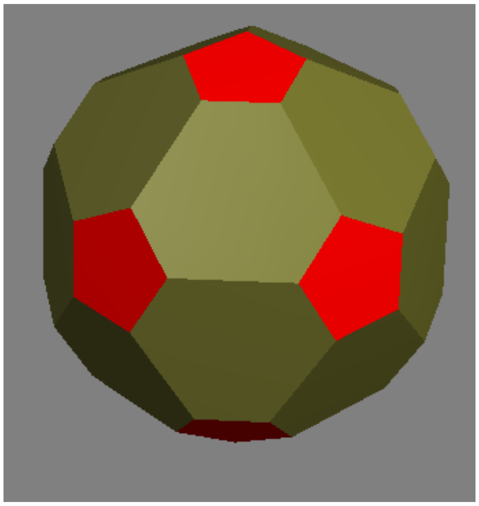
In other cases, the faces themselves can self intersect, such as the Great Truncated Dodecahedron, which has some decagon faces and some pentagram faces:


[37]

Great Truncated Dodecahedron

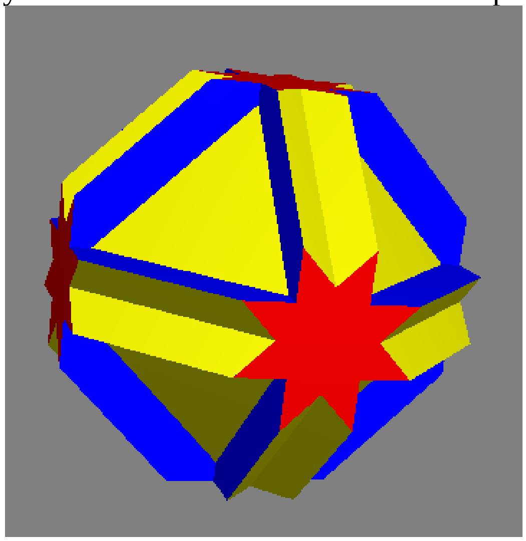
How many uniform star polyhedra are there? We were able to enumerate the regular star polyhedra (and compounds) by faceting the Platonic solids. A logical first step to enumerating the uniform star polyhedra would then be to facet the Archimedean solids. While this does get us partway to the goal, in this case it is not sufficient. Consider for instance the Great Truncated Dodecahedron from earlier. The shape below is its convex hull.

37 In these images, the entire interior of the star faces has been colored in a single color. An alternate color scheme changes from filled to unfilled every time an edge is crossed: this better highlights the edges, but makes counting faces a little harder. As with all representations, there are trade offs.

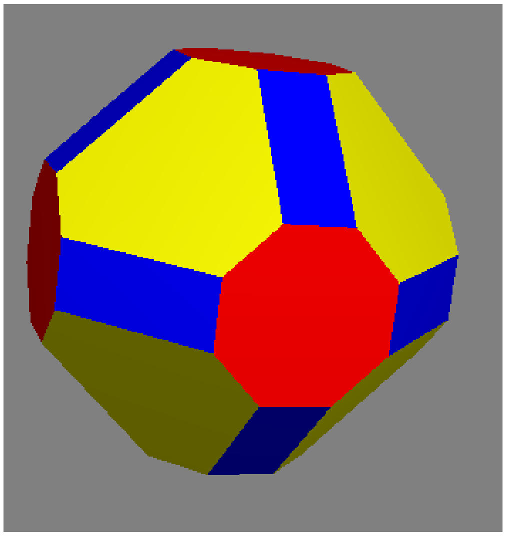


A vertex transitive, but not uniform, polyhedron

If you look carefully, you may notice that while the hexagons are vertex transitive, they are not regular. This truncated icosahedron is somewhat distorted from the Archimedean solid of similar geometry. There are even more extreme examples, like the Cubitruncated Cuboctahedron:



The Cubitruncated Cuboctahedron



Its convex hull: a distorted Archimedean solid

Thus, we can see that faceting the undistorted Archimedean solids will omit some of the uniform star polyhedra. We also do not no *a priori* which distortions will be necessary to find all of the uniform star polyhedra. We need a different method.

