• Start by printing your name in the above box.

• Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.

• Do not detach pages from this exam packet or unstaple the packet.

• Please try to write neatly. Answers which are illegible for the grader can not be given credit.

• No notes, books, calculators, computers, or other electronic aids are allowed.

• Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.

• You have 180 minutes time to complete your work.

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<td><strong>Total:</strong></td>
<td><strong>140</strong></td>
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Problem 1) (20 points) No justifications are necessary

1) T F The dot product between $\langle 1, 1, 2 \rangle \cdot \langle 2, 3, 4 \rangle$ is 13.

2) T F The distance between a line $\vec{r}(t) = \vec{Q} + t\vec{v}$ with unit vector $\vec{v}$ and a point $P$ is given by $|\vec{PQ} \times \vec{v}|$.

3) T F The TNB frame consists of the vectors $\vec{r}_u, \vec{r}_v, \vec{r}_u \times \vec{r}_v$.

4) T F The surface $x^2 + y^2 - z^2 = -1$ is a one sheeted hyperboloid.

5) T F The equation $u_{tt} + uu_x = u_{xx}$ is called the Burgers equation.

6) T F The fundamental theorem of line integrals assures that the line integral of any vector field along a closed loop is zero.

7) T F If $\vec{r}(t)$ is a curve in space then $\nabla \vec{r}(t)$ is a vector perpendicular to the curve.

8) T F If $\nabla f(3, 1) = \langle 0, 0 \rangle$ and $f_{xx}(3, 1) > 0$, then $(3, 1)$ is a local minimum of $f$.

9) T F Given two points $P, Q$ in space and two lines $L, M$ where $L$ goes through $P$ and $M$ goes through $Q$. The distance between $P, Q$ is larger or equal than the distance between the two lines.

10) T F The equation $u_t - u_x = u_{xx}$ is called the heat equation.

11) T F The flux of the curl of $\vec{F}$ through the surface $S$ is positive, where $S$ is the surface $x^2 + y^2 + z^2 = 1$ oriented outwards.

12) T F The dot product between two parallel vectors is always zero.

13) T F $\vec{r}(u, v) = \langle u, u, 0 \rangle$ parametrizes a surface $S$ which is a cylinder.

14) T F The length of the gradient $|\nabla f|$ is always minimal at critical points.

15) T F The triple integral $\int \int \int_E \text{div}(\vec{F}(x, y, z)) \ dx dy dz$ over a sphere $E$ is always zero since the flux of $\vec{F}$ through the boundary surface is zero.

16) T F Assume $\vec{r}(t)$ is a path of length 1 parametrized on $[a, b]$, then $\int_a^b |\vec{r}'(t)| \ dt = 1$.

17) T F If $\vec{r}'(t) = \langle 2t, 1 - 2t \rangle$ and $\vec{r}(0) = \langle 2, 3 \rangle$, then $\vec{r}(t) = \langle 2 + t^2, 3 + t - t^2 \rangle$.

18) T F For any two unit vectors $\vec{v}, \vec{w}$ we have $|\vec{v} \times \vec{w}|^2 + (\vec{v} \cdot \vec{w})^2 = 1$.

19) T F The directional derivative $D_v(f)$ is always perpendicular to the vector $\vec{v}$ and to the surface $f = c$.

20) T F $\langle 1, 0, 0 \rangle \cdot (\langle 1, 1, 0 \rangle \times \langle 0, 0, 1 \rangle) = 1/6$. 

2
Problem 2) (10 points) No justifications are necessary.

Match the following objects.

<table>
<thead>
<tr>
<th>Formula</th>
<th>Enter 1-9</th>
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<tbody>
<tr>
<td>$\rho \leq (1 + \sin(\phi + \theta))$</td>
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</tr>
<tr>
<td>$x^2 + y^2 + z^2 + \sin(x^2y^2z^2) = 1.$</td>
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</tr>
<tr>
<td>$r = t + (2\pi - t) + \cos(5\theta), \ z = 0.$</td>
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<tr>
<td>$\vec{F}(x, y) = \langle x, y^2 \rangle$</td>
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</tr>
<tr>
<td>$(x - 5)^2 + (y - 3)^2 + (z + 1)^2 = 1$</td>
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<tr>
<td>$x^2 + y^2 = 3$</td>
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<tr>
<td>$\vec{r}(t) = \langle \cos(t), \sin(3t), \sin(2t) \rangle$</td>
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<tr>
<td>$\vec{r}(u, v) = \langle (3 + \sin(u) \cos(v)) \cos(u), (3 + \sin(u) \cos(v)) \sin(u), (1 + \sin(u)) \sin(v) \rangle$</td>
<td></td>
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<tr>
<td>$\vec{F}(x, y, z) = \langle x, y^2, z^3 \rangle$</td>
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Problem 3) (10 points) No justifications are necessary

a) (3 points) Matching solids.
b) (3 points) Matching polar regions

\[
\begin{array}{|c|c|}
\hline
\text{Formula} & \text{Enter E,F,G,H} \\
\hline
r = \theta & \hline \\
| \sin(3\theta) | & \hline \\
\sin(\theta)^6 & \hline \\
1 + \cos(3\theta) & \hline \\
\hline
\end{array}
\]

