INTRODUCTION. Topics beyond multi-variable calculus are usually labeled with special names like "linear algebra", "numerical methods", "complex analysis", "differential equations", or "complex analysis" or "complex analysis". Where one would draw the line between calculus and non-calculus topics is not clear but if calculus is about learning the basics of limits, differentiation, integration and summation, then multi-variable calculus is the "black belt" of calculus. Are there other ways to play this sport?

HOW WOULD ALIENS COMPUTE? On another planet, calculus might be taught in a completely different way. The lack of fresh ideas in the current textbook offerings, where all books are essentially clones (*) of each other and where innovation is faked by ejecting new editions every year (which of course has the main purpose to prevent the retail of second hand books), one could think that in the rest of the universe, multi-variable calculus should be taught in the same way, and where chapter 12 of the calculus book is always the same and may look like a calculus text.

Actually, as we want to show here, even the human species has come up with a wealth of different ways to deal with calculus. It is very likely that calculus textbooks would look very different in other parts of our galaxy. This week broke the news that one has discovered a 5’th arm of the Milky way galaxy. It is 77,000 light years long and should increase the chance that there are other textbooks in our home galaxy. In this text, we want to give an idea that the calculus topics are not completely different. Actually, even numbers can be defined differently. John Conway introduced once numbers as pairs \((L|R)\) where \(L\) and \(R\) are sets of numbers defined previously. For example \(\theta(\emptyset) = 0\) and \(\{0\}\) = \(-1, 0, 1\). The advantage of this construction is that it allows to see "numbers" as part of "games". Donald E. Knuth, the giant of a computer scientist, who also designed "TI", a typesetting system in which this text is written, wrote a book called "surreal numbers" in which two student finds themselves on an island. They find a stone with the axioms for a new number system is written and develop from that an entirely new number system which contains the real line and more. The book is a unique case, where mathematical discovery is described as a novel. This is so totally different from what we know traditionally about numbers that one could expect Conway to be an alien himself if there were not many other proofs of his ingenious creativity.

(*) One of the few exceptions is maybe the book of Marsden and Tromba, which is original, precise and well written. It is unfortunately too mathematical for most calculus consumers and suffers from the same disease that other textbooks suffer from, the "calculus prime". A definite counterexample is a book by an amateur original and contains the essential staff. And it comes as a paperback. Together with a Schaum outline volume (also in paperback), it would suffice as a rudimentary textbook combination (and would cost together half and weight one fourth) of the standard door stoppers.

NONSTANDARD CALCULUS. At the time of Leonard Euler, people thought about calculus in a more intuitive, but less formal way. For example \(\frac{1 + x^2}{\sqrt{1 + x^2}}\) with infinitely large \(n\) would be perfectly fine. A modern approach which catches this spirit is "nonstandard analysis", where the notion of "infinitesimal" is given a precise meaning. The simplest approach is to extend the language and introduce infinitesimal as objects which are smaller than all standard objects. We say \(x \sim y\) if \(|x - y|\) is infinitesimal. All numbers are traditionally defined like \(\pi \sqrt{2}\) are "standard". The notion which tells that every bounded sequence has an accumulation point is expressed by the fact that there exits a finite set \(A\) such that all \(x \in \mathbb{R}\) are infinitesimally close to an element in \(A\). The fact that a continuous function on a compact set takes its maximum is seen by taking \(M = \max_{y \in A} f(y)\).

What impressed me as an undergraduate student learning nonstandard calculus (in a special course completely devoted to that subject) was the elegance of the language as well as the compactness in which the entire calculus story could be packed. For example, to express that a function \(f\) is continuous, one would say \(x \sim y\) then \(f(x) \sim f(y)\). This is more intuitive than the Weierstrass definition \(\forall \varepsilon > 0 \exists \delta > 0: |f(x) - f(y)| < \varepsilon \Rightarrow |x - y| < \delta\). Weierstrass definition is understood today primarily by intuition. To illustrate this, Ed Nelson, the founder of one of the nonstandard analysis flavors, asks the meaning of \(\exists \varepsilon > 0 \exists \varepsilon > 0 \delta > 0: |f(x) - f(y)| < \varepsilon \Rightarrow |x - y| < \delta\). To demonstrate how unintuitive this definition really is. The derivative of a function is defined as the standard part of \(f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}\). In order to define the derivative rigorously, one would have to use some of the foundations of mathematics to justify the game. It is also quite a sharp knife and its easy to cut the finger too long and doing mistakes. The name "nonstandard calculus" certainly was not fortunate too. People call it now "infinitesimal calculus". Introducing the subject using names like "hyper-real" and "using ultra filters" certainly did not help to promote the ideas (we usually also don’t teach calculus by introducing Dedekind cuts or completeness) but there are books like by Alain Robert which show that it is possible to teach nonstandard calculus in a natural way.