John Skilling provides this method with a computer-aided search in 1975[38], relying again on group theory. An earlier proof was published in Russian in 1970 by S. P. Sopov. To avoid simply copying down Skilling's entire paper here, a low-level summary will have to suffice.

Theorem: There are 75 uniform polyhedra, excluding prisms and antiprisms

Summary of proof:

38 Skilling (1975) (obviously)

All uniform polytopes are highly symmetric objects, and if we project those symmetries onto the surface of a sphere we can construct a regular polyhedron. The star prisms and antiprisms in this way correspond to improper hosohedra and dihedra, but all the non-prismatic uniform star polyhedra will correspond to Platonic solids, and thus share symmetries with them. Thus it makes sense to look at the symmetries of the Platonic solids.

There are 24 distinct ways to map a tetrahedron onto itself (thus its symmetry group is of *order* 24):
we can take an edge and move it to any one of the six total edges, rotated 180 degrees or not and either reflected or not, for a total of $6 \times 2 \times 2 = 24$ possibilities.

There are 48 distinct ways to map a cube or octahedron onto itself (thus their symmetry group is of order 48): we can take an edge and move it to any one of the twelve total edges, rotated 180 degrees or not and either reflected or not, for a total of $12 \times 2 \times 2 = 48$ possibilities.

Similarly, there are 120 ways to map an icosahedron or dodecahedron onto itself (thus their symmetry group is of order 120): we can take an edge and move it onto any one of 30 edges, rotated 180 degrees or not and reflected or not, for a total of $30 \times 2 \times 2 = 120$ possibilities.

Thus if we fix our axes and planes of symmetry, if we know the coordinates of a single vertex we can generate the either 24, 48 or 120 points available to be the vertices of a polyhedron. Note that since the tetrahedron's symmetries are a subset of that of the cube (as evidenced by the Stella Octangula), we need to only consider the 48-fold and 120-fold cases.

Skilling then constrains the edge length given the position of one of its vertices and the rotation needed to get from there to the edge's other vertex. We set the radius of the circumscribed sphere to be 1. Then note that the angles of rotational symmetry between connected vertices determine the side length of the polyhedron, which (since every face of the polyhedron is edge-transitive) must come out as the same side length when calculated for between any pair of connected vertices. Consider a side emanating from a vertex at position $\vec{r}$ and generated by a rotational symmetry of an angle θ about an axis $\vec{n}$. The length *s* of the side must satisfy $\frac{1}{2}s = [1 - (\vec{r} \cdot \vec{n})^2]^{\frac{1}{2}} \sin(\frac{1}{2}\theta)$
Because each edge lies on two faces, there are at least two (and often more) ways of describing the symmetry between two connected vertices in terms of an axis and an angle. This gives us systems of equations that can be solved for the vertices and edge lengths of the uniform polyhedron. Skilling is able to use symmetries to pare these systems down to a little more than a thousand cases, few enough to be solved by the 1970s era computers, giving a list of 75 uniform polyhedra, excluding prisms and antiprisms. 5 of these are the Platonic solids, and 13 the Archimedean solids, leaving 57 uniform star polyhedra. Further excluding the 4 Kepler-Poinsot solids leaves 53.

The fundamental idea behind Skilling's proof is that if we fix the axes and planes of symmetry (which must be either cubic or icosahedral), then if we choose any arbitrary point on the sphere to be a vertex all other possible vertices are defined. Then we see which such sets of vertices allow for a connected set of edges of all the same length. Once we have our edges, we can use them to build faces and see if any set of those faces make a uniform polyhedron. This essentially allows us to facet not just every Archimedean solid, but every distorted Archimedean solid as well, meaning that our enumeration of the uniform star polyhedra will be complete.

THE VOID: 4D

7. A. Square to A. Cube to A. Tesseract

In this section, we will apply our knowledge of 3D techniques for enumerating polyhedra to the enumeration of 4D polytopes, or polychora. The ultimate goal will be to apply these to the (currently not proven complete) enumeration of the 4D uniform star polytopes, to see how the search so far has been conducted.

