Formal Groups (Lecture 11)

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We begin by recalling our discussion of the Adams-Novikov spectral sequence:

Claim 1. Let $X$ be any spectrum. Then $\text{MU}_*(X)$ is a module over the commutative ring $L = \pi_* \text{MU}$, and can therefore be understood as a quasi-coherent sheaf on the affine scheme $\text{Spec} \ L$ which parametrizes formal group laws (here $L$ denotes the Lazard ring). This quasi-coherent sheaf admits an action of the affine group scheme $G = \text{Spec} \mathbb{Z}[b_1, b_2, \ldots]$ which assigns to each commutative ring $R$ the group $\{g \in R[[t]] : g(t) = t + b_1 t^2 + b_2 t^3 + \cdots\}$, compatible with the action of $G$ on $\text{Spec} \ L$ by the construction

$$(g \in G(R), f(x, y) \in \text{FGL}(R) \subseteq R[[x, y]]) \mapsto gf(g^{-1}(x), g^{-1}(y)) \in \text{FGL}(R) \subseteq R[[x, y]].$$

There is a spectral sequence $\{E^{p,q}_r, d_r\}$, called the Adams-Novikov spectral sequence, with the following properties. If $X$ is connective, then $\{E^{p,q}_r, d_r\}$ converges to a finite filtration of $\pi_{p-q} X$. Moreover, the groups $E^{p,q}_2$ are given by the cohomology groups $H^p(G; \text{MU}_* X)$.

Equivalently, we can think of $E^{p,q}_2$ as the cohomology of the stack $\mathcal{M}^*_{\text{FG}} = \text{Spec} \ L/G$ with coefficients in the sheaf $\mathcal{F}_X$ determined by $\text{MU}_*(X)$ with its $G$-action.

To be more precise, we should observe that the ring $L$ and the ring $\mathbb{Z}[b_1, \ldots]$ are all equipped a canonical grading. In geometric terms, this grading corresponds to an action of the multiplicative group $\mathbb{G}_m$. This group acts on $L$ by the formula

$$(\lambda \in R^\times, f(x, y) \in \text{FGL}(R)) \mapsto \lambda f(\lambda^{-1} x, \lambda^{-1} y).$$

In fact, we can identify both $\mathbb{G}_m$ and $G$ with subgroups of a larger group $G^+$, with $G^+(R) = \{g \in R[[x]] : g(t) = b_0 t + b_1 t^2 + \cdots, b_0 \in R^\times\}$. This group can be identified with a semidirect product of the subgroup $\mathbb{G}_m$ (consisting of those power series with $b_i = 0$ for $i > 0$) and $G$ (consisting of those power series with $b_0 = 1$), and this semidirect product acts on $\text{Spec} \ L$ by substitution.

For any spectrum $X$, $\text{MU}_*(X)$ is a graded $L$-module, and the action of $G$ on $\text{MU}_*(X)$ is compatible with the grading. In the language of algebraic geometry, this means that $\text{MU}_{\text{even}}(X) = \bigoplus_n \text{MU}_{2n}(X)$ can be regarded as a representation of the group $G^+$ compatible with the action of $G^+$ on $\text{Spec} \ L$. In the language of stacks, this means that $\text{MU}_{\text{even}}(X)$ can be regarded as a quasi-coherent sheaf on the quotient stack $\text{Spec} \ L/G^+$.

Definition 2. The quotient stack $\text{Spec} \ L/G^+$ is called the moduli stack of formal groups and will be denoted by $\mathcal{M}_{\text{FG}}$.

To understand $\mathcal{M}_{\text{FG}}$, it will be useful to have a more conceptual way of thinking about formal group laws. Let $R$ be a commutative ring and let $f(x, y) \in R[[x, y]]$ be a formal group law over $R$. We let $\text{Alg}_R$ denote the category of commutative $R$-algebras. We can associate to $f$ a functor $\mathfrak{G}_f : \text{Alg}_R \to \text{Ab}$ from $R$ to the category of abelian groups; namely, we let $\mathfrak{G}_f(A) = \{a \in A : (\exists n) a^n = 0\} \subseteq A$, with the group structure given by $(a, b) \mapsto f(a, b)$. Note that this expression makes sense: though $f$ has infinitely many terms, if $a$ and $b$ are nilpotent then only finitely many terms are nonzero. We will call $\mathfrak{G}_f$ the formal group associated to $f$. 
Remark 3. The condition that \( f \in R[[x, y]] \) define a formal group law is equivalent to the requirement that the above formula defines a group structure on \( \mathcal{G}_f(A) \) for every \( R \)-algebra \( A \).

