We begin by recalling Adams’ variant of the Brown representability theorem:

**Theorem 1** (Adams). Let $E$ be a spectrum and let $h_*$ be a homology theory. Suppose we are given a map of homology theories $\alpha : E_* \to h_*$ (that is, a collection of maps $E_*(X,Y) \to h_*(X,Y)$, depending functorially on a pair of spaces $(Y \subseteq X)$ and compatible with boundary maps). Then there is a map of spectra $\beta : E \to E'$ and an isomorphism of homology theories $E'_* \simeq h'_*$ such that $\alpha$ is given by the composition $E_* \to E'_* \simeq h_*$. 

**Corollary 2** (Adams). Let $E$ and $E'$ be spectra, and let $\alpha : E_* \to E'_*$ be a map between the corresponding homology theories. Then $\alpha$ is induced by a map of spectra $\alpha : E \to E'$. 

**Proof.** Let $h_* = E'_*$. Applying Theorem 1 the evident map $\alpha : E_* \oplus E'_* \to h_*$, we get a spectrum $F$ and a map $E \oplus E' \to F$ inducing $\alpha$. This comes from a pair of spectrum maps $f : E \to F$ and $g : E' \to F$. The map $g$ induces an isomorphism $\pi_* E' = h_*(\ast) = \pi_* F$ and is therefore a homotopy equivalence. Then $\alpha : g^{-1} \circ f$ is the desired map of spectra from $E$ to $E'$. 

**Corollary 3** (Adams). Every homology theory $h_*$ is represented by a spectrum $E$, which is uniquely defined up to (nonunique) homotopy equivalence. 

**Proof.** The existence of $E$ follows from Theorem 1. For the uniqueness, we note that if $E$ and $E'$ are two spectra with $E_* \simeq h_* \simeq E'_*$, then the isomorphism $E_* \simeq E'_*$ is induced by a map of spectra $E \to E'$ (Corollary 2), which is automatically a homotopy equivalence. 

In the situation of Corollary 2, the map $\alpha$ is generally not determined by $\alpha$, even up to homotopy. This is due to the existence of **phantom maps**:

**Definition 4.** Let $f : E \to E'$ be a map of spectra. We say that $f$ is a **phantom** if the underlying map of homology theories $E_* \to E'_*$ is zero: that is, for every space $X$, the map $E_*(X) \to E'_*(X)$ is identically zero.

**Lemma 5.** Let $f : E \to E'$ be a map of spectra. The following conditions are equivalent:

1. The map $f$ is a phantom.
2. For every spectrum $X$, the map $E_*(X) \to E'_*(X)$ is zero.
3. For every finite spectrum $X$, the map $E_*(X) \to E'_*(X)$ is zero.
4. For every finite spectrum $X$, the map $E^*(X) \to E'^*(X)$ is zero.
5. For every finite spectrum $X$ and every map $g : X \to E$, the composition $f \circ g : X \to E'$ is nullhomotopic. 

**Proof.** The implication $(2) \Rightarrow (1)$ is obvious, and the converse follows from the fact that every spectrum $X$ can be written as a filtered colimit $\lim_{\to} \Sigma^{-n} \Omega^{-n} X$. The implication $(2) \Rightarrow (3)$ is obvious, and the converse follows from the fact that every spectrum is a filtered colimit of finite spectra. The equivalence of $(4)$ and $(5)$ follows by Spanier-Whitehead duality, and the equivalence of $(4)$ and $(5)$ is a tautology. 

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Let us now return to the setting of the previous lectures. Let $L \simeq \mathbb{Z}[t_1, \ldots]$ denote the Lazard ring, and let $M$ be a graded $L$-module. Assume that the grading on $M$ is even: that is, $M_k \simeq 0$ for every odd number $k$. In the last lecture, we saw that if $M$ satisfies Landweber’s criterion: that is, if the sequence $v_0 = p, v_1, v_2, \ldots \in L$ is $M$-regular for every prime number $p$, then the construction

$$X \mapsto \text{MU}_*(X) \otimes_L M$$

is a homology theory. It follows from Corollary 3 that this homology theory is represented by a spectrum $E$, which is unique up to homotopy equivalence. We will say that a spectrum $E$ is Landweber-exact if it arises from this construction. Our goal in this lecture is to show that, as an object of the homotopy category of spectra, $E$ is functorially determined by $M$. This is a consequence of the following assertion:

**Theorem 6.** Let $E$ be a Landweber-exact spectrum, and let $E'$ be a spectrum such that $\pi_k E' \simeq 0$ for $k$ odd. Then every phantom map $f : E \to E'$ is nullhomotopic.