DISCRETE SPACE CALCULUS. Many ideas in calculus make sense in a discrete setup, where space is a graph, curves are in the graphs and surfaces are collections of "plaquettes", polygons formed by edges of the graph. Scalar functions on the graph are defined as functions on the vertices. Vector fields are functions defined on the edges, other vector fields are defined as functions defined on plaquettes. The gradient is a function defined on an edge as the difference between the values of \(f\) at the end points.

Consider a network modeled by a planar graph which forms triangles. A scalar function assigns a value \(f_n\) to each node. An area function assigns values \(f_T\) to each triangle \(T\). A vector field assigns values \(F_{Ed}\) to each edge connecting node \(n\) with node \(m\). The gradient of a scalar function is the vector field \(\nabla f = f_n - f_m\). The curl of a vector field \(F\) is attaches to each triangle \((k, m, n)\) the value \(curl^T F_{kmn} = F_{kn} + F_{nm} - F_{mk}\).

It is a measure of the circulation of the field around a triangle. A curve \(\gamma\) in our discrete world is a set of points \(r_{ij} = 1, \ldots, n\) such that nodes \(r_{i+1}\) and \(r_{i+2}\) are adjacent. For a vector field \(F\) and a curve \(\gamma\), the line integral is \(\sum_{i=1}^n F_{r_i r_{i+1}}\). A region \(R\) in the plane is a collection of triangles \(T\). The double integral of an area function \(f_T\) is \(\sum_{R \in T} f_T\). The boundary of a region is the set of edges which are only shared by one triangle. The orientation of \(\gamma\) is as usual. Greens theorem is now almost trivial. Summing up the curl over a region is the line integral along the boundary.

One can push the discretization further by assuming that the functions take values in a finite set. The integral theorems still work in that case too.

QUANTUM MULTIVARIABLE CALCULUS. Quantum calculus is "calculus without taking limits". There are indications that space and time look different at the microscopic scale, the Planck scale of the order. One of the ideas to deal with this situation is to introduce quantum calculus which comes in different types. We discuss q-calculus where the derivative is defined as \(D_q f(x) = f(qx) - f(x)\). There are quantum versions for differentiation. It has solutions \(f = 0\) which is quantum dierential equations \(QODE\). For functions \(f \leq M/\alpha^2\) with \(\alpha \leq 1\), the integral is defined with an anti-derivative, there are definite integrals. The fundamental theorem of\(q\)-calculus \(\int f(x) = \frac{D_q f(x) - f(0)}{\alpha}\) would be a lot more friendly to students there because there are no chain rules. Once, we can differentiate, we can take anti-derivatives. It is denoted by \(f(q^{-1})\). As we know the derivative of \(q^x\), we have \(f' q^x = f(q^x) + \frac{f(qx) - f(x)}{\alpha}\) which is quantum calculus. There are quantum versions for differentiation. One can look at functions on quantum calculus and differential equations can be defined too. A handicap is the lack of a chain rule. For example, to define a line integral, we would have to define \(\int f d\alpha\) in such a way that \(\int f d\alpha = \int (f(qx))\alpha\). In quantum calculus, the naive definition of the length of a curve depends on the parameterization of the curve. Surprises with \(\alpha\)-quantum differential equations \(QODE\) that \(f(0) = 0\) which is \(f(q^{-1})\)\(\alpha\) simplifies to \(f(x) = qf(x)\). It has solutions \(f(x) = qf(x)\). Many results generalize to \(q\)-calculus but the Taylor expansion is a lot more complicated.

