The Wall Finiteness Obstruction (Lecture 2)

August 28, 2014

We begin with the following:

**Question 1.** Let $M$ be a compact manifold. Does $M$ have the homotopy type of a finite CW complex?

Of course, if $M$ is triangulable, then it is actually *homeomorphic* to a finite simplicial complex. This provides an affirmative answer when $M$ is smooth (since any smooth manifold can be triangulated).

For general topological manifolds, Question 1 is not so easy to answer. As a starting point, we note that any (paracompact) topological manifold $M$ has the homotopy type of a (possibly infinite) CW complex $X$: this follows from the theory of absolute neighborhood retracts, which we will review in a subsequent lecture. Of course, we should not expect that $X$ can be chosen finite in the case where $M$ is noncompact (for a counterexample, consider the “surface of infinite genus”).

Let us fix a homotopy equivalence $f : M \to X$, where $X$ is a CW complex. If $M$ is compact, then $f(M)$ is contained in a finite subcomplex $X_0 \subseteq X$.

**Exercise 2.** Prove this.

Let $g$ be a homotopy inverse to $f$. Then the composite map

$$X \xrightarrow{f g} X_0 \hookrightarrow X$$

is homotopic to the identity map from $M$ to itself. This motivates the following:

**Definition 3.** Let $X$ be a CW complex (or, more generally, a space with the homotopy type of a CW complex). We say that $X$ is *finitely dominated* if it is a retract (in the homotopy category of CW complexes) of a finite CW complex $Y$. In other words, $X$ is finitely dominated if there exists a finite CW complex $Y$ and a pair of maps

$$i : X \to Y \quad r : Y \to X$$

such that $r \circ i$ is homotopic to the identity map from $X$ to itself.

Of course, every finite CW complex $X$ is finitely dominated: we can take $Y = X$ and the maps $i$ and $r$ to be the identity. More generally, if $X$ is homotopy equivalent to a finite CW complex, then $X$ is finitely dominated.

**Question 4.** Let $X$ be a finitely dominated CW complex. Does $X$ have the homotopy type of a finite CW complex?

We will see that the answer to Question 4 is “no” in general, but “yes” in many cases (for example, when $X$ is simply connected). Moreover, the *topological* question of deciding whether or not $X$ is finitely dominated will be reduced to an *algebraic* one (the vanishing of a certain $K$-theory class).

**Remark 5.** It is not true in general that every finitely dominated space $X$ has the homotopy type of a finite CW complex. Nevertheless, it can be shown that every compact manifold has this property: that is, Question 1 has an affirmative answer, though we will not establish that in this lecture (a proof is given in Kirby-Siebenmann; if time permits, we’ll discuss a stronger result later in this course).
We begin by summarizing some of the finiteness properties enjoyed by finitely dominated spaces.

**Lemma 6.** Let $X$ be a finitely dominated space. Then:

(a) The set $\pi_0 X$ is finite.

(b) For each base point $x \in X$, the group $\pi_1(X, x)$ is finitely presented.

(c) Let $x \in X$ and let $X^o \subseteq X$ be the path component of $x$. For each abelian group $V$ with an action of the fundamental group $\pi_1(X, x)$, let $H^*(X^o; V)$ denote the cohomology of $X^o$ with coefficients in the local system determined by $V$. Then the construction $V \mapsto H^*(X^o; V)$ commutes with filtered direct limits.

(d) For each $x \in X$ as above, there exists an integer $n$ such that $H^*(X^o; V) \cong 0$ for $* > n$ and any representation $V$ of $\pi_1(X, x)$.

**Proof.** Choose finite CW complex $Y$ and a map $i : X \to Y$ which admits a left homotopy inverse $r$. Note that $\pi_0 Y$ is finite and that the map $\pi_0 X \to \pi_0 Y$ is injective (it has a left inverse given by $r$). This proves (a).

