Classification of Surfaces (Lecture 33)

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In this lecture, we will (belatedly) discuss the classification of 2-manifolds, which we have frequently used in our discussion of 3-manifolds. We begin with the oriented case.

**Theorem 1.** Let $\Sigma$ be a connected compact oriented surface. Then $\Sigma$ can be obtained as a connected sum $T \# T \# \cdots \# T$ of $g$ copies of the torus $T$, for some $g \geq 0$.

The integer $g$ is called the *genus* of the surface $\Sigma$. It is a topological invariant of $\Sigma$: a simple calculation shows that $\chi(\Sigma) = 2 - 2g$.

The proof will require a few preliminaries.

**Lemma 2.** Let $\Sigma$ be a connected compact surface. Then $\chi(\Sigma) \leq 2$, and equality holds if and only if $\Sigma$ is a 2-sphere.

*Proof.* We have $\chi(\Sigma) = b_0 - b_1 + b_2$, where $b_i$ denotes the $i$th Betti number of $\Sigma$. Since $\Sigma$ is connected, we have $b_0 = 1$, and $b_2$ is either 1 or 0 depending on whether $\Sigma$ is orientable or nonorientable. It follows that

\[
\chi(\Sigma) = \begin{cases} 
2 - b_1 & \text{if } \Sigma \text{ is orientable} \\
1 - b_1 & \text{if } \Sigma \text{ is nonorientable}.
\end{cases}
\]

This proves the inequality. If equality holds, then $\Sigma$ must be orientable, and therefore admits a complex structure. As we explained in a previous lecture, a Riemann surface with $\chi(\Sigma) = 2$ must be biholomorphic to the Riemann sphere, and in particular is a topological sphere.

The following can be regarded as a baby version of the loop theorem:

**Lemma 3.** Let $\Sigma$ be a connected surface and let $N \subset \pi_1 \Sigma$ be a proper normal subgroup. Then there is an embedded loop $f : S^1 \to \Sigma$ such that $[f] / \in N$.

*Proof.* Since $N$ is proper, we can choose a closed loop $f : S^1 \to \Sigma$ such that $[f]$ (which is well-defined up to conjugacy) does not belong to $N$. Without loss of generality, we may assume that $f$ is in general position. Then $f$ is an immersion with a finite number $k$ of double points. We will assume that $f$ has been chosen minimally. If $k = 0$, then $f$ is an embedding and we are done. Otherwise, there exist $x, y \in S^1$ with $x \neq y$ but $f(x) = f(y)$. The points $x$ and $y$ partition $S^1$ into two intervals $I_0$ and $I_1$. The restrictions of $f$ to $I_0$ and $I_1$ give two other loops $f_0, f_1 : S^1 \to \Sigma$. Since each of these loops has a smaller number of double points, the minimality of $k$ guarantees that $[f_0], [f_1] \in N$. We now conclude by observing that $[f]$ belongs to the normal subgroup of $\pi_1 \Sigma$ generated by $[f_0]$ and $[f_1]$, and therefore also belongs to $N$, which contradicts our assumption.

We now prove Theorem 1. We proceed by descending induction on $\chi(\Sigma)$. If $\chi(\Sigma) \geq 2$, then Lemma 2 implies that $\chi(\Sigma) = 2$ and $\Sigma$ is a 2-sphere. We may therefore assume that $\chi(\Sigma) = 2 - b_1 < 2$, so that $H_1(\Sigma; \mathbb{Z}) \neq 0$. It follows that the commutator subgroup $[\pi_1 \Sigma, \pi_1 \Sigma]$ is a proper subgroup of $\pi_1 \Sigma$. Using Lemma 3, we can choose an embedded loop $f : S^1 \to \Sigma$ which represents a nontrivial class in $H_1(\Sigma; \mathbb{Z})$. It follows that $f$ must be nonseparating, so that the surface $\Sigma'$ obtained by cutting $\Sigma$ along $f$ is connected.
Let $\Sigma''$ be the closed surface obtained by capping off the boundary circles of $\Sigma'$. A simple calculation shows that
$$\chi(\Sigma'') = 2 + \chi(\Sigma') = 2 + \chi(\Sigma).$$
By the inductive hypothesis, $\Sigma''$ can be realized as a connected sum $T\# T\# \ldots \# T$.