Suppose that we are given two formal group laws \( f, f' \in R[[x, y]] \) and an isomorphism \( \alpha : \mathcal{G}_f \simeq \mathcal{G}_f' \) of the corresponding formal groups. In particular, for every \( R \)-algebra \( A \), \( \alpha \) determines a bijection \( \alpha_A \) from the set \( \{ a \in A : a \text{ is nilpotent} \} \) with itself. To understand this bijection, let us treat the universal case where \( A \) contains an element \( a \) such that \( a^{n+1} = 0 \). This is the truncated polynomial ring \( A = R[t]/t^{n+1} \). In this case, \( \alpha \) carries \( t \) to another nilpotent element, necessarily of the form \( b_0 t + b_1 t^2 + \ldots + b_{n-1} t^n \). Since \( \alpha \) is functorial, it follows that for any commutative \( R \)-algebra \( A \) containing an element \( a \) with \( a^n = 0 \), we have \( \alpha_A(a) = b_0 a + b_1 a^2 + \ldots + b_{n-1} a^n \). In particular, if \( A = R[t]/t^n \), we deduce that \( \alpha_A(t) = b_0 t + b_1 t^2 + \ldots + b_{n-2} t^{n-1} \).

In other words, the coefficients \( b_i \) which appear are independent of \( n \). We conclude that there exists a power series \( g(t) = b_0 t + b_1 t^2 + \ldots \) such that \( \alpha_A(a) = g(a) \) for every commutative ring \( a \). Since \( \alpha \) is a bijection for any \( A \), we conclude that \( g \) is an invertible power series. Since \( \alpha_A \) is a group homomorphism, we deduce that \( g \) satisfies the formula \( f'(g(x), g(y)) = g f(x, y) \): that is, the formal group laws \( f \) and \( f' \) differ by the change-of-variable \( g \).

**Definition 4.** Let \( R \) be a commutative ring. An coordinatizable formal group over \( R \) is a functor \( \mathcal{G} : \text{Alg}_R \to \text{Ab} \) which has the form \( \mathcal{G}_f \), for some formal group law \( f \in R[[x, y]] \).

We regard the coordinatizable formal group laws (and isomorphisms between them) as a subcategory of the category \( \text{Fun}(\text{Alg}_R, \text{Ab}) \) of functors from \( \text{Alg}_R \) to abelian groups. We have just seen that this subcategory admits a less invariant description: it is equivalent to a category whose objects are formal group laws \( f \in R[[x, y]] \), and whose morphisms are maps \( g \) such that \( f'(g(x), g(y)) = g f(x, y) \).

The coordinatizable formal group laws over \( R \) do not satisfy descent in \( R \). Consequently, it is convenient to make the following more general definition:

**Definition 5.** Let \( R \) be a commutative ring. A formal group law over \( R \) is a functor \( \mathcal{G} : \text{Alg}_R \to \text{Ab} \) satisfying the following conditions:

1. The functor \( \mathcal{G} \) is a sheaf with respect to the Zariski topology. In other words, if \( A \) is a commutative \( R \)-algebra with a pair of elements \( x \) and \( y \) such that \( x + y = 1 \), then \( \mathcal{G}(A) \) can be described as the subgroup of \( \mathcal{G}(A[\frac{1}{xy}]) \times \mathcal{G}(A[\frac{1}{x}]) \) consisting of pairs which have the same image in \( \mathcal{G}(A[\frac{1}{xy}]) \).

2. The functor \( \mathcal{G} \) is a coordinatizable formal group law locally with respect to the Zariski topology. That is, we can choose elements \( r_1, r_2, \ldots, r_n \in R \) such that \( r_1 + \cdots + r_n = 1 \), such that each of the composite functors

\[
\text{Alg}_{R[\frac{1}{r_1}]} \to \text{Alg}_R \to \text{Ab}
\]

has the form \( \mathcal{G}_f \) for some formal group law \( f \in R[\frac{1}{r_1}][[x, y]] \).

By definition, the moduli stack of the formal groups \( \mathcal{M}_FG \) is the functor which assigns to each commutative ring \( R \) the category of formal group laws over \( R \) (the morphisms in this category are given by isomorphisms).

There is a canonical map of stacks \( \mathcal{M}_FG = \text{Spec } L/G \to \text{Spec } L/G^+ = \mathcal{M}_FG^+ \). To understand this map (and the failure of general formal groups to be coordinatizable) it is useful to introduce a definition.

**Definition 6.** Let \( \mathcal{G} \) be a formal group over \( R \). The Lie algebra of \( \mathcal{G} \) is the abelian group \( g = \ker(\mathcal{G}(R[t]/(t^2)) \to \mathcal{G}(R)) \).