To prove (b), we note that for each $x \in X$ the induced map $i_* : \pi_1(X, x) \to \pi_1(Y, y)$ is split injective. Since $Y$ is a finite complex, the group $\pi_1(Y, y)$ is finitely presented. It follows that the group $\pi_1(X, x)$ is finitely presented, which proves (b). Let $Y^o$ be the path component of $y$. By composition with a left inverse to $i_*$, we see that representations $V$ of $\pi_1(X, x)$ can be extended functorially to representations of $\pi_1(Y, y)$, which we can regard as local systems on $Y^o$. Using cellular cochains to compute $H^*(Y^o; V)$, we see immediately that the construction $V \mapsto H^*(Y^o; V)$ commutes with filtered direct limits. We can functorially identify $H^*(X^o; V)$ with a direct summand of $H^*(Y^o; V)$, so the construction $V \mapsto H^*(X^o; V)$ also commutes with filtered direct limits. This proves (c), and assertion (d) follows if we take $n = \dim(Y)$.

**Remark 7.** Lemma 6 actually characterizes finitely dominated spaces; we will give a proof at the end of this lecture.

We next show that conditions (a) through (c) guarantee that $X$ behaves approximately like a finite-dimensional CW complex.

**Proposition 8.** Let $X$ be a CW complex which satisfies conditions (a), (b), and (c) of Lemma 6. For each integer $n \geq 0$, there exists a finite CW complex $Z$ of dimension $< n$ and an $(n-1)$-connected map $f : Z \to X$.

We will deduce Proposition 8 from the following more precise statement:

**Lemma 9.** Let $X$ be a CW complex which satisfies conditions (a), (b), and (c) of Lemma 6. Suppose we are given an $(n-1)$-connected map $f : Z \to X$, where $Z$ is a finite CW complex. Then there exists another finite CW complex $Z'$, obtained from $Z$ by attaching finitely many $n$-cells, and an $n$-connected map $f' : Z' \to X$ extending $f$. In particular, we have $\dim(Z') \leq \max\{n, \dim(Z)\}$.

For the remainder of this lecture, it will be convenient to always assume that the space $X$ is connected (we can always handle disconnected spaces by considering each connected component separately). Fix a base point $x \in X$, let $G = \pi_1(X, x)$, and let $\tilde{X}$ be a universal cover of $X$ (so that $G$ acts on $\tilde{X}$ by deck transformations).

**Proof of Lemma 9.** If $n = 0$, then we can either take $Z' = Z$ (if $Z$ is nonempty) or $Z' = *$ (if $Z$ is empty).

We next consider the case $n = 1$. If $Z$ is not connected, then we first enlarge $Z$ by adding 1-cells connecting the different components of $Z$ (the map $f$ extends continuously over this enlargement by virtue of our assumption that $X$ is connected). Without loss of generality, we may assume that there exists a 0-cell $z \in Z$ such that $f(z) = x$. We let $Z'$ be obtained from $Z$ by attaching several loops based at the point $z$, and we define $f$ so that it carries these loops to generators of the group $G = \pi_1(X, x)$. Since $G$ is finitely generated, this only requires finitely many 1-cells.
We now consider the case $n = 2$. Let $z \in Z$ be as above, and consider the group homomorphism \( \phi : \pi_1(Z, z) \to G \) induced by $f$. Since $\pi_1(Z, z)$ is finitely generated and $G$ is finitely presented, the kernel \( \ker(\phi) \) is generated as a normal subgroup by finitely many elements of $\pi_1(Z, z)$. Each of these elements can be represented by a loop in the 1-skeleton of $Z$. We may therefore reduce to the case where $\phi$ is an isomorphism.

We can now treat all of the cases $n \geq 2$ in a uniform manner. For any abelian group $V$ with an action of $G$, we let $H_*(X, Z; V)$ and $H^*(X, Z; V)$ denote the homology and cohomology of $X$ relative to $Z$ with coefficients in $V$. Since $f$ is $(n-1)$-connected, the groups $H_*(X, Z; Z[G])$ vanish for $* < n$. It follows from the universal coefficient theorem that we have canonical isomorphisms

\[
H^n(X, Z; V) \simeq \text{Hom}_{Z[G]}(H_n(X, Z; Z[G]), V).
\]

Since $X$ satisfies condition (c) of Lemma 6 and $Z$ is finite CW complex, the construction $V \mapsto H^n(X, Z; V)$ commutes with filtered direct limits (exercise!)