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INFINITE DIMENSIONAL CALCULUS. Calculus in infinite dimensions is called functional analysis. Functions on infinite dimensional spaces which are also called functionals for which the gradient can be defined. The latter is the analogue of \(\partial f\). One does not always have \(\partial f = f'\). An example of an infinite dimensional space is the set \(X\) of all continuous functions on the unit interval. An example of a two dimensional surface in that space would be \(s(x) = \cos(x) \sin(x)^2 + \sin(x) \sin(x) \cos(x) + \cos(x)/2 + \alpha\). This surface is actually a two dimensional sphere. On this space \(X\) one can define a dot product \(S = \int (f(x)g(x))\). The theory which deals with the problem of extremizing functionals in infinite dimensions is called calculus of variations. There are problems in this field, which actually can be answered within the realm of multivariable calculus. For example: a classical problem to find among all closed regions with boundary of length 1 the one which has maximal area. The solution is the circle. One of the proofs which was found more than 100 years ago uses Greens theorem. One can also look at the problem to find the polygon with \(n\) edges and length 1 which has maximal area. This is a Lagrange extremization problem with regular polygons as solutions. In the limit when the number of points go to infinity, one obtains the isoperimetric inequality. Other problems in the calculus of variations are the search for the shortest path between two points in a hilly region. This shortest path is called a geodesic. Also here, one can find approximate solutions by considering polygons and solving an extremizing problem but direct methods in that theory are better.
GENERALIZED CALCULUS. In physics, one wants to deal with objects which are more general than functions. For example, the vector field $F(x, y) = (-y, x)(x^2 + y^2)$ has its curl concentrated on the origin $(0, 0)$. This is a function which is a distribution, a object with discontinuities. An example of such a Schwartz distribution is a "function" $f$ which is infinite at $0$, zero everywhere else, but which has the property that $f \, dx = 1$ is called the Dirac delta function. Mathematically, one defines distributions as a linear map on a space of smooth "test functions" $\phi$ which decay fast at infinity. One writes $(f, \phi)$ for this. For continuous functions one has $(f, \phi) = \int \limits_0^\infty f(x) \phi(x) \, dx$, for the Dirac distribution one has $(f, \phi) = (0)$. One would define the derivative of a distribution as $(f', \phi) = -\phi'$. For example, for the Heaviside function $H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$ one has $(H', \phi) = \int \limits_0^\infty \phi' \, dx$ and because $(H', \phi) = \int \limits_0^\infty \phi' \, dx = 0$ one has $(H', \phi) = 0$ the Dirac delta function.

FRACTAL CALCULUS. We have dealt in this course with $0$-dimensional objects (points), 1-dimensional objects (curves), 2-dimensional objects (surfaces) as well as 3-dimensional objects (solids). Since more than 100 years, mathematicians also studied fractals, objects with non-integer dimension which are called fractals. An example is the Koch snowflake which is obtained as a limit by repeated stellation of an equilateral triangle of initial arc-length $A = 3$. After one stellation, the height has increased by a factor $4/3$. After $n$ steps, the length of the curve is $A (4/3)^n$. The dimension is $\log(4/3) > 1$. There are things which do no more work here. For example, the length of this curve is infinite. Also the curve has no defined velocity at all places. One can ask whether one could apply Green's theorem still in this case. In some sense, this is possible. After every finite step of this construction one can compute the line integral along the curve and Green's theorem tells that this is a double integral of $\text{curl}(F)$ over the region enclosed by the curve. Since for larger and larger $n$, less and less region is added to the curve, the integral $\int \limits_0^\infty \text{curl}(F) \, dx$ is defined in the limit. So, one can define the line integral along the curve. Closely related to fractal theory is geometric measure theory, which is a generalization of differential geometry to surfaces which are no more smooth. Merging in ideas from generalized functions and differential forms, one defines currents, functionals on smooth differential forms. The theory is useful for studying minimal surfaces. Fractals appear naturally in differential equations as attractors. The most infamous fractal is the Mandelbrot set, which is defined as the set of complex numbers $c$ for which the iteration of the map $f(z) = z^2 + c$ starting with 0 leads to a bounded sequence $0 \rightarrow c \rightarrow c^2 \rightarrow (c^2 + c) \rightarrow \ldots$. The boundary of the Mandelbrot set is actually not a fractal. It is so wiggly that its dimension is actually 2. One can modify the Koch snowflake to get dimension 2 too. If one adds new triangles each time so that the length of the new curve is doubled each time, then the dimension of the Koch curve is 2 too.

COMPLEX CALCULUS. In the calculus is called complex analysis. Many things which are a bit mysterious in the real become more transparent when considered in the complex. For example, complex analysis here solves some problems which in the real and sometimes in the complex plane.

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THE FOUNDATIONS OF CALCULUS. Would it be possible that aliens somewhere else would build up mathematics radically different, by starting with a different axiom system? It is likely. The reason is that already we know that there is not a single way to build up mathematical truth. We have some choice: it came as a shock around the middle of the last century that for any strong enough mathematical theory, one is forced to follow is called ultra strict finitism.