Definition
A pair \((X, d)\) is called a *metric space* if
- \(X\) is a set, whose elements we shall call *points*.
- \(d : X \times X \to \mathbb{R}\) is a function called the *distance function*.
- For any two points \(p, q \in X\),
  1. \(d(p, q) > 0\) if \(p \neq q\) and \(d(p, p) = 0\).
  2. \(d(p, q) = d(q, p)\)
  3. \(d(p, q) \leq d(p, q) + d(q, r)\) for any \(r \in X\)

Condition (c) is called the *triangle inequality*.

Theorem
*Notice that if \((X, d)\) is a metric space and \(Y \subseteq X\) then so is \((Y, d)\).*
$\mathbb{R}^k$ as a Metric Spaces

The most important example of metric spaces, for us, are Euclidean Spaces.

**Definition**
For all $x, y \in \mathbb{R}^k$ let $d(x, y) = |x - y|$. Then $(\mathbb{R}^k, d)$ is a metric space.
Convex Sets

Definition

The segment \((a, b)\) is the set \(\{x : a < x < b\}\). The interval \([a, b] = \{x : a \leq x \leq b\}\). The half open interval \([a, b) = \{x : a \leq x < b\}\) and \((a, b] = \{x : a < x \leq b\}\).

Definition

If \(a_i < b_i\) for \(i = 1 \ldots k\), then \(\{(x_1, \ldots, x_k) : a_i \leq x_i \leq b_i\} \subseteq \mathbb{R}^k\) is called a \(k\)-cell.

So a 1-cell is an interval, a 2-cell is a rectangle, etc.
Balls

Definition
If \((X, d)\) is a metric space, \(x \in X\) and \(r > 0\) is a real then

- \(B(x, r) = \{y \in X : d(x, y) < r\}\) is the open ball of radius \(r\) at \(x\)
- \(\overline{B}(x, r) = \{y \in X : d(x, y) \leq r\}\) is the closed ball of radius \(r\) at \(x\)
Convex Sets

**Definition**
We say a set $E \subseteq \mathbb{R}^k$ is convex if for all $x, y \in E$ and all $0 < \gamma < 1$

$$\gamma x + (1 - \gamma) y \in E$$

**Theorem**
Any ball (open or closed) in $\mathbb{R}^k$ is convex.

**Theorem**
All $k$-cells are convex.
Definitions For a Metric Space

Definition
Let \((X, d)\) be a metric space. All points and sets mentioned are elements or subsets of \(X\).

(a) A \textit{neighborhood} of \(p\) is a set \(N_r(p)\) consisting of all \(q\) such that \(d(p, q) < r\) for some \(r > 0\).

(b) A point \(p\) is a \textit{limit point} of the set \(E\) if every neighborhood of \(p\) contains a point \(q \neq p\) such that \(q \in E\).

(c) If \(p \in E\) and \(p\) is not a limit point of \(E\) then \(p\) is called an \textit{isolated point} of \(E\).

(d) \(E\) is \textit{closed} if every limit point of \(E\) is a point of \(E\).
Definition

(e) A point $p$ is an interior point of $E$ if there is a neighborhood $N$ of $p$ such that $N \subseteq E$.

(f) $E$ is open if every point of $E$ is an interior point of $E$.

(g) The complement of $E$ (denoted by $E^c$) is the set of all points $p \in X$ such that $p \notin E$.

(h) $E$ is perfect if $E$ is closed and if every point of $E$ is a limit point of $E$. 
Definitions For a Metric Space

Definition

(i) $E$ is bounded if there is a real number $M$ and a point $q \in X$ such that $d(p, q) < M$ for all $p \in E$.

(j) $E$ is dense in $X$ if every point of $X$ is a limit point of $E$ or a point of $E$ (or both).

Note that in $\mathbb{R}^1$ neighborhoods are segments, in $\mathbb{R}^2$ neighborhoods are interiors of circles, and in $\mathbb{R}^k$ neighborhoods are interiors of $k$-spheres.
Open Neighborhoods

Theorem

Every neighborhood is open
Limit Points

Theorem
If \( p \) is a limit point of a set \( E \), then every neighborhood of \( p \) contains infinitely many points of \( E \).

Corollary
A finite set has no limit points
Closed Sets:

- $\mathbb{R}^2$. This is also a perfect set.
- The set of all complex numbers $z$ with $|z| \leq 1$. This is also perfect and bounded.
- Any non-empty finite set. This is also bounded.
- The set of integers.

Note the set $E = \{1/n : n \in \mathbb{N}\}$ is not closed (nor perfect) because while it has a limit point ($z = 0$) no point of $E$ is a limit point of $E$. 
Examples of Open Sets

Open Sets:

- $\mathbb{R}^2$. This is also a perfect set.
- The set of all complex numbers $z$ with $|z| < 1$. This is bounded.

Note that the interval $(a, b)$, while open in $\mathbb{R}^1$ is not open when considered as a subset of $\mathbb{R}^2$. 
Theorem

Let \( \{ E_\alpha : \alpha \in I \} \) be a (finite or infinite) collection of sets. Then

\[
(\bigcup_{\alpha \in I} E_\alpha)^c = \bigcap_{\alpha \in I} E_\alpha^c
\]
Opens and Closed Sets

**Theorem**

*A set \( E \) is open if and only if its complement is closed.*

**Corollary**

*A set \( F \) is closed if and only if its complement is open.*
Theorem

(a) For any collections \( \{ G_\alpha : \alpha \in I \} \) of open sets \( \bigcup_{\alpha \in I} G_\alpha \) is open.

(b) For any collections \( \{ F_\alpha : \alpha \in I \} \) of closed sets \( \bigcap_{\alpha \in I} F_\alpha \) is closed.

(c) For any finite collection \( G_1, \ldots, G_n \) of open sets \( \bigcap_{i=1}^n G_n \) is open.

(d) For any finite collection \( F_1, \ldots, F_n \) of closed sets \( \bigcup_{i=1}^n F_n \) is closed.

Let \( E_n = (-1/n, 1/n) \). Then \( \bigcap_{n \in \mathbb{N}} E_n = \{0\} \) which isn’t open. So the intersection of infinitely many open sets may not be open.
Closure

Definition
If $(X, d)$ is a metric space and $E \subseteq X$ then let $E'$ be the set of limit points of $E$ in $X$. The closure of $E$ is $\overline{E} = E \cup E'$.

Theorem
If $(X, d)$ is a metric space and $E \subseteq X$ then
(a) $\overline{E}$ is closed
(b) $E = \overline{E}$ if and only if $E$ is closed.
(c) $\overline{E} \subseteq F$ for every closed set $F \subseteq X$ such that $E \subseteq F$. 
Bounded Closed Sets

**Theorem**

Let $E$ be a non-empty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \overline{E}$. Hence $y \in E$ if $E$ is closed.
Definition
Suppose \((X, d)\) is a metric space with \(E \subset Y \subset X\). We say that \(E\) is open relative to \(Y\) if for all \(p \in E\) there is an \(r_p > 0\) such that for all \(q \in Y\) if \(d(p, q) < r_p\) then \(q \in E\).

So \(E\) is open relative to \(Y\) if and only if \(E\) is open in the metric space \((Y, d)\).

Theorem
Suppose \((X, d)\) is a metric space and \(Y \subseteq X\). A subset \(E\) of \(Y\) is open relative to \(Y\) if and only if \(E = Y \cap G\) for some open subset \(G\) of \(X\).