It follows that the construction

\[
V \mapsto \text{Hom}_{Z[G]}(H_n(X, Z_n; Z[G]), V)
\]

also commutes with filtered direct limits. In particular, $H_n(X, Z; Z[G])$ is finitely generated as a module over $Z[G]$.

Set $\tilde{Z} = Z \times_X \tilde{X}$. Applying the relative Hurewicz theorem to the map $\tilde{Z} \to \tilde{X}$, we deduce that the Hurewicz map

\[
\pi_n(X, Z) \simeq \pi_n(\tilde{X}, \tilde{Z}) \to H_n(\tilde{X}, \tilde{Z}; Z) \simeq H_n(X, Z; Z[G])
\]

is an isomorphism. Consequently, the group $\pi_n(X, Z)$ is finitely generated as a $Z[G]$-module. Each element of $\pi_n(X, Z)$ supplies a recipe for attaching an $n$-cell to $Z$ and extending the definition of $f$ over that $n$-cell. Without loss of generality, we may assume that the relevant attaching maps factor through the $(n-1)$-skeleton of $Z$. Let $Z'$ be the CW complex obtained from $Z$ by attaching $n$-cells corresponding to a set of generators of $\pi_n(X, Z)$, so that $f$ extends to an $n$-connected map $f' : Z' \to X$.

For any space $X$ satisfying conditions (a), (b), and (c) of Lemma 6, Lemma 9 allows us to construct a sequence of better and better approximations to $X$. It is condition (d) that will allow us to stop this construction.

**Definition 10.** Let $X$ be a CW complex and let $n \geq 2$ be an integer. We will say that $X$ has homotopy dimension $\leq n$ if it satisfies condition (d) of Lemma 6: that is, if $H^*(X; \mathcal{L})$ vanishes for $* > n$ and any local system of abelian groups $\mathcal{L}$ on $X$.

**Remark 11.** Definition 10 makes sense for any value of $n$, but is not really the right condition when $n = 0$ and $n = 1$: in those cases, one should also require vanishing for “nonabelian” cohomology.

**Lemma 12.** Let $X$ be a CW complex satisfying the conditions of Lemma 6. Let $Z$ be a finite CW complex of dimension $\leq n-1$ and let $f : Z \to X$ be an $(n-1)$-connected map. If $X$ has homotopy dimension $\leq n$, then the homology group $H_n(X, Z; Z[G])$ is a finitely generated projective $Z[G]$-module.

**Proof.** Let $V$ be any abelian group with an action of $G$, which determines local systems on $X$ and $Z$ which we will also denote by $V$. Since $Z$ is $(n-1)$-dimensional, the local cohomology groups $H^n(Z; V)$ vanish for $* \geq n$. Using the exact sequence

\[
H^{n-1}(Z; V) \to H^n(X, Z; V) \to H^n(X; V),
\]

we see that the groups $H^n(X, Z; V)$ vanish for $* > n$. Any exact sequence of representations $0 \to V' \to V \to V'' \to 0$ gives rise to a long exact sequence

\[
H^n(X, Z; V') \to H^n(X, Z; V) \to H^n(X, Z; V'') \to H^{n+1}(X, Z; V') \simeq 0
\]
It follows that the construction

\[ V \mapsto H^n(X, Z; V) \simeq \text{Hom}_{\mathbb{Z}[G]}(H_n(X, Z; \mathbb{Z}[G]); V) \]

is a right exact functor of \( V \), so that \( H_n(X, Z; \mathbb{Z}[G]) \) is a projective \( \mathbb{Z}[G] \)-module. As in the proof of Lemma 9, it is finitely generated because the construction \( V \mapsto H^n(X, Z; V) \) commutes with filtered direct limits.

\[ \square \]

**Remark 13.** In the situation of Lemma 12, the relative homology groups \( H_*(X, Z; \mathbb{Z}[G]) \) vanish for \( * \neq n \). For \( * < n \), this follows from our connectivity assumption on the map \( f : Z \to X \). On the other hand, suppose there were some \( m > n \) for which \( H_m(X, Z; \mathbb{Z}[G]) \neq 0 \). Choose \( m \) as small as possible and set \( A = H_m(X, Z; \mathbb{Z}[G]) \). Using the projectivity of \( H_n(X, Z; \mathbb{Z}[G]) \), the universal coefficient formula gives

\[ H^m(X, Z; A) = \text{Hom}_{\mathbb{Z}[G]}(H_m(X, Z; \mathbb{Z}[G]), A) = \text{Hom}_{\mathbb{Z}[G]}(A, A) \neq 0. \]

This is impossible, since \( H^m(X; A) \) and \( H^{m-1}(Z; A) \) both vanish.