First, how can we conceive of 4D shapes at all when our universe has only three spacial dimensions? For the sake of this paper, we will simply imagine that in addition to the familiar dimensions of up-down, front-back, and left-right, there are another set (called ana-kata, although these names will not appear much), perpendicular to the other three and which behave exactly like them. We can work out the necessary properties of 4D shapes by analogy to 2D and 3D shapes, and with some additional computation or mathematical thought. Some resources that can help build an intuition of 4D space, and which helped to inspire this paper, include the book *Flatland*[39] and the computer application *4D Toys*[40], as well as the *Stella4D* software used as a source of many images in this paper. The remainder of this paper will assume the reader is comfortable discussing four-dimensional shapes and concepts.

Next, some terminology[41]. A polychoron is a 4D polytope, meaning it is made out of 3d polyhedral *cells* connected in a single closed figure, themselves made of 2d faces, 1d edges and 0d vertices. Two cells meet at each face, while three or more meet at each edge and four or more at each vertex. We say a polychoron is regular if it is vertex-transitive, edge-transitive, face-transitive, and *cell transitive*. The first three are defined similarly to the 3D case, while the last (in parallel to the other cases) means that we can map any cell of the polychoron onto any other cell via a rigid transformation while maintaining how the overall shape lies in 4D space. A polychoron is *uniform* if it is vertex-transitive and all of its 2d faces are regular polygons. This implies that all of its 3D cells are uniform polyhedra.

The prefix *hyper-* is often used to describe a higher dimensional analogue of a lower dimensional shape. For instance, a 4D *hypersphere* is the locus of points in 4D space a fixed distance from a center, making it the 4D analogue of a sphere. Hyper- is sometimes used to describe general n-dimensional versions of shapes and concepts, so one could have 5, 6, or *n* dimensional hyperspheres as well.

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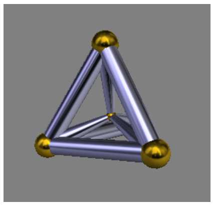
39 Abbott E. A. (1884)

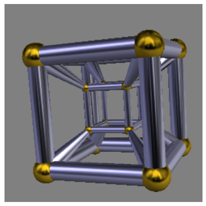
40 Ten Bosch, M. (2017)

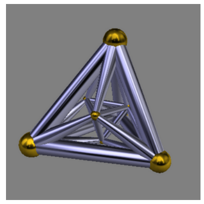
41 Again, definitions can be found in many sources e.g. Olshevsky (2008)

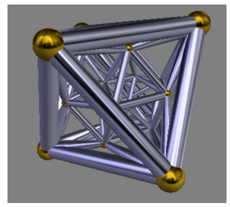
8. Hyper-Plato: The Six Regular Convex Polychora

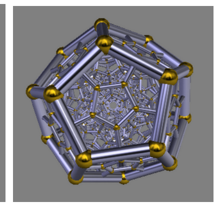
Corresponding to the five Platonic solids, there are six regular convex polychora, which can be described by their number of cells: the 5-cell, 16-cell, and 600-cell whose cells are tetrahedra, the tesseract or 8-cell whose cells are cubes, the 24-cell whose cells are octahedra, and the 120-cell whose cells are dodecahedra.

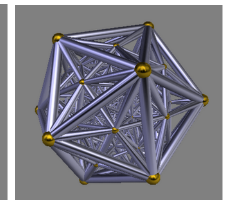

5-cell[42]


8-cell


16-cell

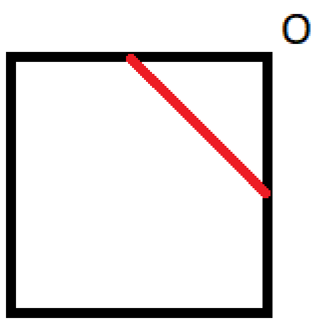

24-cell

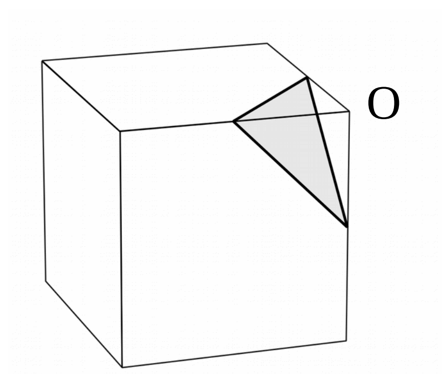

120-cell


600-cell

To prove that there were 5 Platonic solids, we looked at vertex configurations. For the regular convex polychora, we will look at the related notion of a polytope's *vertex figure*: the shape formed at each vertex by the sub-shapes meeting there. Coxeter (in Regular Polytopes)[43] defines this notion recursively:

> "The vertex figure at vertex O of a polygon is the segment joining the mid-points of the two sides through O…The vertex figure of a polyhedron is the polygon whose sides are the vertex figures of all the faces that surround O."





