Math 112: Real Analysis– Homework 5
Due Tuesday, March 8, 2016

(1) Rudin, Chapter 2, Problems 19
(2) Rudin, Chapter 3, Problems 1, 2, 4, 5
(3) Prove that a compact metric space has a countable dense subset

OPTIONAL CHALLENGE PROBLEMS

(C1) Rudin, Chapter 3, Problems 23, 24 ,25. This sequence of problems shows how, given a metric space \((X, d)\) one can construct another metric space \((\hat{X}, \hat{d})\) with the following properties

- there is a map \(\iota : X \hookrightarrow \hat{X}\) – that is, we can view \(X\) as a subset of the new metric space \(\hat{X}\)
- View \(X\) as a subset of \(\hat{X}\), if we restrict the metric \(\hat{d}\) to \(X\), we get back the metric on \(X\). That is,
  \[(\iota(X), \hat{d}|_X) = (X, d)\].
- \((\hat{X}, \hat{d})\) is complete.

This gives one method for constructing the real numbers from the rationals.

(C2) Let \((X, d)\) be a metric space. For any compact subset \(A \subset X\), and any \(\epsilon > 0\) we set

\[B_\epsilon(A) := \bigcup_{p \in A} B_\epsilon(p)\].

This is the “\(\epsilon\)-fattening” of \(A\) (draw a picture). For \(Y, Z\) compact subsets of \(X\) define the \textit{Hausdorff distance} between \(Y\) and \(Z\) by

\[d_H(Y, Z) := \inf \{\epsilon > 0 | Y \subset B_\epsilon(Z), \ Z \subset B_\epsilon(Y)\}\, .\]

(a) Show that \(d_H\) defines a metric on the set \(\hat{X} := \{A \subset X | A\ \text{is compact}\}\).
(b) Define $\tilde{X} := \{ A \subset X | A \text{ is compact} \}$. Show that $(\tilde{X}, d_H)$ is a compact metric space if and only if $(X, d)$ is compact, by using the following hints:

Hint 1 It suffices to show that every infinite subset of $\tilde{X}$ contains a limit point. Let $\{ A_j \}$ be an infinite set of points in $\tilde{X}$. Since $A_j \subset X$ is compact, by problem (3) above, there exists a countable dense subset

$$\{ p^j_k \}_{k \in \mathbb{N}} \subset A_j.$$ 

These points form a sort of “skeleton” of each $A_j$.

Hint 2 Consider the set $E_1 := \{ p^j_1 \}_{j \in \mathbb{N}}$, which is an infinite subset of $X$. Argue that $E_1$ has a limit point $p_1 \in X$. In particular, find a subsequence $j_\ell$ so that

$$p^{j_\ell}_1 \in B_{\frac{1}{\ell}}(p_1)$$

Consider the sequence of sets $\{ A_{j_\ell} \}$. It suffices to show that this set has a limit point. We reindex $j_\ell \rightarrow j$ (for notational simplicity). Then we have a sequence of points $\{ p^j_2 \}$. Repeating the above argument we obtain a limit point $p_2$, and after reindexing we have

$$p^{j_\ell}_1 \in B_{\frac{1}{\ell}}(p_1) \quad p^{j_\ell}_2 \in B_{\frac{1}{\ell}}(p_2)$$

Continue this to find points $E_\infty := \{ p_k \}_{k \in \mathbb{N}}$. Let $A$ be the closure of $E_\infty$. Show that, up to taking a subsequence, $A_{j_\ell}$ converges to $A$ in the metric $d_H$. 

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