Of course, the projective module \( P = H_n(X, Z; \mathbb{Z}[G]) \) depends on the choice of \( (n-1) \)-connected map \( f : Z \to X \). For example, we could enlarge the CW complex \( Z \) by adjoining some several \( (n-1) \)-spheres which map trivially to \( X \); this would have the effect of replacing \( P \) by a direct sum \( P \oplus \mathbb{Z}[G]^r \), where \( r \) is the number of additional spheres. This motivates the following:

**Definition 14.** Let \( R \) be a ring and let \( P \) be a finitely generated \( R \)-module. We say that \( P \) is stably free if \( P \oplus R^n \) is a free \( R \)-module for some \( a \geq 0 \).

**Proposition 15.** Let \( n \geq 3 \) and let \( X \) be a space which satisfies the conditions of Lemma 6 which is of homotopy dimension \( \leq n \). Choose a finite CW complex \( Z \) of dimension \( < n \) and an \( (n-1) \)-connected map \( f : Z \to X \). Then the following conditions are equivalent:

(i) The space \( X \) is homotopy equivalent to a finite CW complex of dimension \( \leq n \).

(ii) The projective module \( P = H_n(X, Z; \mathbb{Z}[G]) \) is stably free.

**Proof.** Suppose first that \( P \) is stably free. As indicated above, we can then alter the definition of \( Z \) (attaching some extra spheres) to arrange that \( P \) is actually free. In this case, we repeat the construction of Lemma 9 but slightly more carefully: we choose a basis for \( H_n(X, Z; \mathbb{Z}[G]) \simeq \pi_n(X, Z) \) and attach only \( n \)-cells corresponding to those basis elements. This produces a map \( f' : Z' \to X \) which is an isomorphism on fundamental groups where the relative homology \( H_*(X, Z; \mathbb{Z}[G]) \simeq H_*(X, Z'; \mathbb{Z}) \) vanishes, so that \( f' \) is a homotopy equivalence by Whitehead’s theorem.

Conversely, suppose that \( X \) is homotopy equivalent to a finite CW complex of dimension \( n \). Without loss of generality we may assume that the map \( f \) is cellular, so that the relative homology \( H_n(X, Z; \mathbb{Z}[G]) \) can be computed by a cellular chain complex of finitely generated free \( \mathbb{Z}[G] \)-modules

\[ 0 \to \mathbb{Z}[G]^r_n \to \mathbb{Z}[G]^r_{n-1} \to \cdots \to \mathbb{Z}[G]^r_0 \to 0. \]

Since this complex is acyclic away from the top degree, it is split exact: that is, it has the form

\[ 0 \to Q_n \oplus Q_{n-1} \to Q_{n-1} \oplus Q_{n-2} \to \cdots \to Q_1 \oplus Q_0 \to Q_0 \to 0. \]

It follows by induction on \( i \) that each \( Q_i \) is stably free; in particular \( P = Q_n \) is stably free. \( \square \)

**Remark 16.** I believe it is an open question whether Proposition 15 is also valid for \( n = 2 \) (the proof given above certainly does not apply).

**Corollary 17.** Let \( X \) be a finitely dominated space which is simply connected. Then \( X \) has the homotopy type of a finite CW complex.
Proof. Every finitely generated projective \( \mathbf{Z} \)-module is free.

Proposition 15 allows us to quantify the failure of finitely dominated spaces to be homotopy equivalent to finite CW complexes.