[xxviii]

The vertex figures of a square and a cube.

Then correspondingly, the vertex figure of a polychoron at a vertex O will be the polyhedron whose faces are the vertex figures of all the cells that surround O.

Theorem: There are six regular convex polychora (Hyper-Platonic solids)

Lemma: The vertex figure of a convex regular polychoron is a Platonic solid

Proof of Lemma: Note that a polytope's vertex figure has a *face* for each 3D cell meeting at a vertex, while a polychoron's dual has a *vertex* for each cell meeting at a vertex. This means that the vertex figure of a polychoron *P* will be the dual of the cells of the dual of *P*. Since the dual of a regular polytope is regular, its cells will be regular and the dual of those cells will be

42 As in the introduction, these are 2d renderings of 3d renderings of 4D polychora. With apologies to René Magritte and his not-a pipe, we shall pretend these and similar images are the objects they depict.

43 Coxeter (1973) pp. 15-16, 68

regular. This fact implies that the vertex figure of any vertex of a convex regular polychoron must be a convex regular polyhedron.

Proof of theorem: A regular polychoron must be made out of identical cells, those cells being Platonic solids. Furthermore, by the lemma, those cells must meet in a vertex figure that is itself a Platonic solid.   A convex polychoron must have a combined solid angle less than 4π steradians at each vertex, and a combined angle at each edge of less than 360 degrees.

The vertex figure of a tetrahedron is a triangle, so regular polytopes containing it must have a vertex figure that is a tetrahedron, octahedron, or icosahedron. Tetrahedra meeting in a tetrahedral figure gives the 5-cell, in an octahedral figure gives the 16 cell, and in an icosahedral figure gives the 600-cell.

The vertex figure of a cube is also a triangle, so regular polytopes containing it must also have a vertex figure of a tetrahedron, octahedron, or icosahedron. Four cubes meeting in a tetrahedron at a vertex gives the 8-cell, also known as the tesseract. Eight cubes meeting in an octahedral vertex figure has a solid angle of exactly 4π steradians, and will tile 3-D space in a cubic honeycomb. This implies that 20 cubes will not be convex.

The vertex figure of an octahedron is a square, thus the only possible configuration is six meeting in the shape of a cube. This configuration makes the 24-cell.

The vertex figure of a dodecahedron is again a triangle. Because eight dodecahedra has a solid angle greater then 4π, the only possible vertex figure is four meeting in the shape of a tetrahedron. This makes the 120-cell.

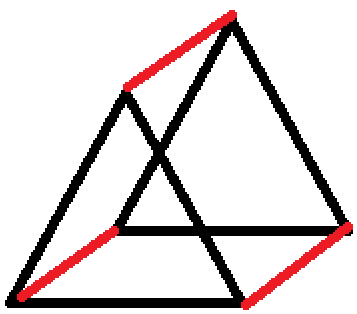
Since the angle between adjacent faces of an icosahedron is more than 120 degrees, three cannot meet at an edge. This implies there is no regular convex polychora with icosahedral cells.

Thus there are precisely six regular convex polychora, and those are the 5-cell, 8-cell (tesseract), 16-cell, 24-cell, 120-cell, and 600-cell.

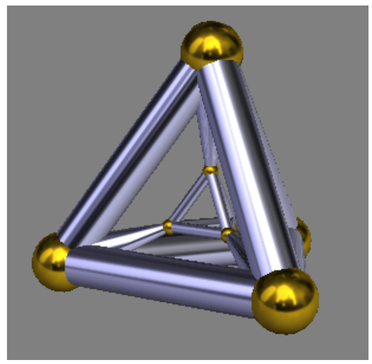
9. Hyper-Archimedes: The convex uniform polychora

Just as with the 3-D case, there are two infinite families of uniform convex polychora, which are essentially 4D analogues of prisms.