E       F       G       H

\[r = \theta\]
\[| \sin(3\theta) |\]
\[\sin(\theta)^6\]
\[1 + \cos(3\theta)\]

\[f(x, y) = 5y - 1\]
\[f(x, y) = x^4 - y^4\]
\[f(x, y) = 3x^2 + y^2\]
\[f(x, y) = y^2 - x^3\]

\[f_{tt} = f_{xx}\]

The equation \(f_{tt} = f_{xx}\) is called:

\[
\begin{array}{|c|}
\hline
\text{Heat equation} & \hline \\
\text{Wave equation} & \hline \\
\text{Transport equation} & \hline \\
\text{Burgers equation} & \hline \\
\hline
\end{array}
\]
Problem 4) (10 points)

a) (3 points) Find a formula for the distance of a point \((x, y, z)\) to the \(xy\)-plane.

b) (3 points) Find a formula for the distance of a point \((x, y, z)\) to the \(z\)-axes.

c) (4 points) Find the surface consisting of all points \((x, y, z)\) for which the distance to the \(z\)-axes is the same than the distance to the \(xy\) plane.

Problem 5) (10 points)

a) (5 points) Estimate \(2.001^3 \cdot 0.9999^4 \cdot 0.999^2\) using linearization.

b) (5 points) Find the tangent plane to the surface \(x^3 y^4 z^2 = 8\) at the point \((2, 1, 1)\).

Problem 6) (10 points)

A **chicken coop** is made of two cubes of length \(x\) and \(y\). The volume of the house is \(f(x, y) = x^3 + y^3\). The surface area is \(g(x, y) = 5x^2 + 3y^2\). Using Lagrange, find the coop of maximal volume if the constraint is \(g = 38\).

Problem 7) (10 points)
The roof of the tower of the Harvard Lovell house has height

\[ f(x, y) = 1 - (x^2 + y^2)^7. \]

Find the volume under the roof above the disc \( x^2 + y^2 \leq 1 \) in the \( xy \)-plane.

**Problem 8) (10 points)**

What is the flux of the curl of the field \( \vec{F}(x, y, z) = \langle 0, z^2 + x^4, x \rangle \) through the shell

\[ \langle s(2 + \cos(t)) \cos(s), s(2 + \cos(t)) \sin(s), 6s + s \sin(t) \rangle, \]

where \( 0 \leq t \leq 2\pi \) and \( 0 \leq s \leq 6\pi \). The shell has the boundary curve

\[ \vec{r}(t) = \langle 6\pi(2 + \cos(t)), 0, 36\pi + 6\pi \sin(t) \rangle. \]

**Problem 9) (10 points)**

At which point does the function

\[ u(x, y) = \frac{2x^3}{3} + 2y^3 - \frac{x^6}{30} - \frac{y^5}{20} \]

have the property that

\[ f(x, y) = u_{xx}(x, y) + u_{yy}(x, y) \]

is extremal. Find and classify all the critical points of \( f(x, y) \).

**Remark you can ignore:** this problem appears in physics. If the function \( u \) is the electric potential, then \( f \) is the charge density. You find the place where the charge density is maximal.

**Problem 10) (10 points)**
The following integral gives the volume of a piece of **Swiss cheese**

\[
\int_0^3 \int_0^{\sqrt{y/3}} \int_{e^{-x^3}}^1 \, dz \, dx \, dy
\]

Find it.

---

**Problem 11** (10 points)

While Mars rover “**Curiosity**” was landing on Mars, a force

\[
\vec{F}(x, y, z) = (\sin(x), y, -30z)
\]

acted on the rover while it was ‘descending on the path

\[
\vec{r}(t) = (1, 2t, 10 - t^2)
\]

Find the line integral

\[
\int_0^2 \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt
\]

---

**Problem 12** (10 points)

Find the area of the **propeller** shaped region enclosed by the figure 8 curve

\[
\vec{r}(t) = (t - t^3, 2t^3 - 2t^5)
\]

parametrized by \(-1 \leq t \leq 1\). To find the total area compute the area of the region \(R\) enclosed by the right loop \(0 \leq t \leq 1\) and multiply by 2.

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**Problem 13** (10 points)
Find the flux of the vector field
\[ \vec{F}(x, y, z) = (-y^7, -x^8, -z + x^5) \]
through the surface given in spherical coordinates as
\[ \rho \leq (\sin(\phi) \cos(\phi) \cos^2(\theta))^{1/3} \]
with \(0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/2\). The surface is oriented outwards.