Definition 18. Let \( R \) be a ring. We let \( K_0(R) \) denote the Grothendieck group of projective \( R \)-modules: that is, the free abelian group generated by symbols \([P]\), where \( P \) is a projective \( R \)-module, modulo the relations

\[[P] = [P'] + [P'']\]

when there exists an isomorphism \( P \cong P' \oplus P'' \).

The construction \( n \mapsto n[R] \) determines a group homomorphism \( \mathbf{Z} \to K_0(R) \). We let \( \widetilde{K}_0(R) \) denote the cokernel of this homomorphism. We refer to \( \widetilde{K}_0(R) \) as the reduced \( K \)-group of \( R \).

Proposition 19. Let \( X \) be a finitely dominated space of homotopy dimension \( \leq n \). Let \( Z \) be a finite CW complex of dimension \( < n \) and let \( f : Z \to X \) be an \((n-1)\)-connected map. Then the image of the class \([H_n(X, Z; \mathbf{Z}[G])]\) in the reduced \( K \)-group \( \widetilde{K}_0(\mathbf{Z}[G]) \) does not depend on the choice of \( Z \) or \( f \).

Proof. Let \( Z' \) be another finite CW complex of dimension \( < n \) equipped with an \((n-1)\)-connected map \( f' : Z' \to X \). We wish to show that \([H_n(X, Z; \mathbf{Z}[G])] = [H_n(X, Z'; \mathbf{Z}[G])]\) in the group \( \widetilde{K}_0(\mathbf{Z}[G]) \).

Starting with the map \( Z \rightarrow X \) and repeatedly applying Lemma 9, we obtain a homotopy commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{f} & Z'' \\
\downarrow g & & \downarrow f' \\
Z' & \xrightarrow{g''} & X
\end{array}
\]

It will therefore suffice to show that we have equalities

\([H_n(X, Z; \mathbf{Z}[G])] = [H_n(X, Z''; \mathbf{Z}[G])]\) \hspace{1cm} \([H_n(X, Z'; \mathbf{Z}[G])] = [H_n(X, Z'; \mathbf{Z}[G])]\).

In other words, we may replace \( Z' \) by \( Z'' \) and thereby reduce to the case where the map \( f \) factors as a composition

\( Z \xrightarrow{g} Z' \xrightarrow{f'} X \).

Note that the map \( g \) is automatically \((n-2)\)-connected. Using Lemma 9, we see that \( g \) factors as a composition

\( Z \xrightarrow{g''} Z^+ \xrightarrow{g'''} Z' \)

where \( g'' \) is \((n-1)\)-connected and \( Z^+ \) is obtained from \( Z \) by attaching finitely many \((n-1)\)-cells. Replacing \( g \) by \( g'' \) or \( g''' \), we are reduced to two special cases:

(a) The map \( g \) is \((n-1)\)-connected. In this case, we have a short exact sequence

\( 0 \to H_n(Z', Z; \mathbf{Z}[G]) \to H_n(X, Z; \mathbf{Z}[G]) \to H_n(X, Z'; \mathbf{Z}[G]) \to 0 \)

(the exactness on the left follows from Remark 13). This sequence splits (since \( H_n(X, Z'; \mathbf{Z}[G]) \) is projective), so we have

\([H_n(X, Z; \mathbf{Z}[G])] = [H_n(X, Z'; \mathbf{Z}[G])] + [H_n(Z', Z; \mathbf{Z}[G])].\)

It will therefore suffice to show that the class \([H_n(Z', Z; \mathbf{Z}[G])]\) vanishes in \( \widetilde{K}_0(\mathbf{Z}[G]) \). This follows from Proposition 15, since \( Z \) is a finite CW complex.
Proposition 22. Let $X$ be a finite CW complex. Choose an integer $n \geq 2$ such that $X$ has homotopy dimension $\leq n$, a CW complex $Z$ of dimension $< n$, and an $(n-1)$-connected map $f : Z \to X$. The Wall finiteness obstruction of $X$ is the element

$$w(X) = (-1)^n[H_n(X, Z; \mathbb{Z}[G])] \in \widetilde{K}_0(\mathbb{Z}[G]).$$

Definition 20. Let $X$ be a finitely dominated space. Choose an integer $n \geq 2$ such that $X$ has homotopy dimension $\leq n$, a CW complex $Z$ of dimension $< n$, and an $(n-1)$-connected map $f : Z \to X$. The Wall finiteness obstruction $w(X)$ is well-defined.