The surface $\Sigma$ can be obtained from $\Sigma''$ by removing small disks $D_x$ and $D_y$ around two points $x, y \in \Sigma''$ (to obtain $\Sigma'$), and then gluing the boundary of these disks together. Without loss of generality, we may assume that $x$ and $y$ are close to one another, so that $D_x$ and $D_y$ are contained in a larger disk $D$. Let $K_0$ be the surface with boundary obtained from $\Sigma''$ by removing the interior of $D$, and let $K_1$ be the surface obtained from $D$ by removing the interiors of $D_x$ and $D_y$ and identifying their boundary. Then $\Sigma = K_0 \coprod_{S^1} K_1$, so we can identify $\Sigma$ with the connected sum of two surfaces $K_0$ and $K_1$ obtained by capping off the boundary circles of $K_0$ and $K_1$. We note that $K_0 \simeq \Sigma''$, and a simple calculation shows that $K = T$ (if we like, we can take this to be a definition of the 2-manifold $T$). We then obtain
$$\Sigma \simeq \Sigma'' \# T \simeq T \# T \# \ldots \# T$$
as desired.

We now treat the case of a nonorientable 2-manifold.

**Theorem 4.** Let $\Sigma$ be a closed connected nonorientable 2-manifold. Then $\Sigma$ can be obtained as a connected sum $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2$ for some $k \geq 1$.

**Remark 5.** In the situation of Theorem 4, the integer $k$ is uniquely determined: a simple calculation of Euler characteristics shows that $\chi(\Sigma) = 2 - k$.

**Warning 6.** A priori, the connected sum $X \# Y$ of two surfaces $X$ and $Y$ is not well-defined: it depends on a choice of identification of the boundary circles of punctured copies of $X$ and $Y$. This issue did not arise in the statement of Theorem 1, because in the orientable case there is a unique choice of identification which allows us to orient $X \# Y$ in a manner compatible with given orientations of $X$ and $Y$ (which we were implicitly using). It also does not matter in the case of Theorem 4, for a different reason: there exists a diffeomorphism of $\mathbb{R}P^2$ which fixes a point $x$ and induces an orientation reversing automorphism of the tangent space at $x$. Namely, we observe that $\mathbb{R}P^2 = (\mathbb{R}^3 - \{0\}) / \mathbb{R}^\times$ carries an action of the orthogonal group $O(3)$: any reflection in $O(3)$ will do the job.

We now prove Theorem 4. The proof proceeds by descending induction on $\chi(\Sigma)$ (which is at most 1, by virtue of Lemma 2). Since $\Sigma$ is nonorientable, the 1st Stiefel-Whitney class $w_1 \in H^1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ induces a nontrivial map $\pi_1 \Sigma \to \mathbb{Z}/2\mathbb{Z}$. Let $N$ be the kernel of this map, so that $N$ is a proper normal subgroup of $\pi_1 \Sigma$. Using Lemma 3, we obtain an embedded loop $f : S^1 \to \Sigma$ such that $[f] \notin N$. Consequently, the restriction of $w_1$ to $S^1$ is nontrivial: this means that the normal bundle to the embedding $S^1 \to \Sigma$ is nontrivial, so that $S^1$ is a one-sided loop in $\Sigma$. Let $K$ be a tubular neighborhood of $S^1$: then $K$ is a Mobius band, whose boundary is another circle $C$. Let $\Sigma'$ be the surface obtained from $\Sigma$ by removing the interior of $K$, and let $\tilde{\Sigma}'$ and $\tilde{K}$ be the closed surfaces obtained by capping off the boundary circles of $K$ and $\Sigma'$. Then $\tilde{K} = \mathbb{R}P^2$ (if you like, you can take this to be the definition of $\mathbb{R}P^2$, and we have $\Sigma \simeq \tilde{\Sigma}' \# \mathbb{R}P^2$. A simple calculation with Euler characteristics shows that $\chi(\Sigma) = \chi(\tilde{\Sigma}') + \chi(\mathbb{R}P^2) - 2 = \chi(\tilde{\Sigma}') - 1$.

There are now two cases to consider. If $\tilde{\Sigma}'$ is nonorientable, then the inductive hypothesis implies that $\tilde{\Sigma}'$ is a connected sum of finitely many copies of $\mathbb{R}P^2$: it then follows that $\Sigma$ is a connected sum of finitely many copies of $\mathbb{R}P^2$. If $\tilde{\Sigma}'$ is orientable, then we apply Theorem 1 to deduce that $\tilde{\Sigma}'$ is a connected sum of $g$ copies of the torus $T$, for some $g \geq 0$. If $g = 0$, then $\tilde{\Sigma}' \simeq S^2$, so that $\Sigma \simeq S^2 \# \mathbb{R}P^2 \simeq \mathbb{R}P^2$. The case $g > 0$ is handled through repeated application of the following Lemma:

**Lemma 7.** There is a diffeomorphism $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 \simeq T \# \mathbb{R}P^2$. 