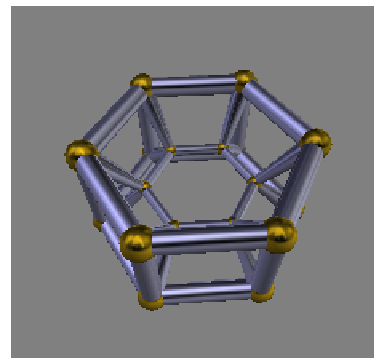
A *n*-gonal prism can be formed in two ways. One is by taking two parallel *n*-gons and connecting them with equal length edges. Another is to take a ring of *n* edges and connect the top and bottom vertices. For the triangular prism below, we can form it by connecting the parallel black triangles with the red edges, or by arranging the red edges in a ring and adding the two black triangles.



A triangular prism



A tetrahedral prism



A (3,6) duoprism

Each of these methods has an analogue in 4d space. First, every 3d polyhedron *P* has a corresponding 4D prism made by taking two parallel copies of *P* and connecting them with equal length edges. This results in figures such as the tetrahedral prism above. Every uniform polyhedron has a uniform polychoron prism, including infinitely many prism prisms and antiprism prisms.

The second method of generating prisms can be generalized into 4D as well. Instead of taking a ring of edges and connecting them, we can take rings of polygons and connect them. The resulting figures are called *duoprisms*[44]. If we take a ring of *m n*-gons and connect them, we get an *(m,n) duoprism*. The same shape will result from a ring of *n m*-gons.

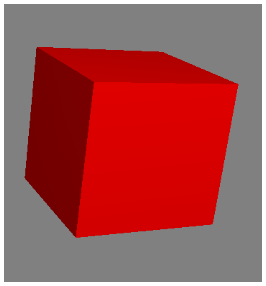
These give us the two infinite families of uniform polychora: the duoprisms and the antiprism prisms. A general uniform polyhedron does not have a uniform polychoron analogue to an antiprism. The *n*-gonal prism prisms will be (4,n) duoprisms, so they need not be counted separately. In addition to these, there are prisms of the 5 Platonic solids and 13 Archimedean solids. Rounding out the list of uniform convex polychora are 47 other convex uniform polychora (including the 6 regular convex polychora), which might be termed the hyper-Archimedean polychora. Note that this list has some overlap: the tesseract is a hyper-Platonic solid, a cubic prism, and (4,4) duoprism, and the octahedron prism is a triangular antiprism prism.

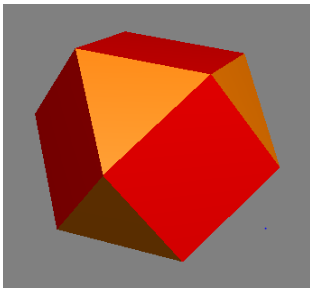
The uniform convex polychora were enumerated by John Conway and Michael Guy in 1965, and the list was proven complete by Marco Möller in his 2004 dissertation. Möller's proof is quite long and involved a computer search of the possible radii for a uniform 4-polytope with a given edge length. It will not be reproduced here due to space constraints. The polychora are listed in *The Symmetries of Things* by Conway, Burgiel, and Goodman-Strauss[45] in the following manner.

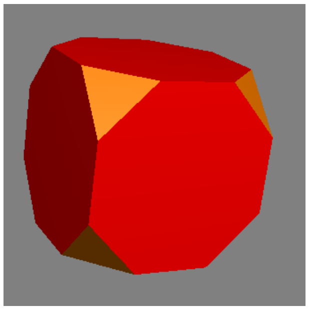
44 Olshevsky (2008). Conway, Burgiel and Goodman-Strauss call these *proprisms* (pp. 391)
45 pp. 389-403

Theorem: There are at least 47 non-prismatic convex uniform polychora

Proof of theorem: We shall list 47 uniform convex polychora in this way: begin with a convex regular polytope. Its *ij-ambo* version can be formed by placing vertices along a line/plane etc. connecting the centers of its *i* and *j* dimensional features. For instance, the 01-ambo tesseract is formed by placing a point *p* along each line between each of a tesseract's vertices *v* and the midpoint of each edge adjacent to *v*, and then forming the convex hull of all the *p*'s. The 0-ambo version of a polytope is just that polytope. Because of duality, the *ij*-ambo version of a n-dimensional polytope *P* is the *(n-1-i)(n-1-j)*-ambo of P's dual.