**Corollary 7.** Let $E$ and $E'$ be Landweber exact spectra. Then every phantom map $f : E \to E'$ is nullhomotopic. In particular, every nontrivial endomorphism of $E$ acts nontrivially on the homology theory $E_*$.  

To prove Theorem 6, we introduce two new notions:

**Definition 8.** We will say that a finite spectrum $X$ is even if the homology groups $H_k(X; \mathbb{Z})$ are free abelian groups, which vanish when $k$ is odd. Equivalently, a finite spectrum $X$ is even if it admits a finite cell decomposition using only even-dimensional cells.

We say that a spectrum $E$ is evenly generated if, for every map $X \to E$ where $X$ is a finite spectrum, there exists a factorization $X \to X' \to E$ where $X'$ is a finite even spectrum.

Theorem 6 is a consequence of the following two assertions:

**Proposition 9.** Every Landweber exact spectrum $E$ is evenly generated.

**Proposition 10.** Let $E$ be an evenly generated spectrum and let $E'$ be a spectrum whose homotopy groups are concentrated in even degrees. Then every phantom map $f : E \to E'$ is null.

We begin by proving Proposition 9. Let $E$ be a Landweber-exact spectrum, associated to a graded $L$-module $M$, and let $f : X \to E$ be a map where $X$ is a finite spectrum. We can associate to $f$ an element of $E^0(X) = E_0(DX) = \text{MU}_0(DX) \otimes_L M = \text{MU}^0(X) \otimes_L M$, which can be written as $\sum c_i m_i$ where $c_i \in \text{MU}^{d_i}(X)$ and $m_i \in M_{d_i}$. Then $f$ factors as a composition

$$X \xrightarrow{(c_i)} \bigoplus \Sigma^{d_i} \text{MU} \xrightarrow{m_i} E.$$  

We may therefore replace $E$ by $\bigoplus \Sigma^{d_i} \text{MU}$: that is, it suffices to prove that $\bigoplus \Sigma^{d_i} \text{MU}$ is evenly generated. Since $M$ is evenly graded, each of the integers $d_i$ is even. We can therefore reduce to showing that $\text{MU}$ itself is evenly generated.

Since $\text{MU} \simeq \varprojlim \text{MU}(n)$, it suffices to show that each $\text{MU}(n)$ is evenly generated. Recall that $\text{MU}(n)$ is the Thom complex of the virtual bundle $\zeta - C^n$, where $\zeta$ is the tautological vector bundle on $BU(n)$. We can write $BU(n) \simeq \varprojlim_m \text{Grass}(n, n + m)$, where Grass$(n, n + m)$ denotes the Grassmannian of $n$-dimensional subspaces of $C^{n+m}$. It follows that $\text{MU}(n)$ is a direct limit of Thom spectra associated to the finite-dimensional Grassmannians Grass$(n, n + m)$. It therefore suffices to show that each of these Thom complexes is an even finite spectrum. We now note that the space Grass$(n, n + m)$ admits a finite cell decomposition with cells of even dimension: for example, we can take the Bruhat decomposition. This proves Proposition 9.
We now prove Proposition 10. Let $E$ be an evenly generated spectrum. We begin by describing the structure of phantom maps from $E$ to other spectra. Let $A$ be a set of representatives for all homotopy equivalence classes of maps $X_\alpha \to E$, where $X_\alpha$ is an even finite spectrum, and form a fiber sequence

$$K \to \bigoplus_{\alpha \in A} X_\alpha \xrightarrow{u} E.$$ 

This sequence is classified by a map $u' : E \to \Sigma(K)$. Since $E$ is evenly generated, every map from a finite spectrum $X$ into $E$ factors through $u'$, so the composite map $X \to E \to \Sigma(K)$ is null: in other words, $u'$ is a phantom map. Conversely, if $f : E \to E'$ is any phantom map, then $f \circ u$ is nullhomotopic, so that $f$ factors as a composition $E \to \Sigma(K) \to E'$. Consequently, to prove Proposition 10, it will suffice to prove that every map $\Sigma(K) \to E'$ is nullhomotopic: that is, that the group $E'_{+1}(K)$ is zero.

Since the homotopy groups of $E'$ are concentrated in even degrees, the Atiyah-Hirzebruch spectral sequence shows that $E'_{+1}(X) \simeq 0$ whenever $X$ is a finite even spectrum. It will therefore suffice to prove the following:

$(\ast)$ The spectrum $K$ is a retract of a direct sum of even finite spectra.