Proof. We have already seen that $w(X)$ does not depend on the map $f : Z \to X$. We now check that it is independent of $n$. Let us temporarily denote $w(X)$ by $w_n(X)$ to emphasize its hypothetical dependence on $n$. Choose any integer $n \geq 2$ such that $X$ has homotopy dimension $\leq n$; we will show that $w_n(X) = w_{n+1}(X)$. To prove this, choose a CW complex $Z$ of dimension $< n$ and an $(n-1)$-connected map $f : Z \to X$. Using Lemma 9, we can extend $f$ to an $n$-connected map $f' : Z' \to X$ where $Z'$ is obtained from $Z$ by attaching finitely many $n$-cells. We then have a (split) short exact sequence

$$0 \to H_{n+1}(X, Z'; \mathbb{Z}[G]) \to H_n(Z, Z'; \mathbb{Z}[G]) \to H_n(X, Z; \mathbb{Z}[G]) \to 0$$

which gives the relation

$$[H_n(X, Z; \mathbb{Z}[G])] + [H_{n+1}(X, Z'; \mathbb{Z}[G])] = [H_n(Z, Z'; \mathbb{Z}[G])] = 0 \in \widetilde{K}_0(\mathbb{Z}[G]).$$

By virtue of Proposition 15, the finiteness obstruction $w(X)$ is zero if and only if $X$ has the homotopy type of a finite CW complex.

We conclude this lecture by tying up a few loose ends. We start with a converse to Lemma 6.

Proposition 22. Let $X$ be a CW complex satisfying conditions (a) through (d) of Lemma 6. Then $X$ is finitely dominated.

Proof Sketch. Write $X$ as a union of finite subcomplexes $X_\alpha$. It will suffice to show that the identity map $id : X \to X$ is homotopic to a map which factors through some $X_\alpha$. We can replace each of the inclusions $X_\alpha \to X$ by a fibration $p_\alpha : E_\alpha \to X$; we wish to show that one of these inclusions has a section.

For each integer $m$, let $\tau_{\leq m}E_\alpha$ denote the $m$th stage in the relative Postnikov tower of $E_\alpha$ over $X$ (so that we have a fibration $p_{\alpha,m} : \tau_{\leq m}E_\alpha \to X$ whose fibers have no homotopy groups above $m$). Suppose we are given a section $s_m$ of some $p_{\alpha,m}$. Note that if $m \geq 1$, then the fiber product

$$(\tau_{\leq m+1}E_\alpha) \times_{E_\alpha} X$$

is a fibration over whose fibers have the form $K(A_{x,\alpha}, m+1)$, where $x \mapsto A_{x,\alpha}$ is a local system of abelian groups on $X$. Consequently, the obstruction to lifting $s_m$ to a section of $p_{\alpha,m+1}$ is measured by a cohomology
class \( \eta(s_m) \in H^{m+2}(X; A_\alpha) \). If \( m + 2 \) is larger than the homotopy dimension of \( X \) (which is finite by assumption), then \( \eta(s_m) \) automatically vanishes, so any section of \( p_{\alpha,m} \) can be lifted to a section of \( p_\alpha \).

We will complete the proof by showing that for every integer \( m \), there exists an index \( \alpha \) such that \( p_{\alpha,m} \) admits a section. The proof proceeds by induction on \( m \). Suppose first that \( m \geq 1 \) and that we are given a section \( s_m \) as above. We claim that it is possible to choose \( \beta \geq \alpha \) such that the image of \( \eta(s_m) \) vanishes in \( H^{m+2}(X; A_\beta) \). In fact, we claim that the direct limit \( \lim_{\beta \geq \alpha} H^{m+2}(X; A_\beta) \) vanishes. Since \( X \) satisfies condition (c) of Lemma 6, it will suffice to show that the direct limit \( \lim_{\beta \geq \alpha} A_\beta \) vanishes as a local system of abelian groups on \( X \). This follows immediately from the fact that \( X \) is a homotopy colimit of the diagram \( \{ E_\alpha \} \).