0-ambo cube


1-ambo cube


01-ambo cube

The following are non-prismatic uniform polychora (duplicate polychora in parentheses):

1-6: 5-cell, 8-cell, 16-cell, 24-cell, 120-cell, 600-cell
7-11: 1-ambo 5-cell, 8-cell, (1-ambo 16-cell is the 24-cell), 24-cell, 120-cell, 600-cell
12-17: 01-ambo 5-cell, 8-cell, 16-cell, 24-cell, 120-cell, 600-cell
18-21: 12-ambo 5 cell, 8-cell (is the 12-ambo 16-cell), 120-cell (is the 12-ambo 600-cell)
22-26: 02-ambo 5-cell, 8-cell, (02-ambo 16-cell is the 1-ambo 24-cell), 24-cell, 120-cell, 600-cell
27-30: 03-ambo 5-cell, 8-cell (is the 03-ambo 16-cell), 120-cell (is the 03-ambo 600-cell)
31-35: 012-ambo 5-cell, 8-cell, (012-ambo 16-cell is the 01-ambo 24-cell), 24-cell, 120-cell, 600-cell
36-41: 013-ambo 5-cell, 8-cell, 16-cell, 24-cell, 120-cell, 600-cell
42-45: 0123-ambo 5-cell, 8-cell (is the 0123-ambo 16-cell), 120-cell (is the 0123-ambo 600-cell)

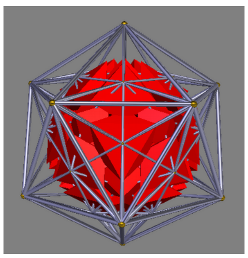
Two additional convex uniform polychora, the semi-snub 24-cell and the grand antiprism, can be described by partial truncations of the 600-cell. Thus, 47 non-prismatic convex uniform polychora exist, in addition to two infinite families of duoprisms and antiprism prisms and the prisms of the Archimedean and Platonic solids.

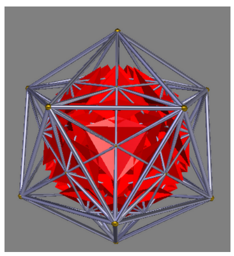
10. Schläfli and Hess: The Ten Regular Star Polychora

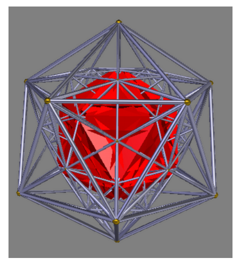
Ludwig Schläfli first described the convex regular polychora, and four out of the ten nonconvex ones as well. Edmund Hess finished the list in an 1883 book[46]. These can be derived from the convex regular polychora by faceting them, analogous to how we derived the Kepler-Poinsot polyhedra from the Platonic Solids. Because the proof is nearly identical we will not reproduce it here. All of these are facetings of the 120-cell or the 600-cell, and therefore share symmetries with that dual pair of polychora.

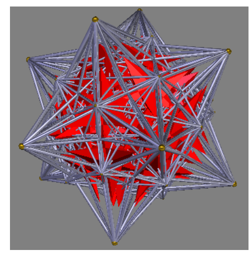
In naming these shapes, we use the same stellating (extending 1d edges) and greatening (extending 2d faces) from the 3d star polyhedra. There is an additional operation in 4d, *aggrandizement*, in which one extends the 3d cells of a polytope until they meet in a new valid polytope[47]. We call an aggrandized polytope "grand".