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Proof. Choose a pair of embedded circles \( C, C' \subset T \) which meet transversely in one point \( x \). Let us identify \( T \# \mathbb{R}P^2 \) with the 2-manifold obtained from \( T \) by removing a small disk \( D \) around \( x \), and gluing on a Mobius band \( K \) along the boundary \( \partial D \). Then \( C - C \cap D \) and \( C' - C' \cap D \) can be extended to nonintersecting embedded loops \( \overline{C} \) and \( \overline{C}' \) on \( T \# \mathbb{R}P^2 \), both of which are one-sided. Using the preceding arguments, we deduce that there exists a decomposition

\[
T \# \mathbb{R}P^2 \simeq (\mathbb{R}P^2 \# \mathbb{R}P^2) \# \Sigma,
\]

where \( \Sigma \) is the surface obtained by removing tubular neighborhoods of \( \overline{C} \) and \( \overline{C}' \) and capping of their boundary components. A simple calculation shows that \( \chi(\Sigma) = 1 \), so that \( \Sigma \) must be nonorientable: we therefore have \( \Sigma \simeq \mathbb{R}P^2 \# \Sigma' \). Then \( \chi(\Sigma') = 2 \), so that \( \Sigma' \) is a 2-sphere (Lemma 2). It follows that \( \Sigma \simeq \mathbb{R}P^2 \) so that

\[
T \# \mathbb{R}P^2 \simeq \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2
\]

as desired. \( \Box \)

Remark 8. In the next few lectures, we will need to understand not only closed 2-manifolds, but also 2-manifolds with boundary. However, it is easy to extend the above classification: the boundary of a (compact) 2-manifold \( \Sigma \) is a compact 1-manifold, hence a union of finitely many circles. If we let \( \Sigma' \) be the 2-manifold obtained by capping off these boundary circles, then \( \Sigma' \) is diffeomorphic to a 2-manifold of the form

\[
T \# T \# \ldots \# T \quad \mathbb{R}P^2 \# \mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2,
\]

and \( \Sigma \) is obtained from \( \Sigma' \) by removing small disks around finitely many points.

Remark 9. Let \( \Sigma \) be a compact connected 2-manifold (possibly nonorientable or with boundary). The properties of \( \Sigma \) depend strongly on the sign of the Euler characteristic \( \chi(\Sigma) \). It is therefore convenient to list the possibilities for \( \Sigma \) when \( \chi \) is nonnegative:

- If \( \chi(\Sigma) = 2 \), then \( \Sigma \simeq S^2 \) (Lemma 2).
- If \( \chi(\Sigma) = 1 \), then either \( \Sigma \simeq \mathbb{R}P^2 \) or \( \Sigma \simeq D^2 \).
- If \( \chi(\Sigma) = 0 \), there are several possibilities. If \( \Sigma \) is orientable, then either \( \Sigma \simeq T \) or \( \Sigma \) is a twice-punctured sphere (an annulus \( S^1 \times [0, 1] \)). Each of these possibilities has a nonorientable analogue: if \( \Sigma \) is nonorientable and has boundary, then it is diffeomorphic to a punctured copy of \( \mathbb{R}P^2 \): this is a Mobius band, given by a nonorientable \([0, 1]\)-bundle over \( S^1 \). If \( \Sigma \) is nonorientable and closed, then it is diffeomorphic to the Klein bottle \( \mathbb{R}P^2 \# \mathbb{R}P^2 \). This 2-manifold can be viewed as obtained by gluing together two Mobius bands along their boundary, which realizes it as a nonorientable \( S^1 \)-bundle over \( S^1 \) (alternatively, one can start with the surface \( \Sigma \) which is a nonorientable \( S^1 \)-bundle over \( S^1 \); then \( \chi(\Sigma) = 0 \) so that Theorem 4 guarantees a diffeomorphism \( \Sigma \simeq \mathbb{R}P^2 \# \mathbb{R}P^2 \).
- If \( \chi < 0 \), then we are in the “generic case”.