It remains to treat small values of \( m \). Let us begin with the case \( m = 0 \), so that each \( \tau_{\leq m} E_\alpha \) can be regarded as a covering space of \( X \). Let \( S_\alpha \) denote the fiber over the base point \( x \in X \), so that each \( S_\alpha \) is a set with an action of the group \( G \). To choose a section of \( \tau_{\leq m} E_\alpha \), we must show that \( S_\alpha \) contains an element which is fixed by \( G \). Because \( G \) is finitely generated, passage to \( G \)-invariants commutes with filtered direct limits. It will therefore suffice to show that the direct limit \( \lim S_\alpha \) contains an element which is fixed by \( G \). This is clear, since \( \lim S_\alpha \) consists of a single point (it is \( \pi_0 \) of the homotopy fiber of the identity map \( X \to X \)).

We conclude by treating the case \( m = 1 \). Let us assume that there exists an index \( \alpha \) and that we have chosen a section of the map \( \tau_{\leq 1} E_\alpha \to X \). For each \( \beta \geq \alpha \), let \( E_\beta \) denote the fiber product \( \tau_{\leq 1} E_\beta \times_{\tau_{\leq 0} E_\beta} X \). The projection map \( q_\beta : E_\beta \to X \) is a fibration whose fibers have the form \( K(\Pi; 1) \). Let \( G_\beta \) denote the fundamental group of \( E_\beta \), so that we have \( \lim G_\beta \simeq G \). Since \( G \) is finitely presented, it follows that the natural map \( G_\beta \to G \) admits a section for \( \beta \) sufficiently large. We are therefore reduced to finding a section of the induced map \( E_\beta \times_{BG_\beta} BG \to X \). This is a fibration whose fibers are of the form \( K(\Pi; 1) \), where \( \Pi \) is abelian, and is therefore classified by an element of \( H^2(X; L_\beta) \) for some local system of abelian groups \( L_\beta \) on \( X \). As before, we have \( \lim G_\beta = 0 \) so (by virtue of condition (c)) the direct limit \( \lim H^2(X; L_\beta) \) vanishes, and therefore the fibration is trivial for \( \beta \) sufficiently large.

\[ \square \]

**Remark 23.** Let \( G \) be a finitely presented group. Then every class \( \eta \in \tilde{K}_0(Z[G]) \) arises as the Wall finiteness obstruction of some finitely dominated space \( X \) with \( \pi_1 X = G \). To see this, we first choose a connected finite 2-dimensional CW complex \( X_0 \) with \( \pi_1 X_0 = G \). Let \( \eta = [P] \) where \( P \) is a finitely generated projective \( Z[G] \)-module, so that \( P \) appears as a direct summand of some free \( Z[G] \)-module \( Z[G]^r \). Then \( P \) is the image of an idempotent map \( e : Z[G]^r \to Z[G]^r \). Choose an even integer \( n \geq 2 \) and let \( Y \) be the CW complex obtained from \( X_0 \) by adding \( r n \)-cells with trivial attaching maps. Using the relative Hurewicz theorem we deduce that the relative homotopy group \( \pi_n(Y, X_0) \) is isomorphic to the free module \( Z[G]^r \). We can therefore choose a map \( \pi : Y \to Y \) which is the identity on \( X_0 \) and which induces the idempotent endomorphism \( e \) of \( H_n(Y, X_0; Z[G]) \simeq Z[G]^r \). Using the fact that \( \pi \) is an idempotent in the homotopy category, one can show that the homotopy colimit \( X \) of the diagram

\[
\cdots \xrightarrow{\pi} Y \xrightarrow{\pi} Y \xrightarrow{\pi} \cdots
\]

satisfies the conditions of Lemma 6 and is therefore a finitely dominated space of homotopy dimension \( \leq n \); a simple calculation shows that the composite map \( X_0 \hookrightarrow Y \to X \) is \((n-1)\)-connected and that the relative homology \( H_n(X, X_0; Z[G]) \) is isomorphic to \( P \).