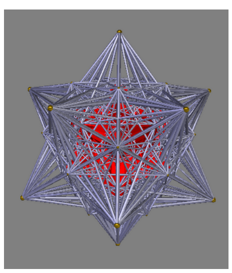
The ten Schläfli-Hess polychora are:

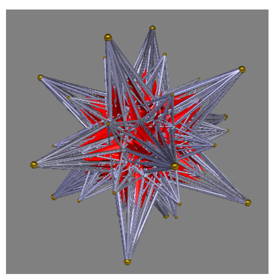

Icosahedral 120-cell

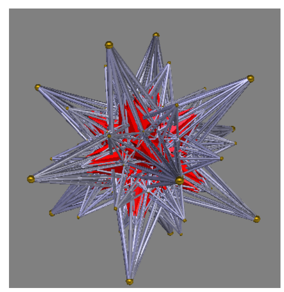

Great 120-cell

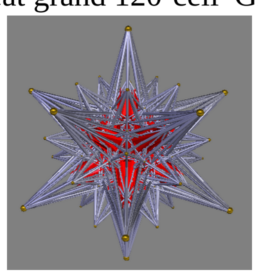

Grand 120-cell

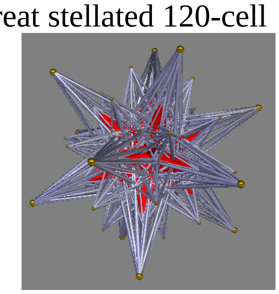

Small stellated 120-cell

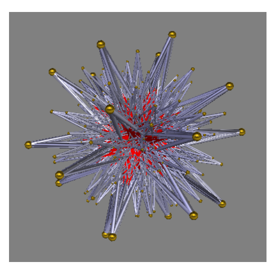

Great grand 120-cell


Great stellated 120-cell


Grand stellated 120-cell


Great icosahedral 120-cell


Grand 600-cell


Great grand stellated 120-cell

46 "Regular Polychoron", *Wolfram Mathworld*
47 Olshevsky (2008)

11. 1849 and counting: The Uniform Star Polychora

We come at last to the final grouping of polytopes this paper will consider: the uniform star polychora. These are four-dimensional polytopes that are vertex transitive and have regular faces, and they are allowed to self-intersect. As of the last count kept by Jonathan Bowers[48], there are *1849* of these. We will examine the techniques used to find them, all of which are analogous to techniques used for the smaller categories.

First, we can get some of the uniform star polychora by modifying other polychora. This includes stellation, greatening, or aggrandizement of convex uniform polychora. It also includes 4D-analogues of polyhedral transformations like truncation and cantellation, and a version that cuts faces as well called *runcintaion*. Jonathan Bowers' acronym-based naming scheme relies on this method of generating polychora: first we describe a shape using its transformations, then we abbreviate that name into something pronounceable. For instance, the rectified five-cell (pentachoron) becomes the acronym RP pronounced "rap".[49]

We can also get some uniform star polyhedra by faceting the convex uniform polychora, although as with the 3D case this will not provide a complete list because the convex hull of a uniform star polychoron might be a distortion of a hyper-Archimedean solid.

A very fruitful method has been that of vertex figures and configurations, the method we used to classify the convex regular polychora. Robert Webb's Stella4D software, the source of many of this paper's images, generates polychora directly from the cells that meet at each of its vertices and the figure that they make. Jonathan Bowers and George Olshevsky on their respective websites have searched in great detail using this method, and are responsible for most of the known uniform star polychora. There are unfortunately no obvious bounds to which vertex figures must be considered, especially when the prisms and antiprisms are involved.

Conway, Burgiel, and Goodman-Strauss's -ambo naming scheme, which we used for the convex uniform polychora, is mathematically equivalent to Wythoff's kaleidoscopic method of finding (hyper-)spherical tilings using reflections around a basic triangle, or in the 4D case a tetrahedron[50]. While this only directly is capable of searching for convex shapes, it has been extended to the 3d nonconvex case, and might be further usable in the 4d nonconvex case. Examples like the grand antiprism, however, show that Wythoff's method is not capable of directly constructing every uniform polytope.

What about Skilling's algebraic technique to find the nonconvex uniform polyhedra? This seems like it would be the most directly applicable method to the question at hand, but we run into a problem of scale. Remember the icosahedron (and its dual the dodecahedron) has 120 symmetries that need to be checked. The 600-cell (and dual the 120-cell) has a whopping *14400*: a given tetrahedral cell of the 600 cell can be mapped onto any of the 600 cells in one of 24 ways, giving 14400 symmetries. Since four of those symmetries are needed to describe a vertex and an edge length, our unpruned list of possible edge arrangements to examine is 14400 choose 4 or nearly 1.8 quadrillion.