Note that if \( \mathcal{G} = \mathcal{G}_f \) for some formal group law \( f \), we get a group isomorphism \( g \simeq tR[t]/(t^2) \simeq R \) (since \( f(x, y) = x + y \) to order 2). In fact, \( g \) is not just an abelian group: for each \( \lambda \in R \), the equation \( t \mapsto \lambda t \) determines a map from \( R[t]/(t^2) \) to itself, which induces a group homomorphism \( g \to g \). When \( \mathcal{G} \) is coordinatizable, this is the usual action of \( R \) on itself by multiplication. It follows by descent that the above formula always determines an action of \( R \) on \( g \). Since \( g \simeq R \) locally for the Zariski topology, we deduce that \( g \) is an invertible \( R \)-module: that is, it determines a line bundle on the affine scheme \( \text{Spec } R \).
Proposition 7.  

(1) A formal group $\mathcal{G}$ over $R$ is coordinatizable if and only if its Lie algebra $\mathfrak{g}$ is isomorphic to $R$.

(2) The quotient stack $\mathcal{M}^\mathfrak{g}_{FG}$ parametrizes pairs $(\mathcal{G}, \alpha)$, where $\mathcal{G}$ is a formal group and $\alpha: \mathfrak{g} \simeq R$ is a trivialization of its Lie algebra.

Proof. We have already established that $\mathfrak{g} \simeq R$ when $\mathcal{G}$ is coordinatizable. Conversely, fix an isomorphism $\mathfrak{g} \simeq R$. After localizing Spec $R$, the group $\mathcal{G}$ becomes coordinatizable: that is, we can write $\mathcal{G} \simeq \mathcal{G}_f$ for some $f \in R[[x,y]]$. Modifying $f$ by the action of $\mathbb{G}_m$, we may assume that this isomorphism is compatible with our trivialization of $\mathfrak{g}$. The trouble is that these isomorphisms might not glue. The obstruction to gluing them determines a cocycle representing a class in $H^1_{\text{Zar}}(\text{Spec } R, G)$. We claim that this group vanishes. This is because the group $G$ is an iterated extension of copies of the additive group $(A \in \text{Alg}_R) \mapsto (A, +)$, which has no cohomology on affine schemes.

Assertion (2) is just a translation of the following observation: if $f, f' \in R[[x,y]]$ are formal group laws, then an isomorphism of formal groups $\mathcal{G}_f \simeq \mathcal{G}_{f'}$ respects the trivializations of the Lie algebras of $\mathfrak{g}_f$ and $\mathfrak{g}_{f'}$ if and only if it is given by a power series of the form $g(t) = t + b_1 t^2 + \cdots$ (a power series of the form $g(t) = b_0 t + \cdots$ acts on the Lie algebras by multiplication by the scalar $b_0$).

We can think of the assignment $(R, \mathcal{G}) \mapsto \mathfrak{g}^{-1}$ as defining a line bundle $\omega$ on the moduli stack $\mathcal{M}^\mathfrak{g}_{FG}$. In fact, $\mathcal{M}^\mathfrak{g}_{FG}$ is just the total space of $\omega$ with the zero section removed (equivalently, the moduli stack of trivializations of $\omega$).

We can now be a little bit more precise about the $E_2$-term of the Adams-Novikov spectral sequence. Translating our gradings into algebraic geometry, we get the following result:

Claim 8. For any spectrum $X$, the bordism groups $\text{MU}_{\text{even}}(X)$ form a module over the Lazard ring $L \simeq \pi_* \text{MU}$ which carries a compatible action of the group scheme $G^+$, and therefore determines a sheaf $\mathcal{F}^{\text{even}}$ on $\mathcal{M}_{FG} = \text{Spec } L/G^+$. The $E_2$-term of the Adams-Novikov spectral sequence satisfies

$$E_{2a,b}^2 = H^b(\mathcal{M}_{FG}; \mathcal{F}^{\text{even}} \otimes \omega^a).$$

Similarly, the odd homotopy groups $\text{MU}_{\text{odd}}(X)$ determine a sheaf $\mathcal{F}^{\text{odd}}$ on $\mathcal{M}_{FG}$ satisfying

$$E_{2a+1,b}^2 = H^b(\mathcal{M}_{FG}; \mathcal{F}^{\text{odd}} \otimes \omega^a).$$

In order to exploit Claim 8, we will need to understand the structure of the moduli stack $\mathcal{M}_{FG}$. This will be our goal in the next lecture.