To prove $(\ast)$, we will compare the cofiber sequence

$$K \to \bigoplus_{\alpha \in A} X_\alpha \to E$$

with another cofiber sequence of spectra. Let $B$ be the collection of triples $(\alpha, \alpha', f)$, where $\alpha, \alpha' \in A$ and $f$ ranges over all homotopy classes of maps fitting into a commutative diagram

$$
\begin{array}{ccc}
X_\alpha & \xrightarrow{f} & X_{\alpha'} \\
\downarrow \downarrow & & \downarrow \downarrow \\
& E.
\end{array}
$$

For each $\beta = (\alpha, \alpha', f) \in B$, we let $Y_\beta = X_\alpha$. We have a canonical map $\phi : \bigoplus_{\beta \in B} Y_\beta \to \bigoplus_{\alpha \in A} X_\alpha$, whose restriction to $Y_\beta$ for $\beta = (\alpha, \alpha', f)$ given by the difference of the maps $Y_\beta = X_\alpha \to \bigoplus_{\alpha \in A} X_\alpha$ and

$$Y_\beta = X_\alpha \xrightarrow{f} X_{\alpha'} \to \bigoplus_{\alpha \in A} X_\alpha.$$

Let $F$ be the cofiber of the map $\phi$. By construction, we have a map of fiber sequences

$$
\begin{array}{ccc}
\bigoplus_{\beta \in B} Y_\beta & \to & \bigoplus_{\alpha \in A} X_\alpha \xrightarrow{u} F \\
\downarrow & & \downarrow \\
K & \to & \bigoplus_{\alpha \in A} X_\alpha \to E.
\end{array}
$$

We now construct a map of spectra $q : E \to F$. By Corollary 2, it will suffice to define a map of homology theories $E_* \to F_*$. We will give a map $E_*(X) \to F_*(X)$ defined for every spectrum $X$. Since homology theories commute with filtered colimits, it will suffice to consider the case where $X$ is a finite spectrum. Replacing $X$ by its Spanier-Whitehead dual, we are reduced to the problem of producing a map $q(f) : X \to F$ for every map of spectra $f : X \to E$ for $X$ finite.

Here is our construction. Since $E$ is evenly generated, every map $f : X \to E$ factors through some map $X \xrightarrow{f'} X_{\alpha'} \to E$ for $\alpha' \in A$. We define $q(f)$ to be the composite map $X \xrightarrow{f'} X_{\alpha'} \to \bigoplus_{\alpha \in A} X_\alpha \to F$. We must show that this construction is well-defined; that is, it does not depend on the choice of $f'$. To this end,
suppose we are given another factorization of \( f : X \xrightarrow{f'} X_{\alpha''} \to E \), where \( \alpha'' \in A \). Let \( Y \) denote the pushout \( X_{\alpha'} \coprod_X X_{\alpha''} \). Then \( Y \) is a finite spectrum, and our data gives a canonical map \( Y \to E \). Since \( E \) is evenly generated, this map factors as a composition

\[
Y \xrightarrow{g} X_{\alpha} \to E
\]

for some \( \alpha \in A \). Let \( h' \) denote the composite map \( X_{\alpha'} \to X' \xrightarrow{g} X_{\alpha} \) and let \( h' \) be defined similarly. Then \((\alpha', \alpha, h)\) and \((\alpha'', \alpha, h)\) can be identified with elements of \( B \). It follows that the composite maps

\[
X \to X_{\alpha'} \to \bigoplus_{\alpha \in A} X_{\alpha} \to F
\]

\[
X \to X_{\alpha''} \to \bigoplus_{\alpha \in A} X_{\alpha} \to F
\]

both coincide with the map

\[
X \to Y \xrightarrow{g} X_{\alpha} \to \bigoplus_{\alpha \in A} X_{\alpha} \to F,
\]

which proves that \( q \) is well-defined.

We now have a larger commutative diagram of fiber sequences

\[
\begin{array}{ccc}
K & \longrightarrow & \bigoplus_{\alpha \in A} X_{\alpha} \longrightarrow E \\
\downarrow & & \downarrow \\
\bigoplus_{\beta \in B} Y_{\beta} & \longrightarrow & \bigoplus_{\alpha \in A} X_{\alpha} \longrightarrow F \\
\downarrow & & \downarrow \\
K & \longrightarrow & \bigoplus_{\alpha \in A} \longrightarrow E.
\end{array}
\]

The right vertical composition induces the identity map on the underlying homology theory \( E_* \); that is, it differs from \( \text{id}_E \) by a phantom map. In particular, it is an equivalence, so that the left vertical composition is an equivalence of \( K \) with itself. It follows that \( K \) is a retract of \( \bigoplus_{\beta \in B} Y_{\beta} \), which proves \((\ast)\).