This same number, 14400 choose four, could also be arrived at by looking for the four points needed to define a 3D-cell of a polytope with 600-cell symmetry. This makes sense because of duality:

48 http://www.polytope.net/hedrondude/polychora.htm
49 Ibid
50 Conway, Burgiel, Goodman-Strauss (2008) pp. 389

instead of defining the typical vertex of a uniform polytope, we could be defining the typical *face* of a dual-uniform polytope. Since not all the faces of dual-uniform polytopes are tetrahedrons, this will necessarily have a good deal of redundancy, as several combinations of four points could define the same 3D hyperplane that in turn defines the generic 3D cell.

Obviously, just as with Skilling's proof, there may be a good deal of room to pare this massive number down to something more manageable. We could also first examine the tamer 120 symmetries of the 5-cell, 384 of the 8 and 16-cells, and 1152 of the 24-cell. Even with today's best computer technology, however, a purely brute force solution is not feasible.

12. Next steps

The obvious immediate next goal is to complete a classification of the uniform star polychora, or at least establishing an upper bound on their number smaller than 1.8 quadrillion[51]. Skilling managed to pair down a theoretical 120 choose 3 = 280,840 to only 894. If we could get a similar reduction of half of the orders of magnitude, we may be looking only at several hundred million systems of five equations, which should be quite solvable by modern computers. The methods by which we are able to pare down may be somewhat more complex however, for instance given that 4d-objects have planes of rotational symmetry rather than merely axes.

Möller's method of searching by radii, which solved to the convex uniform 4d case, may also be a fruitful avenue of pursuit, although it may require the involvement of a mathematician with a higher level of competency in German than the author of this present paper.

Another approach may be to look in more detail at Wythoff's method, which we only covered briefly and informally in this paper. In the 3d case, only a single uniform non-convex polyhedron failed to be generatable by Wythoff's method, and all such known non-Wythoffian polytopes are simple alterations of Wythoffian ones. Even a full enumeration of Wythoffian uniform star polychora would be a major step forward.

In addition to stellation, greatening and aggrandizement, which were covered in this paper, other transformations that take one polytope to another could provide sources of new polytopes to study, for instance truncation, cantellation, and runcination, which are operations made by cutting off vertices, edges, and faces respectively.

We could even hope for divine inspiration: Jonathan Bowers recalls an anecdote in which the acronym "Iquipadah" popped into his mind while at church, and later he found a shape matching that acronym – the inverted quasiprismadosis hexadecachoron[52]. This may not prove to be a repeatable method of search.

Besides classifying the 4D star polytopes, we could look at how to depict them in 2D or 3D figures. Specifically, there are several different valid methods of filling and coloring the faces that will lead to different representations.

We could also look at yet higher numbers of dimensions. In each of the dimensions 5 and above there are only 3 regular polytopes, all of which are convex: the *simplexes*, which are the equivalent of triangles, tetrahedra, and 5-cells; the *hypercubes*, which are the equivalent of squares, cubes, and tesseracts; and the *orthoplexes*, which are the duals of the hypercubes and therefore equivalents of squares, octahedra and 16-cells[53]. There are of course many interesting uniform cases to consider, and even enumerating the convex uniform cases remains incomplete in five dimensions.

Additionally, a further examination of mirror-regular compounds might be warranted, as they do not seem adequately covered in the literature. Specifically, while it is clear that askew compounds of polygons are mirror-regular, they don't (at the time of writing) appear illustrated on lists of regular star

51 Even though this was technically only an upper bound on the ones with 600-cell symmetry, because of the hyper-exponential nature of combinatorics these account for the vast majority of possible cases, so 14400 choose 4 + 1152 choose 4 + 384 choose 4 + 120 choose 4 is still under 1.8 quadrillion.

52 Bowers (2006-2020)

53 Coxeter (1973) pp. 136

polygons and compounds that can be found on sites like Wikipedia. There also does not seem to yet be an analysis of which of the 4D compounds are mirror-regular versus rotationally regular, and it may turn out that there are other useful distinctions to be made beyond these.

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